Technical Appendix to Accompany
“Endogenous Growth Through
Investment-Specific Technological Change”

Gregory W. Huffman∗

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1 Derivation of various results

It is necessary to derive various formulae that are present in the text of the paper. This will be done in a series of results. First of all, it will be useful to let \( \Omega = V_t I_t^{-\alpha} \), \( \phi = V_t^{\alpha-1} k_{t+1} \) and both of these are constant in the steady-state. Obviously there is a link between the two, and this can be written as follows:

\[
\Omega = \phi^{-\alpha} (1+g) \left[ (1+g)^{1/\alpha} - 1 + \delta \right]^{-\alpha}.
\]

Throughout the following analysis it will be assumed that \( \theta = \left( \frac{a-1}{\alpha} \right) \), which is a necessary condition for balanced growth.

Then throughout remainder of this appendix, a series of the results will be derived.

**Result #1**

We have the following characterization for the price of capital

\[
q_t = B \left[ (1-\theta) I_t^{\rho/(1-\theta)} + \theta V_t^\rho \right]^{1-\rho} \left( I_t^{1-\rho} - I_t^{\rho-1} \right)
\]

\[
= B \left[ (1-\theta) + \theta V_t^\rho I_t^{-\rho/(1-\theta)} \right]^{1-\rho} \left( I_t^{1-\rho} - I_t^{\rho-1} \right) \left( I_t^{1-\rho} - I_t^{\rho-1} \right)
\]

*Vanderbilt University*
\[
\begin{align*}
\frac{q_{t+1}}{q_t} &= \left( \frac{I_{t+1}}{I_t} \right)^{\frac{1}{r}} = \left[ (1 + g)^{1/\alpha} \right]^{1/r} = (1 + g)^{\frac{1}{\alpha}} = (1 + g)^{\theta}.
\end{align*}
\]

**Result #2**

This latter result implies that

\[
\frac{q_{t+1}}{q_t} = \left( \frac{I_{t+1}}{I_t} \right)^{\frac{1}{r}} = \left[ (1 + g)^{1/\alpha} \right]^{1/r} = (1 + g)^{\frac{1}{\alpha}} = (1 + g)^{\theta}.
\]

**Result #3**

We also have the following for the return to research spending

\[
R_t = -\theta B V_t^{\rho - 1} \left[ (1 - \theta) I_t^{\rho/(1-\theta)} + \theta V_t^{\rho} \right]^{1-\rho} - \theta B V_t^{\rho - 1} B \left[ (1 - \theta) + \theta V_t^{\rho} I_t^{\rho/(1-\theta)} \right]^{1-\rho} \left( I_t^{1-\rho} - \rho \right)
\]

\[
= -\theta B \left[ (1 - \theta) + \theta \left( V_t I_t^{\alpha} \right)^{\rho} \left( I_t^{1-\rho} \right) \right]^{1-\rho} \left( I_t^{1-\rho} \right) V_t^{\rho - 1}
\]

\[
= -\theta B \left[ (1 - \theta) + \theta \left( V_t I_t^{\alpha} \right)^{\rho} \left( V_t I_t^{1-\rho} \right) \right]^{\rho - 1}
\]

\[
= -\theta B \left[ (1 - \theta) + \theta \left( V_t I_t^{\alpha} \right)^{\rho} \left( V_t I_t^{1-\rho} \right) \right]^{\rho - 1}.
\]

**Result #4**

We also have

\[
(\alpha A) \frac{k_{t+1}^{\alpha-1}}{q_t} = k_{t+1}^{\alpha-1} \left( \frac{\alpha A}{B} \right) \left[ (1 - \theta) I_t^{\rho/(1-\theta)} + \theta V_t^{\rho} \right]^{\frac{\rho - 1}{\rho}} \left( I_t^{1-\rho} \right)
\]

\[
= k_{t+1}^{\alpha-1} \left( \frac{\alpha A}{B} \right) \left[ (1 - \theta) + \theta V_t^{\rho} I_t^{\rho/(1-\theta)} \right]^{\frac{\rho - 1}{\rho}} \left( I_t^{1-\rho} \right) k_{t+1}^{\alpha-1}
\]

\[
= \left( \frac{\alpha A}{B} \right) \left[ (1 - \theta) + \theta \left( V_t I_t^{\alpha} \right) \right]^{\frac{\rho - 1}{\rho}} \left( I_t^{1-\rho} \right) k_{t+1}^{\alpha-1}
\]

