On Fragmented Markets

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Abstract

Centralized markets reduce the costs of search for buyers and sellers. More importantly, their ‘thickness’ increases the chance of order execution at competitive prices. In spite of the incentives to consolidate, some markets, securities markets being the most notable, have fragmented into multiple trading venues. We argue in this paper that fragmentation is an unavoidable feature of any centralized exchange except in certain special circumstances.¹

1 Introduction

A centralized exchange reduces the costs of clearing, settlement and search compared to a market consisting of multiple trading venues. Were these costs to decline because of technological innovation, a centralized exchange should still dominate a fragmented market, because traders would prefer the venue that offers the highest probability of order execution and the most competitive prices. Each additional trader on an exchange reduces the execution risk for other potential traders, attracting more traders. This positive feedback should encourage trade to be concentrated in a single exchange.

In spite of the incentives to consolidate, some markets, securities markets being the most notable, have spawned multiple trading venues.² Traditional exchanges now

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²For fragmentation in labor markets see Roth and Xing (1994).
face a host of competitors such as ECNS (electronic communication networks), ATS (alternative trading systems) and the trading desks of broker/dealer firms. These venues use a variety of pricing rules, may not broadcast the bids they receive and in some cases allow traders to restrict who they will transact with. Madhavan (2000) calls this the network externality puzzle and writes: “despite strong arguments for consolidation, many markets are fragmented and remain so for long periods of time.”

A variety of explanations, summarized below (not entirely mutually exclusive), have been offered for why centralized markets fragment.

1. Regulation: Fragmentation enhances efficiency because competition between exchanges forces them to narrow their bid-ask spreads (e.g. Pagano (1989), Biais, Martimort, and Rochet (2000)). Fragmentation in securities markets can be traced to regulation in the 80’s and 90’s designed to limit the abuse of market power by operators of centralized exchanges.  

2. Heterogenous Preferences: Alternative trading venues arise to cater to traders who differ in their preferences for order size, anonymity and likelihood of execution (Harris (1993), Ambrus and Argenziano (2009) and Petrella (2009)).

3. Congestion: As a market becomes thicker, the time to select, evaluate and process offers lengthens during which time prices may change. This encourages participants to transact earlier, fragmenting the market in time (see Roth and Xing (1994)).

4. Informational: Traders seek out alternative venues so as to conceal private information (see Madhavan (1995)). Alternatively, other venues spring up to attract uninformed traders from the incumbent exchange (Easley (1996)).

We don’t consider the reasons on this list to be fundamental, because they can all be eliminated, in principle, by a suitable (but possibly impractical) mechanism. In the first case, the operator could be mandated to implement the constrained efficient mechanism. In the remaining cases, a mechanism that allowed agents to use a richer message space to communicate preferences could be employed.

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4Congestion can also cause fragmented markets to persist as agents trade-off thickness in one venue for less competition in another (Ellison and Fudenberg (2003), Ellison, Fudenberg and Mobius (2004)).
This paper argues that fragmentation is an unavoidable feature of any centralized exchange. Further, efficiency of the exchange does not prevent fragmentation. Within the model in which we make this point, the reasons for fragmentation listed above do not apply. Our setting is the standard model of bilateral trade (Myerson and Satterthwaite (1983)) where each seller has one unit of a homogeneous good and each buyer is interested in purchasing at most one unit of the same good. The private type of each buyer is their marginal value for the good and the private type of each seller is the opportunity cost of their endowment. Thus, agents are all interested in the same order size. Holding prices equal, they are indifferent about who they trade with. There is no common values component in the private information of agents making them equally informed (or uninformed). Trade takes place in a single time period, so the timing of trades is irrelevant.

We model trade within the incumbent exchange as being conducted via an individually rational, incentive compatible, budget balanced mechanism. We are interested in when the incumbent mechanism is stable in the sense that no subset of agents has an incentive to deviate and trade among themselves using a different mechanism, called the blocking mechanism. Formalizing this idea raises two conceptual difficulties. First, the decision to participate in the blocking mechanism reveals something about one's type which should be incorporated into the beliefs of potential counterparties. Second, payoffs in the incumbent mechanism will depend on the equilibrium played in that mechanism which can be affected by the presence of a blocking mechanism. For this reason we consider two related notions of blocking that depend on the equilibrium being ‘played’ in the incumbent mechanism. The first, from Peivandi (2013), assumes that agents in the incumbent mechanism play a dominant strategy equilibrium. The second, introduced in this paper, assumes that the agents play a Bayesian equilibrium of the incumbent mechanism. We distinguish between them by calling the first D-blocking and the second B-blocking. They differ from blocking notions used in the theory of co-operative games by allowing agents to condition their beliefs about counterparties based on which mechanism they are participating in. Roughly speaking, an incumbent mechanism is blocked by a coalition of agents and a blocking mechanism if the blocking mechanism gives to each member of the blocking coalition, for a critical subset of their types, at least as much surplus as they would obtain if they remained in the incumbent mechanism. Furthermore, no agent with a type outside of the critical subset of types will participate in the blocking mechanism. Two features of this notion of blocking make it different from similar notions in the literature (see Section 3). First, each agent
commits to participating in either the incumbent mechanism or the alternative, but not both, before knowing the outcome of each. Second, each agent recognizes that a decision by a counterparty to defect from the incumbent mechanism (or not) reveals some information about the counterparty’s type which should be used. We argue that the conditions under which a mechanism is immune to blocking are very restrictive. From this we conclude that centralized markets are vulnerable to fragmentation.

