MODERN ALGEBRA −2016

You have 3 hours to complete this exam. You should solve at least 5 *problems of the following 7 problems. Each problem is to be worked on a separate sheet of paper with the problem number clearly listed at the top of the page.*

1. Does there exist a non-abelian group of order n^2 for
(a) $n = 4$,
(b) $n = 11$,
(c) $n = 35$?

2. Let A be a free abelian group of rank 2016.
(a) How many pairwise non-isomorphic subgroups are there in A ?
(b) Prove that every abelian group of order $\leq 2^{2016}$ is a homomorphic image of A.

3. Is it true that
(a) every prime element of a domain is irreducible ?
(b) every irreducible element of a domain is prime ?
Explain please.

4. Let K and L be finite Galois fields of orders 3^n and 3^{2n}, respectively, where n is a positive odd integer. Prove that L is isomorphic to $K[\theta]$, where θ is a root of the polynomial $x^2 + 1$.

5. Let E be a splitting field of an irreducible polynomial $f(x) \in \mathbf{Q}$, and $Gal(E/\mathbf{Q})$ is isomorphic to the quaternion group of order 8.
(a) Prove that the degree of $f(x)$ is greater than 4.
(b) Prove that $\deg f(x) = 8$.
(c) Prove that every subfield $L \subset Q$ is a splitting field of some polynomial over $\mathbf{Q}$.

6. Do there exist non-isomorphic finite groups with isomorphic complex group algebras ? Explain please.

7. Prove that there exist non-equivalent 3-dimensional, irreducible, complex representations of the symmetric group S_4.