\[
= \left( \frac{\alpha A}{B} \right) \left[ (1 - \theta) + \theta \left( I_t^{\rho} \right) \right]^{\frac{\rho - 1}{\rho}} \left( I_t^{1-\rho} \right) k_{t+1}^{\alpha-1}
\]

\[
= \left( \frac{\alpha A}{B} \right) \left[ (1 - \theta) + \theta \left( I_t^{\rho} \right) \right]^{\frac{\rho - 1}{\rho}} \left[ k_{t+1} - (1 - \delta) k_t \right]^{1-\alpha} k_{t+1}^{\alpha-1}
\]
\[
\Psi(i_t, V_t) = \left(\frac{\alpha A}{B}\right) \left[(1 - \theta) + \theta (\Omega)^{\rho} \right]^{\frac{\alpha - 1}{\rho}} \left[1 - (1 - \delta)/(1 + g_k)\right]^{1-\alpha}
\]

\[
\frac{\Psi(i_t, V_t)}{y} = \frac{B \left[(1 - \theta) I_t^{\rho} + \theta (V_t)^{\rho}\right]^{1/\rho}}{A k_t^{\alpha}}
\]

\[
= \frac{B \left[(1 - \theta) + \theta (V_t I_t^{-\alpha})^{\rho}\right]^{1/\rho} I_t^{\alpha}}{A k_t^{\alpha}}
\]

\[
= \frac{B \left[(1 - \theta) + \theta (\Omega)^{\rho} \right]^{1/\rho} [k_{t+1} - (1 - \delta)k_t]^{\alpha}}{A k_t^{\alpha}}
\]

\[
= \frac{B \left[(1 - \theta) + \theta (\Omega)^{\rho} \right]^{1/\rho} \left[(1 + g)\right]^{1/\alpha - (1 - \delta)^{\alpha}}}{A}.
\]

**Result #5**

The investment-output ratio is calculated as follows:

\[
\Psi(i_t, V_t) = \left(\frac{\alpha A}{B}\right) \left[(1 - \theta) + \theta (\Omega)^{\rho} \right]^{\frac{\alpha - 1}{\rho}} \left[1 - (1 - \delta)/(1 + g_k)\right]^{1-\alpha}
\]

\[
\frac{\Psi(i_t, V_t)}{y} = \left(\frac{\alpha A}{B}\right) \left[(1 - \theta) + \theta (\Omega)^{\rho} \right]^{\frac{\alpha - 1}{\rho}} \left[1 - (1 - \delta)/(1 + g_k)\right]^{1-\alpha} (1 + g)^{\frac{\alpha - 1}{\alpha}}
\]

\[
= \left(\frac{\alpha A}{B}\right) \left[(1 - \theta) + \theta (\Omega)^{\rho} \right]^{\frac{\alpha - 1}{\rho}} \left[\phi^{\alpha \theta - \alpha} [1 + g^{1/\alpha} - (1 - \delta)^{\alpha}]^{1 + g}\right]^{\frac{\alpha - 1}{\alpha}}
\]

\[
= \left(\frac{\alpha A}{B}\right) \left[(1 - \theta) + \theta (\Omega)^{\rho} \right]^{\frac{\alpha - 1}{\rho}} \left[\phi^{\alpha} [1 + g^{1/\alpha} - (1 - \delta)^{\alpha}]^{1 + g}\right]^{\frac{\alpha - 1}{\alpha} \phi^{\alpha - 1}}
\]

\[
= \left(\frac{\alpha A}{B}\right) \left[(1 - \theta) + \theta (\Omega)^{\rho} \right]^{\frac{\alpha - 1}{\rho}} \left[\Omega^{\frac{\alpha - 1}{\alpha} \phi^{\alpha - 1}}\right]
\]

**Result #6**

The research-output ratio is calculated as follows. Let

\[
I_t = k_{t+1} - (1 - \delta)k_t
\]

\[
= \left[(1 + g)^{1/\alpha} - (1 - \delta)\right]k_t
\]

so that

\[
\frac{V_t}{Ak_t^{\alpha}} = \frac{V_t k_t^{-\alpha}}{A}
\]
\[
\frac{1}{\alpha} \left( V_t k_t^{-\alpha} \right) \left[ (1 + g)^{1/\alpha} - (1 - \delta) \right]^\alpha \\
= \frac{\Omega}{\alpha} \left[ (1 + g)^{1/\alpha} - (1 - \delta) \right]^\alpha
\]

