Topology Preliminary Exam

August 12, 2025

Instructions:

- Write **only your PIN** (and NOT your name) on your exam booklet.
- You have **3 hours** to complete this exam.
- Work **exactly 6** of the following **7** problems. Clearly indicate which problems you are submitting for grading.
- Each problem is to be worked on a separate page with the problem number clearly listed at the top of the page.
- All problems carry the same value.
- *If you must use a main theorem from the course, be sure to state it correctly when you use it.*

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1. Let $p\colon X \to Y$ be a covering map with Y Hausdorff. Prove that X is compact if and only if Y is compact and $p^{-1}(y)$ is is finite for each $y \in Y$.

2. Let $X_1, X_2, \dots$ be topological spaces, and consider their Cartesian product $Y = \prod_{n\in\mathbb{N}} X_n$.
   (a) **State** and **Prove** the *Universal Property* of the product topology on Y.
   (b) Now fix a topological space X and let $Y = \prod_{n\in\mathbb{N}} X$ be the infinite product. Prove that the diagonal map $f\colon X \to Y$ given by $f(x) = (x, x, x, \dots)$ is continuous.
   (c) Give an example of a space X such that part (b) fails when the product $Y = \prod_{n\in\mathbb{N}} X$ is instead given the *box topology.*

3. Suppose G is a group acting by homeomorphisms on a topological space X.
   (a) Prove the quotient map $X \to X/G$ to the orbit space is an open map.
   (b) Prove that X/G is Hausdorff if and only if the orbit relation

$$\mathcal{O} = \{(x, y) \mid y = g \cdot x \text{ for some } g \in G\} \subset X \times X$$

   is closed in $X \times X$.

4. Use tools from algebraic topology to prove the *Brouwer fixed point theorem*: If

$$\mathbb{D}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

denotes the closed unit ball of dimension $n \geq 1$, then any continuous map $f\colon \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

5. Suppose a group G acts as a covering space action on a simply connected space X, so that the quotient $p\colon X \to X/G$ is a covering map. Set $Y = X/G$. Fix a basepoint $x_0 \in X$, and set $y_0 = p(x_0) \in Y$. We may identify G with the fundamental group $\pi_1(Y, y_0)$ by associating $g \in G$ to the homotopy class of the loop $p \circ \gamma$, where $\gamma\colon [0,1] \to X$ is any path from x_0 to $g \cdot x_0$. Now observe that $G \cong \pi_1(Y, y_0)$ acts on $p^{-1}(y_0) = G \cdot x_0$ in two ways: 1) by deck transformations, and 2) by lifting loops (that is: $[\gamma] \in \pi_1(Y, y_0)$ sends $z \in p^{-1}(y_0)$ to the other endpoint of a lift $\tilde{\gamma}$ of γ based at z).
   (a) Show that if G is abelian, then these two actions of G on $p^{-1}(y_0)$ agree.
   (b) Give an example where this fails if G is not abelian.

Definition: The *connected sum* of two n-manifolds M_1, M_2 is the space $M_1 \# M_2$ formed as follows: Let U_i be an embedded open ball in M_i (i.e. the image of an embedding $\mathbb{B}^n \hookrightarrow M_i$ of the open unit ball). The complement $N_i = M_i \setminus U_i$ is then a manifold with boundary $\partial N_i \cong \mathbb{S}^{n-1}$, and we form $M_1 \# M_2$ by gluing N_1 and N_2 together via a (orientation reversing) homeomorphism $\partial N_1 \to \partial N_1$ of their boundaries.

6. Let $T = \mathbb{S}^1 \times \mathbb{S}^1$ the 2–torus and $P = \mathbb{R}P^2$ the projective plane. Then let $X = T \# P$ be their connected sum.
   (a) Use the van Kampen Theorem to compute a presentation for $\pi_1(X)$.
   (b) Use the Mayer–Vietoris Theorem to compute the homology groups $H_i(X)$ for $i \geq 0$.

7. Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the 2–sphere, equipped with the smooth atlas of consisting of two stereographic projection charts $\varphi_\pm\colon U_\pm \to \mathbb{R}^2$, defined on the complements $U_\pm = \mathbb{S}^2 \setminus \{(0, 0, \pm 1)\}$ of the north and south poles, where the charts $\varphi_\pm$ and their inverses $\psi_\pm = \varphi_\pm^{-1}$ are given by the explicit formulas

$$\varphi_\pm(x, y, z) = \frac{(x, y)}{1 - \pm z} \quad \text{for } (x, y, z) \in U_\pm$$
$$\psi_\pm(u, v) = \frac{(2u, 2v, \pm(u^2 + v^2 - 1))}{u^2 + v^2 + 1}, \quad \text{for } (u, v) \in \mathbb{R}^2.$$

   (a) Let $q = (1, 0) \in \mathbb{R}^2$ and $p = \psi_+(p) = (1, 0, 0) \in \mathbb{S}^2$. Consider the tangent vector $\lambda = \frac{\partial}{\partial u}|_q - \frac{\partial}{\partial v}|_q$ at q, and let $\mu = (d\psi_+)(\lambda) \in T_p\mathbb{S}^2$ be its image under the differential of ψ_+. Let $f\colon \mathbb{S}^2 \to \mathbb{R}$ be the function given by $f(x, y, z) = xyz$. Calculate $\mu(f)$ (that is, the value that the derivation μ assigns to the smooth function f).
   (b) Let $E = \{(x, y, z) \in \mathbb{S}^2 \mid z = 0\}$ be the equator. Use the *Regular Value Theorem* to prove that E is a submanifold of $\mathbb{S}^2$.
   (c) Is the vector $\mu = (d\psi_+)(\lambda) \in T_p\mathbb{S}^2$ above tangent to the submanifold E?