We offer two sets of results. In the first, we restrict attention to deterministic mechanisms that are ex-post individually rational (EIR) and ex-post (weakly) budget balanced (EBB), features enjoyed by many observed trading rules. We do not specify a particular mechanism but consider all mechanisms that are robust to the beliefs of agents. Like Hagerty and Rogerson (1987) we model this by requiring the mechanism to be dominant strategy incentive compatible (DSIC). This is often totuted as a desirable feature of many mechanisms. Dominant strategy incentive compatibility does not exclude the possibility that the mechanism can depend on the designer’s beliefs. For example, the designer could select a single price at which all trade must take place a-priori that depends on the designer’s beliefs about the distribution of types of the agents. We show that any EIR, EBB and DSIC mechanism, there is a distribution over types, for which this mechanism can be ‘D-blocked’ by another EIR, EBB and DSIC mechanism.

If the distribution of buyer and seller types satisfies the monotone hazard rate condition, there is only one EIR, EBB and DSIC mechanism immune to D-blocking. It is a posted price mechanism: a price $p$ is fixed a-priori, and a pair of buyer and seller who wish to transact do so at price $p$.\(^5\) However, any fixed price $p$ will not suffice. It must be tailored to the underlying distribution of types. Therefore, the only EIR, EBB and DSIC mechanisms immune to D-blocking must be sensitive to the underlying distribution of types. One might conjecture that the posted price that is immune to blocking (when it exists) is one which maximizes ex-ante efficiency. This is not the case and contravenes the intuition that greater efficiency should discourage blocking.

If one desires a mechanism to be robust to the beliefs of the designer as well, then, no mechanism (within the class considered) is immune to D-blocking. We show that every mechanism (in the class) can be D-blocked by a simple mechanism called a positive spread posted price mechanism. In such a mechanism, two prices $p_1 \leq p_2$ are posted. If buyer and seller agree to trade, the seller is paid $p_1$ and the buyer pays $p_2$. The spread

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\(^5\)One may interpret this as a reason for why posted price mechanisms are widely used in practice, see Einav et al. (2013).
of a posted price mechanism is $p_2 - p_1$ and this is what the designer pockets. A posted price mechanism is one where $p_1 = p_2 = p$. Thus, every EIR, EBB and DSIC mechanism can be D-blocked by a mechanism that gives the operator of the blocking mechanism positive expected profit. In other words, the tendency to fragmentation cannot be hindered unless the incumbent mechanism is a money pump for its participants. This is the first reason for our assertion that fragmentation is an unavoidable feature of a centralized exchange.

Our second result focuses on the double bid auction. With one buyer and seller, the price is set between the bid and ask (provided they cross). It is observed in practice, satisfies EIR and EBB but is not DSIC. It is not even Bayesian incentive compatible (BIC). The analysis of blocking in this case is more subtle. The Bayesian equilibrium played in the incumbent mechanism depends on the alternative mechanism that agents can participate in. Its presence changes the beliefs of agents about their counterparties in the incumbent mechanism. We show that there is a Bayesian equilibrium of the double auction that cannot be B-blocked by a positive spread posted price mechanism. We think this result of interest for two reasons. First, it shows that the rules of the mechanism alone do not determine its stability but equilibrium played. Hence, two markets using the same pricing and allocation rule need not share the same vulnerability to blocking. Second, it shows that a non-DSIC mechanism is, in one sense, more robust than a DSIC mechanism. However, it also shows that agents must settle on exactly the right equilibrium to prevent fragmentation.

In the case when types are uniformly distributed in $[0, 1]$ it is well known that there is a constrained efficient equilibrium of the double bid auction. However, the equilibrium that prevents blocking is not constrained efficient. This runs counter to the intuition that a mechanism that produces efficient outcomes is robust to blocking.

In the next section of this paper we introduce notation and give a precise definition of D-blocking. Subsequently we contrast D-blocking with prior notions of the core of games with incomplete information. The subsequent section states and prove the main results concerning D-blocking. In section 5 we introduce B-blocking and its application to the double bid auction. Section 6 concludes.
2 D-blocking

Let $N = \{1, 2, 3, ..., n\}$ be the set of agents. The value of agent $i$ for a unit of the good is $v_i \in V_i$ where $V_i \subset \mathbb{R}^+$ is bounded. Each $v_i$ is the private information of agent $i \in N$ and is independently distributed. Each agent $i$ has an endowment $\omega_i \in \{0, 1\}$ of the good, which is common knowledge. If $\omega_i = 1$, then agent $i$ is a seller and if $\omega_i = 0$ agent $i$ is a buyer. Agent preferences are quasilinear, that is, agent $i$’s payoff from receiving a quantity $q$ of the good (interpret as probability) for a payment of $t$ is $qv_i - t$.

A direct mechanism is defined by an allocation rule and a payment rule. The allocation rule maps profiles of reports of the private information of agents to an allocation of the good. If $Q$ is the allocation rule, denote the component of $Q$ that corresponds to agent $i$’s allocation by $q_i$. Thus, $q_i : \prod_{i \in N} V_i \to \mathbb{R}^+$. As agent $i$ has an endowment of $\omega_i$ we require that an allocation rule be feasible in the sense that for all $i \in N$ and all profiles $v \in \prod_{i \in N} V_i$ that

1. $1 \geq q_i(v) + \omega_i \geq 0$, and,

2. $\sum_{i \in N} q_{i}(v) = 0$.

The payment rule maps each profile $v \in \prod_{i \in N} V_i$ to a per-unit price each agent must make. If $P$ is the payment rule, the component of $P$ that corresponds to agent $i$’s per-unit payment is denoted $p_i$. Thus, $p_i : \prod_{i \in N} V_i \to \mathbb{R}^+$.

