Topics for Graduate Preliminary Exam in Algebra

1 Group theory

1. Groups: definitions, basic properties, and examples.
2. Subgroups and normal subgroups. Examples: normalizers, centralizers, and centers. Subgroups generated by a subset. Subgroups of cyclic groups.
3. Symmetric groups. Cycle decomposition and parity of permutations. Alternating groups.
4. Lagrange's theorem and its applications (Little Fermat theorem and Euler's theorem).
5. Homomorphisms of groups and their properties. Monomorphisms, epimorphisms, isomorphisms, and automorphisms. Classification of cyclic groups up to isomorphism.
6. Kernels of homomorphisms. Cosets and quotient groups. Isomorphism theorems.
7. Direct products and direct sums of groups. Basic properties and examples.
8. Group actions on sets. Stabilizers. Transitive, free, and regular actions. Cayley's theorem.
9. The orbit-stabilizer theorem and its applications to finite p-groups.
10. Sylow theorems. Examples of Sylow subgroups in cyclic groups, $GL_n(\mathbb{Z}_p)$, and S_p for a prime p.
11. Extensions of groups. Split extensions. Classification of groups of order pq.
12. Abelian groups. Free abelian groups and their bases. Subgroups of free abelian groups.
13. The decomposition theorem for finitely generated abelian groups.
14. Simple groups. Simplicity of A_n for $n \geq 5$ and projective special linear groups.
15. Solvable groups and their basic properties. Solvability of triangular matrix groups. The Feit–Thompson theorem (without proof).
16. Free groups: constructive definition and universal property.
17. Group presentations.

2 Basic ring and field theory

1. Rings and subrings. Basic properties and examples. Zero divisors and units.
2. Integral domains and fields. Characteristic. Fields of fractions.
3. Ring homomorphisms, ideals, and quotient rings. Isomorphism theorems for rings.
4. Polynomial rings. Polynomials over a field and the division algorithm. Applications: Bézout's theorem, the number of roots of a polynomial over a field, polynomials versus polynomial functions. Multiplicative groups of finite fields.
5. Divisibility in rings. Greatest common divisors and least common multiples. Computing the gcd of polynomials over a field. Application: all ideals in $K[x]$, where K is a field, are principal.
6. Reducible and irreducible polynomials over a field. Factorization into irreducible polynomials. Quotients by ideals generated by irreducible polynomials.
7. Irreducibility over $\mathbb{Z}$ and $\mathbb{Q}$. Gauss's lemma and Eisenstein's criterion.
8. The fundamental theorem of algebra. Classification of irreducible polynomials over $\mathbb{R}$.

3 Field extensions and Galois theory

1. Algebraic and transcendental extensions. Classification of simple extensions.
2. Splitting fields of polynomials: existence and uniqueness.
3. Classification of finite fields and their subfields. Frobenius automorphism.
4. Algebraic closures. Existence and uniqueness.
5. Ruler-and-compass constructions. Doubling the cube and constructing the regular pentagon.
6. Normal extensions: equivalent definitions and examples. Galois groups.
7. Separable polynomials. Separability over finite fields and fields of characteristic 0.
8. Separable extensions and Artin's primitive element theorem.
9. Galois extensions. Galois groups of polynomials. Orders of Galois groups of finite extensions.
10. The fundamental theorem of Galois theory.
11. Solvability in radicals. Unsolvability of general equations of degree 5.

4 Advanced topics in ring theory

1. Euclidean domains. Examples: polynomial rings and Gaussian integers.
2. Principal ideal domains and their relationship to Euclidean domains.
3. Irreducible and prime elements in integral domains. Associates. Unique factorization domains. Examples.
4. Polynomial rings over UFDs. Relationships between PIDs and UFDs.
5. Noetherian rings. The Hilbert basis theorem.
6. Finite division rings. Little Wedderburn theorem.

5 Modules

1. Modules over unital rings. Submodules. Basic properties and examples. Homomorphisms, kernels, and quotients.
2. Free modules and their submodules. Torsion-free modules over PIDs.
3. Decomposition theorem for finitely generated modules over PIDs.
4. Normal forms. Minimal and characteristic polynomials of matrices. Cayley–Hamilton theorem.

6 Associative algebras

1. Associative algebras over fields. Subalgebras, ideals, homomorphisms, and quotients.
2. Dimension of an algebra. Finite dimensional algebras and their matrix representations.
3. Division algebras. Finite dimensional division algebras and zero divisors. Finite dimensional division algebras over algebraically closed fields.
4. Real division algebras. Quaternions. Frobenius Theorem.
5. Central simple algebras. Simplicity of matrix algebras. The Wedderburn–Artin Theorem.

Recommended textbooks

Most topics are covered in one of the following books:

[1] J. Rotman, *A First Course in Abstract Algebra*, Pearson, 2005.

[2] J. B. Fraleigh, *A First Course in Abstract Algebra*, Pearson, 2002.

[3] J. Gallian, *Contemporary Abstract Algebra*, Brooks/Cole, 2012.

Additional useful references:

[4] S. Lang, *Algebra*, Springer, 2005.

[5] B. L. van der Waerden, *Modern Algebra*, Frederick Ungar Publishing Co., 1949.