Topics for Graduate Preliminary Exam in Topology

1 Basic Topology

1. Definition of topology; open and closed sets. Basic examples: trivial and discrete topologies, finite or countable complement topology.
2. Continuous functions; homeomorphisms; topological embeddings.
3. Metric Spaces and the metric topology; Euclidean space $\mathbb{R}^n$.
4. Subspace topology; universal property of subspace topology; glueing lemma. Key examples: intervals, spheres, open and closed balls in $\mathbb{R}^n$.
5. Closure, interior and boundary; limit points. convergent sequences.
6. Hausdorff spaces and their key properties.
7. Bases, subbases, and neighborhood bases.
8. Countability axioms: First and second countability; separability; Lindelöf property. Relationships between these.
9. Topological manifolds and manifolds with boundary. Key examples: open subsets of Euclidean space; graphs of functions, spheres, balls.

2 Constructing topological spaces

1. Product topology: Universal property and canonical projections. Contrast with box topology. Interaction with subspace topology.
2. Disjoint union: Universal property and canonical inclusions.
3. Quotient topology: Universal property and induced maps out of quotient spaces. Equivalence relations. Example quotients: circle, torus, projective space, collapse spaces, cone on a topological space, wedge sums.
4. Attaching spaces (glueing one space to another along a map). Example: double of a manifold with boundary.

3 Topological Groups

1. Definition of topological groups. Examples Euclidean space, nonzero complex numbers, circle, torus, general linear group, orthogonal group, unitary group.
2. Group actions. Types of actions: continuous, transitive, free.
3. Orbit space and quotient by group actions. Space of cosets.

4 Connectedness

1. Connected topological spaces. Key example: intervals.
2. Image of a connected space under a continuous map. Intermediate Value Theorem.
3. Path-connectedness. Non-example: topologist's sine curve
4. Unions, products, and quotients of (path) connected spaces.
5. Connected components and path components.
6. Local path-connectedness and its implications for components and path components. Key example: Manifolds.

5 Compactness

1. Compact topological spaces. Key example: closed intervals
2. Image of a compact space under a continuous map. Extreme value theorem.
3. Separation lemma for compact subsets of a Hausdorff spaced. Relationship between closed and compact.
4. Quotients, finite disjoint unions, and finite products of compact spaces; tube lemma.
5. Heine–Borel Theorem.
6. Tychonoff theorem on infinite products
7. Sequential and limit point compactness; implications between these and compactness.
8. Closed map lemma
9. Local compactness. Key example: manifolds
10. Baire Category Theorem
11. Paracompactness and compact exhaustions. Key example: manifolds.

12. Normal spaces and the Urysohn lemma. Bump functions.
13. Partitions of unity. Application: embedding compact manifolds in Euclidean space.
14. Proper maps; the proper closed map lemma.

6 Smooth manifolds

1. Smooth maps between open subset in Euclidean spaces.
2. Atlas of coordinate charts; transition functions. Smooth atlas, and smooth structures.
3. Smooth manifolds. Examples: open subsets, spheres, vector spaces, projective space.
4. Smooth manifolds with boundary. Example: closed balls.
5. Smooth maps between smooth manifolds. Coordinate representations; the ring of smooth functions; diffeomorphisms.
6. Smooth partitions of unity and smooth bump functions.
7. Tangent vectors: geometric tangent space; directional derivatives; derivations.
8. The tangent space to a manifold at a point; coordinate bases.
9. Differential of a smooth map; chain rule; coordinate representations and Jacobian matrix.
10. Tangent bundle: its natural smooth structure; global differential of a smooth map.
11. Velocity vectors of curves, geometric interpretation of tangent vectors.
12. Submersions and immersions. Examples: canonical submersion and canonical immersion.
13. The Inverse Function Theorem, and local diffeomorphisms. Local immersion and local submersion theorems.
14. Submanifolds and smooth embeddings.
15. Level sets of functions and the Regular Value Theorem.
16. Lie groups. Examples: Euclidean space, circle, torus, special linear groups, orthogonal group.
17. Vector fields and the Lie bracket.

7 Foundations for algebraic topology

1. Categories and functors: definition and key examples.
2. Homotopy: homotopic maps; homotopy equivalence; contractible spaces; deformation retractions.
3. Cell complexes. Examples: graphs; surfaces; spheres and $\mathbb{S}^\infty$, projective spaces and $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$. Quotient by collapsing a subcomplex.
4. Constructions: suspension; join; attaching spaces; mapping cylinder and mapping cone.
5. Homotopy extension property, and conditions for homotopy equivalence.

8 Fundamental group and covering spaces

1. Paths; homotopy of paths and concatenation of paths.
2. Fundamental group of a topological space; induced maps; functoriality and homotopy invariance.
3. Change of basepoint isomorphism.
4. Covering spaces; lifts; homotopy lifting property and unique path lifting.
5. The fundamental group of the circle. Applications: fundamental theorem of algebra; Brouwer fixed point theorem, Borsuk–Ulam theorem.
6. Van Kampen's theorem. Free products and amalgamated products / pushouts and their universal properties. Fundamental groups of closed surfaces.
7. The universal cover: construction and key properties.
8. Classification of covering spaces: Correspondence between covers and subgroups of fundamental group; the lifting criterion; dependence on basepoint and conjugate subgroups.
9. Deck transformation and normal covering spaces. Covering space actions.

9 Homology

1. Essential algebraic tools: chain complexes; chain maps; chain homotopy; exact sequences; split short exact sequences; the five lemma.
2. Δ–complexes and simplicial homology: the simplicial chain complex and boundary map.
3. Singular homology of topological spaces: singular simplices and the singular chain complex.

4. Functoriality and induced homomorphisms; homotopy invariance.
5. Short exact sequences of chain complexes and associated long exact sequence of homology groups.
6. Reduced homology; relative homology, the long exact sequence of a pair.
7. Excision and barycentric subdivision. Applications: local homology and invariance of domain; relationship between relative homology and homology of the quotient.
8. Mayer–Vietoris sequences.
9. Degree of maps between spheres, and local degree.
10. Equivalence of simplicial and singular homology.
11. Cellular homology and the cellular chain complex of a CW complex. Euler characteristic
12. Homology with coefficients
13. Key examples: Spheres; closed surfaces; wedge sums; suspensions; real and complex projective spaces; Moore spaces; Lens spaces

10 Cohomology

1. The dual cochain complex of a chain complex.
2. Universal coefficient theorem.
3. Cohomology of spaces: simplicial, singular, reduced, relative, and cellular.
4. Geometric interpretation of cochains, cocycles, and coboundaries, particularly in simplicial case and in low degree.
5. Cohomological analogs of main homology results: LES of a pair; excision; Mayer-Vietoris.
6. Cup product; interaction with induced maps; relative cup product. Key examples: closed surfaces, Moore spaces; projective spaces.
7. The cohomology ring; algebra of graded rings; polynomial rings.
8. Künneth formula for the cohomology ring of a product; tensor products and tensor products of graded rings. Examples: Tori and products of spheres.
9. Poincaré duality; local orientations and orientable manifolds; the fundamental class; the cap product. Key example: closed surfaces.

Recommended textbooks

Most topics are covered in one of the following books:

[1] John M. Lee, *Introduction to Topological Manifolds*, Springer, 2011.

[2] John M. Lee, *Introduction to Smooth Manifolds*, Springer, 2012.

[3] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2001.

Additional useful references:

[4] James Munkres, *Topology*, Prentice Hall, 2000.