We now define dominant strategy incentive compatibility. Let $v = (v_i, v_{-i})$ and $\hat{v} = (\hat{v}_i, v_{-i})$ be two profiles of valuations in $\prod_{i \in N} V_i$. Observe that $\hat{v}$ differs from $v$ in that agent $i$ only changes the report of his marginal value. Agents can misreport their marginal value or opportunity cost but not their role as buyer or seller. Note that we only need to impose incentive compatibility on deviations from profiles that result in feasible outcomes. The mechanism $(Q, P)$ is DSIC if for all $v$ and $\hat{v}$:

$$q_i(v)(v_i - p_i(v)) \geq q_i(\hat{v})(v_i - p_i(\hat{v})).$$

Mechanism $(Q, P)$ is ex-post individually rational if for all profiles $v \in \prod_{i \in N} V_i$ and all $i \in N$

$$q_i(v)(v_i - p_i(v)) \geq 0.$$
Mechanism \((Q, P)\) is (weakly) ex-post budget balanced if for all \(v \in \prod_{i \in N} V_i\)

\[
\sum_{i \in N} p_i(v)q_i(v) \geq 0.
\]

In mechanism \((Q, P)\), the utility that agent \(i \in N\) under profile \(v\) enjoys is

\[
u_i(v, Q, P) = q_i(v)(v_i - p_i(v)).
\]

The expected utility that agent \(i \in N\) enjoys when her type is \(v_i\) is

\[
E_{v_i}[u_i\{v_i, v_{-i}\}, Q, P].
\]

Fix a subset \(A \subseteq N\) of the agents and for each \(i \in A\) a positive measure subset \(V'_i \subseteq V_i\). Call \(V'_i\) the critical set of types for agent \(i\). For each \(i \in A\) let \(T_i\) be the event that each agent \(j \in A \setminus \{i\}\) has a type in \(V'_j\). We say the set \(A\) D-blocks \((Q, P)\) with respect to \(\Pi_{i \in A} V'_i\) if there exists a feasible, DSIC, EIR mechanism \((\hat{Q}, \hat{P})\):

\[
\hat{p}_i : \prod_{i \in A} V_i \rightarrow \mathbb{R}^+ \\
\hat{q}_i : \prod_{i \in A} V_i \rightarrow \mathbb{R}^+ \forall i \in A
\]

that satisfies five conditions listed below. To interpret them it is helpful to imagine that before participating in the mechanism \((Q, P)\), each agent in \(A\) (and only \(A\)) is invited to participate in \((\hat{Q}, \hat{P})\). If every agent in \(A\) accepts the invitation, then, the agents in \(A\) enjoy the outcome delivered by \((\hat{Q}, \hat{P})\). If at least one of the agents in \(A\) declines the invitation, then all agents are required to participate in \((Q, P)\), in this case we say the block is unsuccessful. The agents now face a Bayesian game in which they must first decide which of the two mechanism to participate in and subsequently what to report in their chosen mechanism. As each mechanisms is DSIC we assume truthful reporting. Under these conditions, the decision of each agent in \(A\) to participate (or not) in \((\hat{Q}, \hat{P})\) must be a Bayesian equilibrium.

1. If \(v_i \in V'_i\), then,

\[
E_{v_i}[u_i\{v_i, v_{-i}\}, Q, P]|T_i] \leq E_{v_i}[u_i\{v_i, v_{A\setminus\{i\}}\}, \hat{Q}, \hat{P})|T_i]| \forall i \in A
\]
2. If $v_i \notin V'_i$ then,
\[
E_{-i}[u_i(\{v_i, v_{-i}\}, Q, P)|T_i] \geq E_{-i}[u_i(\{v_i, v_{A\setminus\{i\}}\}, \hat{Q}, \hat{P})|T_i].
\] (2)

3. For all $\bar{v} \in \prod_{i \in A} V'_i$
\[
\sum_{i \in A} \hat{q}_i(\bar{v}) = 0.
\] (3)

4. 
\[
\forall \bar{v} \in \prod_{i \in A} V'_i \sum_{i \in A} \hat{q}_i(\bar{v})\hat{p}_i(\bar{v}) \geq 0
\] (4)

5. 
\[
E[\sum_{i \in A} \hat{q}_i(\bar{v})\hat{p}_i(\bar{v})|\bar{v} \in \prod_{i \in A} V'_i] > 0.
\] (5)

Condition (1) and (2) state that if each $i \in A$ has a type in $V'_i$, then every agent in $A$ choosing to participate in $(\hat{Q}, \hat{P})$ is an equilibrium. Condition (3) ensures that the sum of the net trades is zero. Condition (4) states that the mechanism is weakly ex-post budget balanced. Condition (5) requires that on some profile, the D-blocking mechanism generates a positive surplus. There is a technical and a substantive reason for this condition. First, it eliminates the possibility that of a mechanisms being D-blocked by itself. Second, there must be a strict incentive for someone to offer the D-blocking mechanism.

3 Prior Notions of Blocking

Immunity to D-blocking (or B-blocking) in the sense defined is analogous to the notion of the core of a co-operative game. The important point of departure of this paper is the presence of asymmetric information. There have been various proposals for how to extend the notion of the core to games of incomplete information. They differ in the information that agents considering blocking condition on. We summarize those, that like this paper, make use of the incentive compatibility of some mechanism. To describe them it is useful to imagine two mediators. Mediator 1 proposes a Bayesian incentive compatible mechanism to all the agents in $N$. It is important to note that the mechanism proposed by Mediator 1 is Bayesian incentive compatible under the assumption that reports made in this mechanism do not influence the outcomes in
other mechanisms. For the given profile of types, let $x$ be the resulting allocation and call this the status quo.