**Result #7**

The ratio of research spending to output is calculated as follows. The law of motion for research knowledge is the following:

\[
V_{t+1} = V_t + v_t
\]

so

\[
\frac{V_{t+1}}{V_t} = (1 + g) = 1 + \frac{v_t}{V_t}
\]

and hence

\[
\frac{v_t}{V_t} = g
\]

Therefore we have

\[
\frac{v_t}{Ak_t^\alpha} = \frac{v_t}{V_t} \frac{V_t}{Ak_t^\alpha} = g \left( \frac{\Omega}{A} \right) \left[ (1 + g)^{1/\alpha} - (1 - \delta) \right]^\alpha.
\]

**2 The Model Without Taxes**

Now the first optimization condition for the model without taxes can then be re-written as follows:

\[
\left[ i_t \left( \frac{\rho}{1-\sigma} \right)^{-1} \right] B \left[ (1 - \theta)i_t \left( \frac{\rho}{1-\sigma} \right)^{1/\rho} + \theta(V_t)^{\rho} \right]^{(1/\rho)-1} (c_t)^{-\sigma} =
\]

\[
\beta(c_{t+1})^{-\sigma} \left[ A \alpha k_{t+1}^{\alpha-1} + (1 - \delta) \left[ (c_{t+1})^{(\frac{\rho}{\sigma-\rho})^{-1}} \right] B \left[ (1 - \theta)i_t \left( \frac{\rho}{1-\sigma} \right)^{1/\rho} + \theta(V_{t+1})^{\rho} \right]^{(1/\rho)-1} \right]
\]

or

\[
(c_t)^{-\sigma} = \beta(c_{t+1})^{-\sigma} \left[ \frac{A \alpha k_{t+1}^{\alpha-1}}{q_t} + (1 - \delta) \left( \frac{q_{t+1}}{q_t} \right) \right]
\]
where
\[
q_t \equiv \frac{\partial \Psi(i_t, V_t)}{\partial i_t} = \left[ i_t^{(\rho)^{-1}} \right] B \left[ (1 - \theta) i_t^{(\rho^\alpha - 1)} + \theta(V_t)^\rho \right]^{(1/\rho) - 1}
\]

This can condition can then be written as
\[
(1 + g)^\sigma = \beta \left[ \frac{A \alpha k_{t+1}^\rho}{q_t} + (1 - \delta) \left( \frac{q_{t+1}}{q_t} \right) \right]
\]

Using results 2 and 4 yields
\[
(1 + g)^\sigma = \beta \left[ \left( \frac{\alpha A}{B} \right) [(1 - \theta) + \theta (\Omega)]^{\rho^\alpha - 1} \left[ \Omega \right]^{\rho^\alpha - 1} + (1 - \delta)(1 + g)^\rho \right] (1)
\]

The second optimization condition can be written as follows:
\[
(c_t)^{-\sigma} = \sum_{i=1}^\infty \beta^i (c_{t+i})^{-\sigma} (-B\theta)(V_{t+i}^{\rho^\alpha - 1}) \left[ (1 - \theta) i_{t+i}^{(\rho^\alpha - 1)} + \theta(V_{t+i})^\rho \right]^{(1/\rho) - 1}
\]

where
\[
R_{t+i} = (-B\theta)(V_{t+i}^{\rho^\alpha - 1}) \left[ (1 - \theta) i_{t+i}^{(\rho^\alpha - 1)} + \theta(V_{t+i})^\rho \right]^{(1/\rho) - 1}.
\]

Using result 3, this can be further re-written as follows
\[
1 = \sum_{i=1}^\infty \beta^i (1 + g)^{-(i\sigma)} (-B\theta) [(1 - \theta) + \theta (\Omega)]^{\rho^\alpha} (\Omega)^{\rho^\alpha - 1}
\]
or
\[
1 = \frac{\beta(1 + g)^{-(i\sigma)} (-B\theta) [(1 - \theta) + \theta (\Omega)]^{\rho^\alpha} (\Omega)^{\rho^\alpha - 1}}{1 - \beta(1 + g)^{\sigma}}. (2)
\]
3 The Model With Taxes

