MATH 250 — Introduction to Mathematical Logic.

Most intermediate and advanced mathematics courses involve proving theorems. Mathematical logic is a formal analysis of proofs. If starts by formalizing language. For instance, the everyday English, the statement

> if it is sunny in the morning when I leave my house, then I do not take along my umbrella

is equivalent to

> if I have my umbrella with me during the day, then it was not sunny in the morning when I left my house.

This equivalence can be represented symbolically in a formal language as $(A \to \neg B) \leftrightarrow (B \to \neg A)$ where the associations of symbols to meanings are:

$$\neg \text{ (not)}, \qquad \to \text{ (implies)}, \qquad \leftrightarrow \text{ (is equivalent to)}.$$

Formal statements like this (along with rules used to deduce other statements) are referred to as *syntax*. The meanings behind the symbols, such as A means "I have my umbrella with me during the day" and can either be true or false, are referred to as *semantics*. Mathematical logic studies the syntactic approach, the semantic approach, and the connection between the two.

- Syntactic logic is concerned with *deductions*, or formal proofs, transforming one or more formulas into other formulas, with the aid of *axioms* (assumed formulas). The *theorems* are the deduced formulas, i.e., those that can be derived starting from just the axioms.
- Semantic logic is concerned with the evaluation of formulas. For instance, in the classical two-valued semantics, the only possible values are false and true, which can be represented numerically by 0 and 1. In a fuzzy logic, typically, the values are all the numbers in the interval $[0, 1]$, with 1 being the only value that is "completely true." In a relevant logic or a comparative logic, one might use for values all the integers $(\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots)$, with higher values being "more true"; there is no "truest" value. For the values in constructive logic one might use sets of numbers — more specifically, the open subsets of the real line; larger sets are "more true," and the entire line is the only set that is "completely true." A typical question in semantic logic is, which formulas are always true?
- Mathematical modeling is concerned with pairing up syntactic logic with semantic logic. Typical questions about these pairings are:
  - Are things that can be proved syntactically, always true semantically? (*soundness*)
  - If something is always true semantically, can it be proved syntactically? (*completeness*)

The examples used to convey the concepts laid out above –syntax, semantics, deduction, axioms, theorems, soundness, completeness – will depend on the tastes and pedagogical philosophy of your instructor.