Mediator 2 now enters and offers an incentive compatible mechanism (possibly randomized) that selects a subset $S$ of agents and an allocation for the agents in $S$. Mediator 2 succeeds in blocking the status quo if the interim expected utility of each agent in $S$ under the proposed mechanism is at least as large as the utility she would enjoy in the status quo with strict inequality for at least one agent. Keep in mind that Mediator 2’s mechanism can choose the status quo, $x$. This means that incentive compatibility in Mediator 2’s mechanism assumes truthful reporting to Mediator 1.

The various notions of blocking differ in the restrictions they place on Mediator 2. Dutta and Vohra (2005), for example, restrict Mediator 2 to randomizing between an allocation for a single subset $S$ and the status quo. Like the notion used here, there is a subset of agents and a critical subset of their types. The mechanism proposed by Mediator 1 is blocked if each agent within the subset has a type within the critical set of types such that the relevant agents are the only ones that prefer the alternative to the status quo. Dutta and Vohra (2005) call this a credible objection. In other words, a coalition has a credible objection if it can identify an informational event such that the types of agents involved in the event are the only ones that prefer the alternative proposed to the status quo, given that the other types behave as prescribed in the objection. Myerson (2007), using the virtual utility construct, proposes a blocking notion that, in addition to the credibility requirements, considers random coalition formation and random allocations for each coalition. Serrano and Vohra (2007) use the test of coalitional voting in an incomplete information environment to formalize a notion of resilience of a mechanism. The concept makes it possible to define core concepts with endogenous information transmission among members of a coalition. Finally, Liu et al. (2014) study the implications of common knowledge of stability of a two-sided match when one side of the market has incomplete information about the other side.

This paper differs from prior work in that agents must decide simultaneously which mechanism to participate in. For the application we have in mind we think this timing is more natural. It means, unlike prior notions, that when an agent contemplates deviating he cannot know what the outcome of the incumbent mechanism will be. Furthermore, when contemplating which mechanism to participate in, he must condition his beliefs on the participation choices of his counterparty. Moreover, our notion of blocking does not assume that in the presence of Mediator 2, agents report truthfully to Mediator 1. This is important because when agents update their information about the types of
other agents, they may not truthfully report to Mediator 1.

4 Vulnerability to D-blocking

Here we prove that the only EIR, EBB and DSIC mechanism immune to D-blocking (under restrictions on the distribution of types) is a posted price mechanism. To provide some intuition, restrict attention to one buyer and seller and to positive (or zero) spread posted price mechanisms. Consider a positive spread posted price mechanism with a spread of $\delta > 0$. It is easy to see that such a mechanism can always be D-blocked by a positive spread posted price mechanism with a smaller spread. Thus, the only mechanisms (within the class considered) that might be immune to D-blocking are posted price mechanisms. But, what should the posted price should be? Let agent 1 be the seller with an opportunity cost of $c \in [0, 1]$ and $\omega_1 = 1$ and agent 2 the buyer with a value of $v \in [0, 1]$ and $\omega_2 = 0$. Assume $v$ and $c$ are private information distributed independently with atomless density functions $g(c)$ and $f(v)$ respectively. Denote the corresponding distribution functions by $G$ and $F$. Endowments are common knowledge.

Let $p \in [0, 1]$ be the posted price of the incumbent mechanism. Consider a positive spread posted price mechanism $(p', p'')$ with $p' < p''$ as a possible D-blocking mechanism.

As there are only two agents (one buyer and one seller), the D-blocking coalition will consist of just these two agents. It remains to identify a critical set of types. We can do this by ’reverse’ engineering. There are three cases:

1. **Case 1:** $p' < p'' < p$ : A buyer with type $v \geq p''$ strictly prefers the D-blocking mechanism conditioned on a seller being present. Thus, the critical set of types of the buyer will be $[1, p'']$. Now, we find the critical set of types for the seller that would make them prefer the D-blocking mechanism. A seller with type $c < p'$ will join the D-blocking mechanism only if:

$$(p-c)Pr(v \geq p|v \geq p'') \leq (p' - c) \Rightarrow \frac{1 - F(p)}{1 - F(p'')} \leq \frac{p' - c}{p-c}.$$ 

The right hand side is maximized at $c = 0$, therefore the posted price $p$ cannot be D-blocked by the positive spread posted price mechanism $(p', p'')$ if the following holds:

$$\frac{1 - F(p)}{1 - F(p'')} > \frac{p'}{p}.$$ 

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This is equivalent to \( p(1 - F(p)) > p'(1 - F(p')) \). Therefore, if for all \( p' < p \),

\[
p(1 - F(p)) > p'(1 - F(p')), \tag{6}
\]

the posted price mechanism cannot be blocked with prices lower than \( p \).

2. **Case 2**: \( p < p' < p'' \): In this case the seller with opportunity cost \( c < p' \) joins the D-blocking mechanism conditional on a buyer being present. A buyer with type \( v > p'' \) joins the D-blocking mechanism if

\[
(v - p)Pr(c \leq p | c \leq p') \leq v - p''.
\]

As in Case 1 this does not happen if:

\[
\forall p' > p \ (1 - G(p)) > (1 - p')G(p'). \tag{7}
\]

3. **Case 3**: \( p' < p < p'' \): In this case no agent will join the D-blocking mechanism.