The first equilibrium condition for the model with taxes can be written as follows:

\[ q_t (1 - \tau_i) (c_t)^{-\sigma} = \beta (c_{t+1})^{-\sigma} [r_{t+1} (1 - \tau_k) + (\tau_k \delta q_t) + (1 - \delta) (1 - \tau_i) q_{t+1}] . \]

This can be further re-written as follows:

\[
(1 + g)^\sigma = \beta \left[ \frac{r_{t+1} (1 - \tau_k)}{q_t (1 - \tau_i)} + \left( \frac{\tau_k \delta}{1 - \tau_i} \right) + (1 - \delta) \left( \frac{q_{t+1}}{q_t} \right) \right].
\]

Using results 2 and 4, this can be re-written as follows:

\[
(1 + g)^\sigma = \beta \left[ \left( \frac{A^\alpha}{B^\alpha} \right) \left( \frac{1 - \tau_k}{1 - \tau_i} \right) \phi^{\alpha-1} \Omega^{\frac{\alpha}{\alpha-1}} [(1 - \theta) + \theta \Omega]^{1-1/\rho} \right] \]

Let

\[ R_{t+i} = (1 - \tau_i) (-\theta B) (V_{t+i}^\rho)^{-1} \left[ (1 - \theta) \frac{i_{t+i}^{\rho - 1}}{i_{t+i}^\rho} + \theta (V_{t+i})^\rho \right]^{(1/\rho) - 1}. \]

With this in mind, the other optimization condition can be written as

\[
(1 - \tau_r) = \sum_{i=1}^{\infty} \beta^i \left( \frac{c_{t+i}}{c_t} \right)^{-\sigma} R_{t+i}
= \sum_{i=1}^{\infty} \beta^i [(1 + g)^i]^{-\sigma} R_{t+i}
= \sum_{i=1}^{\infty} \beta^i (1 + g)^{-i\sigma} R_{t+i}
= \frac{\beta (1 + g)^{-\sigma} R}{1 - \beta (1 + g)^{-\sigma}}
= \frac{\beta (1 + g)^{-\sigma} (1 - \tau_i) (-\theta B) [(1 - \theta) + \theta (\Omega)^{1/\rho}]^{\frac{1}{\rho} \sigma} (\Omega)^{-1}}{1 - \beta (1 + g)^{-\sigma}} \tag{4}
\]

4 Computing an equilibrium for the economy

The equilibrium conditions for the economy without distortions can be reduced to a few equations. Equation 1 is as follows:
\[(1 + g)^\alpha = \beta \left[ \left( \frac{\alpha A}{B} \right) [(1 - \theta) + \theta (\Omega)^{\rho}]^\frac{\alpha - 1}{\rho} [\Omega]^\frac{\alpha - 1}{\rho} \phi^{\alpha - 1} + (1 - \delta)(1 + g)^\rho \right] \] (5)

Equation 2 is as follows

\[ 1 = \frac{\beta (1 + g)^{-\sigma} (-B\theta) [(1 - \theta) + \theta (\Omega)^{\rho}]^{1-\rho}}{1 - \beta (1 + g)^{-\sigma}} \] (6)

The condition defining \( \Omega \) is again

\[ \Omega = \phi^{-\alpha} (1 + g) \left[ (1 + g)^{1/\alpha} - (1 - \delta) \right]^{-\alpha}. \] (7)

From result 5, the ratio of investment to output is written as follows:

\[ \frac{B [(1 - \theta) + \theta (\Omega)^{\rho}]^{1/\rho} [(1 + g)^{1/\alpha} - (1 - \delta)]^{\alpha}}{A} = \frac{Iy^*}{\phi}. \] (8)

Let \( g^* \), and \( Iy^* \) respectively, denote the desired level of the output growth rate, and the investment to output ratio.

Now with the values of \( g^* \), and \( Iy^* \) held constant, equations 5 through 8 become 4 equations in 4 unknowns: \((A, B, \Omega, \phi)\).

In the version of the model with taxes, the values of the tax parameters, and of \( g^* \), and \( Iy^* \) are taken as constants. Then equations 3, 4, 7, and 8 are 4 equation in the four unknowns \((A, B, \Omega, \phi)\).