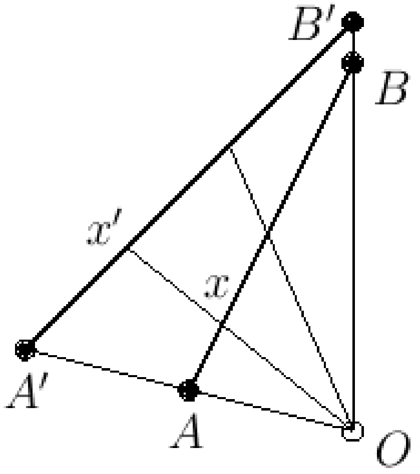
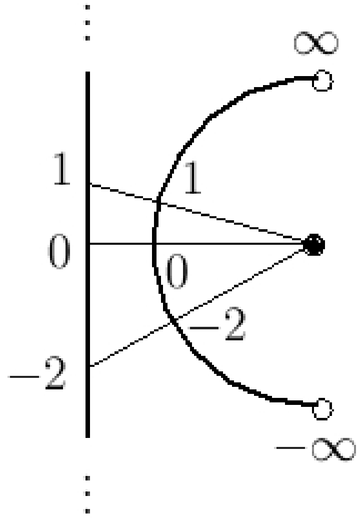
MATH 280 — Set Theory (and metric spaces) — some highlights of the course:

Cardinality: We say that the sequence $1, 2, 3, \ldots$ never gets to infinity; thus we are considering a "potential infinity." The notion of an "actual" or "completed infinity" is conceptually harder. This difficulty is part of why the ancient Greek mathematicians turned to geometry rather than algebra; numerical approximations to π or $\sqrt{2}$ "never quite get there." The difficulty was finally overcome by Georg Cantor near the end of the 19th century. In modern notation, all we really need to do is add a pair of braces: the set $\{1, 2, 3, \ldots\}$ is just *one* object, a set with infinitely many members, so it is a completed infinity. But Cantor did a lot more than add braces. Some of his proofs are presented in Math 280.

Two sets are said to *have the same cardinality* if they can be put into one-to-one correspondence. For instance, the first illustration below shows that any two line segments AB and $A'B'$ (including their endpoints) have the same cardinality, regardless of length. The second illustration shows that a semicircle (with endpoints omitted) with finite length has the same cardinality as an entire line (with infinite length).





A set is called *countable* if its cardinality is less than or equal to that of the integers. Surprising, the set $\mathbb{Q}$ of all rational numbers is countable, but the set $\mathbb{R}$ of all real numbers is not. Thus $|\mathbb{N}| = |\mathbb{Q}| < |\mathbb{R}|$. Moreover, any set is smaller in cardinality than its powerset. Hence $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$, so there are *infinitely many* different kinds of infinity! (But most mathematicians have little use for any but the two lowest infinities.) Cantor conjectured, but could not prove, the *Continuum Hypothesis* (there are no cardinalities between between $|\mathbb{N}|$ and $|\mathbb{R}|$) and the *Generalized Continuum Hypothesis* (when X is an infinite set, there are no cardinalities between between $|X|$ and $|\mathcal{P}(X)|$). Are these true? The answer is not simply "yes" or "no"; it depends on what kind of set theory we choose to use. The two halves of that answer were determined by Gödel in the 1930's and Cohen in the 1960's. Their proofs are beyond the scope of Math 280, but at least we can discuss their conclusions.

The Axiom of Choice and related topics: "The Axiom of Choice is obviously true; the Well Ordering Principle is obviously false; and who can tell about Zorn's Lemma?" That's a *joke.* In the setting of ordinary set theory, all three of those principles are mathematically equivalent – i.e., if we assume any one of those principles, we can use it to prove the other two. (The proofs are sometimes given in Math 280.) However, human intuition does not always follow what is mathematically correct. The Axiom of Choice agrees with the intuition of most mathematicians; the Well Ordering

Principle is contrary to the intuition of most mathematicians; and Zorn's Lemma is so complicated that most mathematicians are not able to form any intuitive opinion about it.

The Axiom of Choice, in one of its forms, says that

> If C is a collection of nonempty sets, then we can choose a member from each set in that collection. In other words, there exists a function f defined on C with the property that, for each set S in the collection, $f(S)$ is a member of S.

In some ways that sounds obvious: just *pick any member* from each nonempty set. But that is not an explicit rule, procedure, or algorithm. In some cases an explicit rule is available (e.g., if we're working with nonempty subsets of $\mathbb{N}$, just pick the *lowest* member of each set), but in other cases we cannot find such an f (e.g., if we're working with nonempty subsets of $\mathbb{R}$). In those cases, does f "exist"? Philosophically, constructivists reject the Axiom of Choice. When we accept the Axiom of Choice (as most mathematicians do), this merely means we are agreeing to the *convention* that we shall permit ourselves to use a choice function f in proofs, *as if* it "exists" in some sense, without concerning ourselves about what kind of existence it has or whether we can find it. This simplifies many explanations in mathematics, but it also has some bizarre consequences — e.g., the existence (but not the construction!) of the paradoxical Banach-Tarski decomposition of the sphere.

The real numbers: What are they, really? A chief difference between $\mathbb{Q}$ and $\mathbb{R}$ is the presence of "holes" in the rationals. For instance,

$$3, \qquad 3.1, \qquad 3.14, \qquad 3.141, \qquad 3.1415, \qquad 3.14159, \qquad 3.141592, \qquad \ldots$$

is a Cauchy sequence of rationals — i.e., a sequence of rational numbers that get closer and closer together — but there isn't any rational number π that they are getting closer *to*. There is a hole in the rational number system, right where π "ought to be." By filling in all the holes, we get the reals, the *completion* of the rationals.

Metric spaces: A *metric* on a set X is a function d giving a *distance* $d(a,b)$ between each two members $a, b \in X$. This is a nonnegative real number, satisfying these three axioms:

$$d(a,b) = 0 \;\Leftrightarrow\; a = b, \qquad d(a,b) = d(b,a); \qquad d(a,b) \le d(a,c) + d(z,c).$$

Different metrics are used for different purposes. For instance, on $\mathbb{R}^2$, the Euclidean plane, here are metrics measuring (respectively) distance as the taxicab drives and as the crow flies:

$$d_1\Big((x_1,y_1),\,(x_2,y_2)\Big) = |x_1 - x_2| + |y_1 - y_2|, \qquad d_2\Big((x_1,y_1),\,(x_2,y_2)\Big) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}.$$

Here are three metrics on the set $C[0,1] = \{$continuous functions from $[0,1]$ to $\mathbb{R}\}$:

$$D_1(f,g) = \int_0^1 |f(t) - g(t)|\,dt, \qquad D_2(f,g) = \sqrt{\int_0^1 |f(t) - g(t)|^2\,dt}, \qquad D_\infty(f,g) = \max_{t\in[0,1]} |f(t) - g(t)|.$$

Metric spaces are the most elementary of topological spaces, and we study some of their topological properties in Math 280. For instance, a sequence (x_n) is *Cauchy* if $\lim_{\min\{m,n\}\to\infty} d(x_m, x_n) = 0$; a metric space is *complete* if every Cauchy sequence in X is convergent. $C[0,1]$ is complete with D_∞ but not with D_1 or D_2. (In fact, the completion of $\Big(C[0,1], D_1\Big)$ turns out to be $L^1[0,1]$, the space of *Lebesgue integrable* functions, but generally those are not studied until the first year of graduate mathematics.)