This analysis shows that under suitable restrictions on the distribution function of types, there exists a posted price mechanism that cannot be D-blocked by any positive spread posted price mechanism. For example, equations (6) and (7) imply that if functions \( x(1 - F(x)) \) and \( (1 - x)G(x) \) are convex and \( \arg\max_{x \in [0,1]} x(1 - F(x)) > \arg\max_{x \in [0,1]} (1 - x)G(x) \), any posted price mechanism with a price

\[
p \in [\arg\max_{x \in [0,1]} (1 - x)G(x), \arg\max_{x \in [0,1]} x(1 - F(x))]\]

is immune to D-blocking.

Posted price mechanisms, as shown by Hagerty and Rogerson (1987), are the only deterministic, dominant strategy incentive compatible, (weakly) ex-post budget balanced and ex-post individually rational mechanisms. In this sense, the simple argument above goes a long way to establishing our result for the case of one buyer and one seller. Below, we characterize all (not necessarily deterministic) EIR, EBB and DSIC mechanisms that are immune to D-blocking by a positive spread posted price mechanism. We then use this characterization to determine which posted price mechanisms are immune to D-blocking.
4.1 Bilateral Trade

Recall that the hazard rate of the buyer is defined as \( v - \frac{1-F(v)}{f(v)} \) while the hazard rate of the seller is defined as \( c + \frac{G(c)}{g(c)} \).

**Theorem 1.** Assume that the hazard rate of both buyer and seller are increasing. Suppose there exists \( p \in [0, 1] \) such that \( p - \frac{1-F(p)}{f(p)} \) and \( 1 - p - \frac{G(p)}{g(p)} \) are both negative. Then, a posted price mechanism with price \( p \) is immune to D-blocking by a positive spread posted price mechanism.

**Proof.** Let \( \mathcal{M} \) be any EBB and DSIC mechanism for the case of bilateral trade. Denote by \( u_b(v, c) \) and \( u_s(v, c) \) the buyer’s and seller’s payoff, respectively under \( \mathcal{M} \). We first identify conditions under which \( \mathcal{M} \) is immune to D-blocking by a positive spread posted price mechanism.

**Lemma 2.** \( \mathcal{M} \) is immune to D-blocking by a positive spread posted price mechanism if and only if for all \( 0 \leq y < x \leq 1 \) the following holds:

\[
E[u_b(x, c)|c \leq y] + E[u_s(v, y)|v \geq x] \geq x - y. \tag{8}
\]

If for some \( 0 \leq y < x \leq 1 \) inequality (8) is violated, we construct a posted price blocking mechanism. Let \( V_b = [x, 1] \) and \( V_s = [0, y] \) be the critical set of types for buyer and the seller respectively. As inequality (8) is violated, there exists \( 0 \leq p_1 < p_2 \leq 1 \) such that the following holds:

\[
E[u_b(x, c)|c \leq y] = x - p_2, \tag{9}
\]
\[
E[u_s(v, y)|v \geq x] = p_1 - y. \tag{10}
\]

For a candidate D-blocking mechanism we choose the positive spread posted price mechanism with prices \( (p_1, p_2) \). This mechanism is clearly dominant strategy incentive compatible and budget balanced. We now verify that all types in the critical set weakly prefer the D-blocking mechanism to the mechanism \( \mathcal{M} \).

Let \( a(v, c) \) be the probability of trade in \( \mathcal{M} \) when the the profile of types is \( (v, c) \). Recall, from Myerson and Satterthwaite (1983) that \( u_b(\alpha, \beta) = \int_0^\alpha a(t, \beta)dt \) and \( u_s(\alpha, \beta) = \int_\beta^1 a(\alpha, t)dt \). Therefore, for all \( 1 \geq v' \geq x \) and \( y \geq c' \geq 0 \) the following holds:

\[
E[u_b(v', c)|c \leq y] = E[u_b(x, c)|c \leq y] + \int_x^{v'} E[a(v, c)|c \leq y]dv
\]

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\[
E[u_b(x,c)\mid c \leq y] + (v' - x) = v' - p_2,
\]
(11)

\[
E[u_s(v,c')\mid v \geq x] = E[u_s(v,y)\mid v \geq x] + \int_{c'}^{y} E[a(v,c)\mid v \geq x] dc
\]
\[
\leq E[u_s(v,y)\mid v \geq x] + (y - c') = p_1 - c'.
\]
(12)

Equations (11) and (12) ensure that all types in the critical set weakly prefer the D-blocking mechanism to \( M \). It is straightforward to check that when an agent's type is outside the critical set, this agent does not prefer the blocking mechanism to \( M \).

To prove the reverse we show that if there is a positive spread posted price D-blocking mechanism, inequality (8) is violated for some \( 0 \leq y < x \leq 1 \). Let \( 0 \leq p_1 < p_2 \leq 1 \) be the prices in the D-blocking mechanism and \( V_b \) and \( V_s \) be the associated critical set of types. As the sets \( V_b \) and \( V_s \) have positive measure, there exists \( x \geq p_2 \) and \( y \leq p_1 \) such that \( x \in V_b \) and \( y \in V_s \). For all such \( x, y \) the following must hold:

\[
E[u_b(x,c)\mid c \in V_s] \leq E[(x - p_2) I_{c \leq p_1} \mid c \in V_s].
\]
(13)

The left hand side of (13) is the expected payoff to the buyer when she participates in \( M \) knowing that the seller has a type in the critical set \( V_s \). The right hand side is the expected payoff to the buyer when she chooses to participate in the D-blocking mechanism conditional on the seller’s type being in the critical set and the seller participating in the D-blocking mechanism. A similar observation yields:

\[
E[u_s(v,y)\mid v \in V_b] \leq E[(p_1 - y) I_{v \geq p_2} \mid v \in V_b].
\]
(14)

Thus, rewriting inequality (13) yields:

\[
\frac{\int_{c \in V_s} u_b(x,c) g(c) dc}{Pr(c \in V_s)} \leq \frac{(x - p_2) Pr(V_s \cap [0, p_1])}{Pr(c \in V_s)}
\]
\[
\iff \frac{\int_{c \in V_s} u_b(x,c) g(c) dc}{Pr(V_s \cap [0, p_1])} \leq x - p_2
\]
\[
\iff \frac{\int_{c \in V_s \cap [0, p_1]} u_b(x,c) g(c) dc}{Pr(V_s \cap [0, p_1])} \leq x - p_2
\]
\[
\iff E[u_b(x,c)\mid c \in V_s \cap [0, p_1]] \leq x - p_2
\]
(15)

Similarly, the following inequality holds:

\[
E[u_s(v,y)\mid v \in V_b \cap [p_2, 1]] \leq p_1 - y.
\]
(16)
Inequalities (15) and (16) allow us to assume $V_b \subseteq [p_2, 1]$ and $V_s \subseteq [0, p_1]$. Let $x^* = \inf V_b$ and $y^* = \sup V_s$. As the distribution of types is atomless, we may assume $x^* \in V_b$ and $y^* \in V_s$. The following inequalities hold:

$$E[u_b(x^*, c) | c \in V_s] \leq x^* - p_2,$$  \hspace{1cm} (17)

$$E[u_s(v, y^*) | v \in V_b] \leq p_1 - y^*.$$  \hspace{1cm} (18)

Note that the payoff to a seller with type $c \in V_s \cap [p_1, 1]$ is zero in the D-blocking mechanism. Therefore, if a seller has type in $c \in V_s \cap [p_1, 1]$, it must receive a payoff of zero in $M$, i.e., almost surely $\forall v \in V_b$ $u_s(v, c) = 0$. Similarly with a buyer whose type is in $V_b \cap [p_2, 1]$. If $a(x^*, y)$ is constant for all $y \leq y^*$, then $u_b(x^*, c) = u_b(x^*, c')$ for any two $c, c' \in V_s$. It follows from (17) that $u_b(x^*, c) = x^* - p_2$ for all $c \leq y^*$. Hence

$$E[u_b(x^*, c) | c \leq y^*] \leq x^* - p_2,$$  \hspace{1cm} (19)

Similarly, if $a(x, y^*)$ is constant for all $x \geq x^*$ we deduce that

$$E[u_s(v, y^*) | v \geq x^*] \leq p_1 - y^*.$$  \hspace{1cm} (20)

Thus, if $a(x, y)$ is constant in the relevant ranges, the proof is complete. Suppose, for a contradiction, this is not true. Consider the case $x > x^*$ (a similar argument applies when $y < y^*$). For all $x > x^*$ the following holds:

$$E[u_b(x, c) | c \in V_c] = E[u_b(x^*, c) | c \in V_c] + \int_{x^*}^x E[a(s, c) | c \in V_c] ds \leq (x^* - p_2) + (x - x^*) = x - p_2$$  \hspace{1cm} (21)

If inequality (21) holds with equality for any $\bar{x} > x^*$, it must be the case that for all $x > x^*$ and almost all $c \in V_c$, $a(x, c) = 1$. To see why, not that equality for $x = \bar{x}$ implies that $a(x, c) = 1$ for all $x^* < x \leq \bar{x}$. However, $a(\cdot, c)$ is monotone in its first component by dominant strategy incentive compatibility. Therefore, $a(x, c) = 1$ for all $x > x^*$. This means that $a(x, c)$ is constant and (19) applies.

Suppose then that inequality (21) is strict for all $x > x^*$. Therefore, $E[u_b(x, c) | c \in V_c] < x - p_2$ for all $x > x^*$. Hence, $x \in V_b$ for all $x > x^*$. A similar argument shows
that \( y \in V_s \) for all \( y < y^* \). This proves the lemma.

Consider a posted price mechanism that selects a price according to density \( h(p) \). Lemma 8 implies that this mechanism is D-blocked by a positive spread posted price mechanism if for all \( 1 \geq x > y \geq 0 \) the following holds:

\[
\int_y^x (x - p)h(p)dp + \frac{\int_y^p (x - p)g(c)h(p)dc}{G(y)}
+ \int_y^x (p - y)h(p)dp + \frac{\int_p^1 (p - y)h(p)f(v)dv}{1 - F(x)}
\geq x - y
\tag{22}
\]

The right hand side of inequality (22) can be rewritten as follows:

\[
(x - y)(H(x) - H(y)) + \int_y^1 (p - y)\frac{1 - F(p)}{1 - F(x)}h(p)dp + \int_y^1 G(p)\frac{G(p)}{G(y)}h(p)dp
\]

Using integration by parts inequality (22) can be written as follows:

\[
\int_0^y \int_x^1 H(v)(v - \frac{1 - F(v)}{f(v)} - y)f(v)g(c)dc + \int_0^y \int_x^1 H(c + \frac{G(c)}{g(c)} - x)f(v)g(c)dc
\geq (x - y)G(y)(1 - F(x))\tag{23}
\]

Inequality (23) provides a necessary and sufficient condition for immunity of a trade mechanisms to D-blocking by a positive spread posted price mechanism.

To complete the proof consider a randomized posted price mechanism that randomizes only over prices for which both hazard rates are negative. Note that if \( H(v) = 1 \) for all \( v \geq x \) and \( H(v) = 0 \) for all \( v \leq y \), then inequality (23) holds with equality. Such a randomized posted price mechanism sets \( H(v) < 1 \) in the first part of the integral only if \( v - \frac{1 - F(v)}{f(v)} \leq 0 \) and it sets \( H(v) > 0 \) in the second integral only if \( \frac{G(c)}{g(c)} - 1 \geq 0 \). Note that

\[
v - \frac{1 - F(v)}{f(v)} \leq 0 \Rightarrow v - \frac{1 - F(v)}{f(v)} - y \leq 0 \text{ and } \frac{G(c)}{g(c)} - 1 \geq 0 \Rightarrow \frac{G(c)}{g(c)} - x \geq 0.
\]

Therefore, inequality (23) holds for this mechanism. This proves the theorem. \qed
4.2 The General Case

We now allow for the possibility of more than one buyer and seller.

**Theorem 3.** Fix a EIR, EBB and DSIC mechanism that is robust to the beliefs of the designer. For this mechanism there is an atomless distribution over types under which the mechanism can be D-blocked by a group of agents.

**Proof.** Suppose the mechanism cannot be D-blocked under any atomless distribution over types. We show that such a mechanism must be ex-post efficient. The theorem follows from the fact that such a mechanism does not exist. Let $I \subset N$ be the set of sellers and $J \subset N$ be the set of buyers. Consider a profile of valuations $x = (x_I, x_J) \in \prod_{i \in N} V_i$. Let $I' \subseteq I$ and $J' \subseteq J$ be the subset of the sellers and buyer that should trade in an efficient allocation. Note that $|I'|=|J'|$. Let $T_i$ be the event that the types of the sellers in $I' \setminus \{i\}$ are below the $x_{I\setminus i}$ and the type of buyers in $J' \setminus i$ are below $x_{J\setminus i}$ for all agents in $I' \cup J'$. Formally,

$$T_i = \{ v \in \prod_{i \in N} V_i | \forall k \in I' \setminus \{i\} v_k \leq x_k \text{ and } \forall k \in J' \setminus \{i\} v_k \geq x_k \}.$$

If the following inequality is violated one can construct a D-blocking mechanism as in the the special case:

$$\sum_{i \in I' \cup J'} E[u_i(x_i, v_{-i})|T_i] \geq \sum_{k \in J'} x_k - \sum_{k \in I'} x_k. \quad (24)$$

Inequality (24) must hold for all possible atomless distributions. Consider a sequence of the atomless distributions that converge to the distribution that puts probability one on the event that the type profile is $x$. Therefore, the following must hold:

$$\sum_{i \in I' \cup J'} E[u_i(x)|T_i] \geq \sum_{k \in J'} x_k - \sum_{k \in I'} x_k. \quad (25)$$

Inequality (25) implies that the mechanism must be efficient. \qed

5 B-blocking

In this section we modify the notion of D-blocking to account for the possibility that agents play a Bayesian equilibrium of the incumbent mechanism rather than just a dom-
inant strategy equilibrium. In a Bayesian equilibrium, the distribution of types matters. The presence of an alternative mechanism changes the distribution of types participating in the incumbent mechanism and therefore the equilibrium played there. Denote the incumbent mechanism, not necessarily direct, by \((M, P, Q)\) where \(M = \prod_{i \in N} M_i\) is the message space. Denote the blocking mechanism by \((\hat{P}, \hat{Q})\). The payoff of agent \(i\) with type \(v_i\) when all agents send message profile \(m\) is denoted \(u_i(v_i, \{m\}, Q, P)\). We use the notation \((m_i(v_i))_{i \in N}\) for a Bayesian Equilibrium of the incumbent mechanism.

If agents participating in the incumbent mechanism believe that the type of agents in the coalition are in the prescribed subset of types, then, any Baysian equilibrium of the incumbent mechanism should be consistent with this belief. Therefore, any Bayesian equilibrium must satisfy,

\[
\forall i \in N \text{ and } v_i \in V_i \quad m_i(v_i) = \arg\max_{m_i \in M_i} E[u_i(v_i, \{m_i, m_{-i}(v_{-i})\}, Q, P) | T_i].
\]

The agents in \(A\) and a mechanism \((\hat{P}, \hat{Q})\) B-block the incumbent mechanism if there exists a non-zero measure subset of types \((V'_i)_{i \in A}\) (the critical subset) such that the following inequalities hold for all consistent Bayesian equilibria of the incumbent mechanism, \(m_i(v_i)\),

1. If \(v_i \in V'_i\), then,

\[
E_{-i}[u_i(v_i, \{m_i(v_i), v_{-i}\}, Q, P) | T_i] \leq E_{-i}[u_i(\{v_i, v_{A\setminus \{i\}}\}, \hat{Q}, \hat{P}) | T_i] \quad \forall i \in A \tag{26}
\]

\[
T_i = \{(v_k)_{k \in A} | v_k \in V'_k, \forall k \in A \setminus \{i\}\} \tag{27}
\]

2. If \(v_i \notin V'_i\) then,

\[
E_{-i}[u_i(v_i, \{m_i(v_i), v_{-i}\}, Q, P) | T_i] \geq E_{-i}[u_i(\{v_i, v_{A\setminus \{i\}}\}, \hat{Q}, \hat{P}) | T_i]. \tag{28}
\]

3. For all \(\bar{v} \in \prod_{i \in A} V'_i\),

\[
\sum_{i \in A} \hat{q}_i(\bar{v}) = 0. \tag{29}
\]

4.

\[
E\left[\sum_{i \in A} \hat{q}_i(\bar{v})\hat{p}_i(\bar{v}) | \bar{v} \in \prod_{i \in A} V'_i\right] > 0. \tag{30}
\]

We can interpret these conditions in terms of the following “exit” game: agents in \(A\) simultaneously decide to join \((\hat{P}, \hat{Q})\) or not. Equations (26) and (28) imply that the
“exit” game has a Bayesian equilibrium where agents in $A$ choose the exit option if their types are in the critical subset of types. Equation (29) is the market clearing condition. Finally, (30) requires that that the blocking mechanism generate positive expected surplus (condition on types being in the critical set). The main difference between D-blocking and B-blocking is that in B-blocking agents may report messages different from those reported when there was no alternative mechanism.

5.1 Double Auctions

Double Auctions are widely used in practice. Here is how we define them: assume there are $n$ buyers and $m$ sellers, buyers report bids, $b_i$’s, and sellers report asks, $c_i$’s. Bids and asks are positive real numbers. Change the agents indices such that $b_1 \geq b_2 \geq b_3 \geq \ldots \geq b_n$ and $c_1 \leq c_2 \leq c_3 \leq \ldots \leq c_m$, let $k$ be the largest index and $k'$ be the smallest index such that $b_k > c_k$ and $b_{k'} < c_{k'}$. All buyers and sellers with indices no less than $k$ and a subset of buyers and sellers with indices in $\{k + 1, \ldots, k' - 1\}$ trade at price $p(b_k, c_k, \ldots, b_{k'-1}, c_{k'-1})$. If the number of trading buyers (seller) is more than the number of trading sellers (buyers), then sellers (buyers) with the market clearing probability.

Note that our definition of B-blocking requires the mechanism to be blocked for all consistent Bayesian equilibria of the incumbent mechanism. The following result rationalizes this definition:

**Theorem 4.** For any double auction and any ex-post budget balanced incumbent mechanism, there exists a Bayesian Nash Equilibrium of the double auction that can be blocked by the incumbent mechanism, i.e. equations (26), (28), (29) and (30) are satisfied.

**Proof.** Note that all double auctions have a no trade equilibria where for all types, trade occurs with zero probability. If the no trade equilibrium occurs, then all agents in for all of their types prefer to participate in the incumbent mechanism.

Because of Theorem (4), in the definition of B-blocking we require the alternative mechanism to block the incumbent mechanism for all consistent Bayesian Equilibria of the incumbent mechanism. This issue does not arise in the DSIC case, because, dominant strategy equilibrium are not affected by the existence of a blocking mechanism.
We show despite Theorem (3), the double auction is immune to B-blocking by positive-spread posted price mechanisms. The reason for focusing on positive-spread posted price mechanisms is stated below.

**Theorem 5.** Every deterministic, dominant strategy incentive compatible, (weakly) ex-post budget balanced and ex-post individually rational mechanism that generates positive expected profits can be implemented as a positive spread posted price mechanism.

The proof follows Hagerty and Rogerson (1987) and so is omitted.

**Theorem 6.** Assume there is one buyer and one seller, then all double bid auctions are immune against B-blocking by positive-spread posted price mechanisms.

**Proof.** Consider a potential positive-spread posted-price mechanism with prices $p$ and $p'$ such that $p < p'$. Suppose, for a contradiction, that it B-blocks the double auction. We show the set of buyer types who visit the blocking mechanism is $[x, 1]$ and the set of seller types is $[0, y]$, for some $x$ and $y$ such that $p' < x$ and $y < p$. Set $V'_b$ and $V'_s$ to be the type of agents. Let $x = \inf\{v|v \in V'_b\}$ and $y = \sup\{s|s \in V'_s\}$. If $V'_b \neq [x, 1]$, there exists $x' > x$ such that $x' \notin V'_b$. In that case, the following inequalities must hold:

$$E_s[u_i(x, \{m_b(x), m_s(s)\}, Q, P)|s \in V'_s] \leq x - p', \quad (31)$$

$$E_s[u_i(x', \{m_b(x'), m_s(s)\}, Q, P)|s \in V'_s] > x' - p. \quad (32)$$

Inequalities (31) and (32) imply:

$$E_s[u_i(x', \{m_b(x'), m_s(s)\}, Q, P)|s \in V'_s] - E_s[u_i(x, \{m_b(x), m_s(s)\}, Q, P)|s \in V'_s] > x' - x. \quad (33)$$

Set $a(v)$ to be the probability that buyer with type $v$ trades when the seller’s type is in $V'_s$. The right-hand side of inequality (33) is equal to $\int_x^{x'} a(v)dv$. Note that since $a(v) \leq 1$, $\int_x^{x'} a(v)dv \leq x' - x$ which contradicts with (33). The contradiction proves $V'_b = [x, 1]$, similar proof shows $V'_s = [0, y]$.

Consider a Bayesian equilibrium where the buyer and the seller both report the same bid $p''$, such that $p < p'' < p'$. Note that this equilibrium is consistent, however, equation (26) is violated.

$$\square$$

Assume one buyer and one seller with types selected from the uniform distribution over $[0, 1]$. Consider the double bid auction that selects the mid-point between bid and
the ask. It is well known that there is an equilibrium of this double bid auction that is constrained efficient. This equilibrium is not the one described in the proof of Theorem above. Hence, efficiency is not a shield against fragmentation. If anything, it suggests that the threat of an alternative mechanism induces inefficiency in the incumbent mechanism.

6 Conclusion

Our analysis shows that the conditions under which an incumbent mechanism are immune to blocking are both restrictive and fragile. For this reason we argue that centralized markets are vulnerable to fragmentation. Interestingly, when a mechanism cannot be blocked, it is not the case that the mechanism implements the (constrained) efficient outcome. This runs counter to the intuition that a centralized exchange is vulnerable when there is some underlying inefficiency that can be exploited.

References


