Mechanism Design without Quasilinearity

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Abstract

This paper studies a model of mechanism design when agents’ preferences over transfers need not be quasilinear. In a general model of non-quasilinearity, we characterize dominant strategy incentive compatible mechanisms using a monotonicity property. We also establish a revenue uniqueness result: for every dominant strategy implementable allocation rule, there is a unique payment rule that can implement it. These results apply to a wide variety of problems (single object auction, multiple object auction, public good provision etc.) under suitable richness of type space. If the richness of type space is relaxed, we provide a Myerson-like characterization if there are only two alternatives. We apply this result to study various problems in private values single object auction setting.
1 Introduction

Starting with the seminal works of Vickrey (1961) and Myerson (1981), standard models of mechanism design with transfers assume that agents have quasilinear preferences over transfers. This is highlighted in two recent books on this topic (Vohra, 2011; Borgers et al., 2015), which extensively describe the research frontier of mechanism design theory under the quasilinearity assumption. However, there are ample reasons to believe that agents in real-life have non-quasilinear preferences. Classical examples are high-valued auctions (such as spectrum auctions), where bidders usually have budget constraints and often borrow from banks to pay the deficit amount (Cramton, 1997). Such budget constraints make the preferences over transfers non-quasilinear. More generally, income effects in many problems lead to non-quasilinear preferences.

We analyze a model of mechanism design with transfers where preferences of agents over transfers need not be quasilinear. In particular, agents have classical (continuous and monotonic in transfers) preferences (types) over the entire set of consumption bundles - a consumption bundle consists of an alternative and a transfer amount. Such classical preferences need not be quasilinear - the set of quasilinear preferences are precisely those classical preferences whose indifference curves are parallel to each other.

We provide two broad classes of results. In the first class of results, we assume enough richness in our type space. Our assumption on richness still allows our results to be applied to a variety of important problems: single object auction, multiple object auction, public good provision problem etc. If the type space has enough richness, we provide a simple monotonicity condition which along with the taxation property is necessary and sufficient for a mechanism to be (dominant strategy) incentive compatible.\footnote{The taxation property simply requires that if the allocation decision at two types is the same, then the payment decision must also be the same.} Further, we establish a revenue uniqueness result: for every implementable allocation rule, there is a unique payment rule such that the corresponding mechanism is incentive compatible. Though we do not have revenue equivalence in our model, our results can be interpreted as a counterpart of the monotonicity and revenue equivalence results for the quasilinear type spaces.\footnote{We discuss the literature in detail later, but remind the reader that such characterization results in the quasilinear type spaces form the core for any optimization exercise (for instance, expected revenue maximization).} Though it is a folklore that revenue equivalence fails without quasilinearity, our results make the nature of failure precise.

Our second set of results work for problems where each agent has two possible alterna-
tives to be allocated: for instance, in a single object auction problem, every agent is either allocated the object or not. Other examples include monopoly pricing, bilateral trade, deciding provision of a single public good etc. For these problems, we can relax our richness of type space to an appropriate notion of convexity. Under such convex preferences, we provide an analogue of the characterization in Myerson (1981). However, we observe that revenue equivalence may not hold in general in these models. In the quasilinear type space, with two alternatives, we can always normalize the value of one of the alternatives to zero, and this reduces the problem to a one-dimensional type space. Hence, our two-alternative result can be interpreted as an extension of this one-dimensional result to non-quasilinear preferences. However, note that with non-quasilinear preferences, even with two alternatives, the type of an agent is still infinite-dimensional - an agent’s type is his preference over the entire (infinite) set of consumption bundles. Even then, we are able to apply this result to some specific problems.  

Our first application of this result is a problem where a seller is selling an object to a single buyer with non-quasilinear preferences. We derive the optimal (expected revenue maximizing and individually rational) incentive compatible mechanism for this problem. Even though the preference of the buyer is non-quasilinear, the optimal mechanism is similar to the quasilinear case. In the optimal mechanism, the seller posts a (monopoly) reserve price (based on the prior) and the buyer buys the object if his willingness to pay is higher than the reserve price. If the willingness to pay is less than the reserve price, then the buyer does not buy the object and no transfers are made. We emphasize that these results are obtained even though we do not have revenue equivalence in this framework.

We then analyze the problem of finding an expected revenue maximizing mechanism when there are multiple buyers. Unlike the single buyer case, this problem becomes intractable to analyze. In a restricted class of mechanisms, we show some qualitative properties of the optimal mechanism by using our two-alternative characterization.

Finally, we apply our result to design incentive compatible and anonymous mechanisms for allocating a single indivisible object among a set of agents. Anonymity requires a minimal amount of fairness. It is a compelling desiderata in fair allocation literature (Thomson, 2016). Under a mild condition on the payment of losing agents (agents who are not allocated the object), we provide a complete characterization of incentive compatible and anonymous mechanisms. These mechanisms are analogues of the Vickrey auction with agent-specific reserve prices - see a related auction and its analysis in Yamashita (2015) for the quasilinear-

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3 Leaving aside few exceptions, majority of applications in the quasilinear type space are in models with two alternatives - see Borgers et al. (2015) for numerous examples.
ear model. Thus, we use our result to provide a characterization of a well-known class of mechanisms.

The rest of the paper is organized as follows. We introduce our model in Section 2. We define our type space and our main results in Section 3. We present specific results for the two alternatives case in Section 3.1. We compare our results to results with quasilinear preferences in Section 3.2. Section 4 presents various applications of our results. We discuss the connection of our work with the literature in Section 5. We conclude in Section 6. All our missing proofs are in the Appendix at the end.

2 The Model

Let $A$ be a finite set of alternatives. The (consumption) bundles are $Z = A \times \mathbb{R}$, where a typical element $z = (a, t)$ corresponds to alternative $a \in A$ and transfer $t \in \mathbb{R}$. Throughout the paper, $t$ will be interpreted as the amount paid by an agent to the designer, i.e., a negative $t$ will indicate that the agent receives a transfer of $-t$.

There is a single agent and his preference over $Z$ is defined by a preference ordering over $Z$. The single agent assumption is for convenience and all our results extend to the multiple agents case with dominant strategy as the solution concept. For any preference ordering $R$ over $Z$, we denote its strict part as $P$ and the indifference part as $I$. A preference ordering $R$ over $Z$ is classical if it satisfies the following assumptions:

1. **Money monotonicity (M).** for every $t > t'$ and for every $a \in A$, we have $(a, t') \succ (a, t)$.

2. **Continuity (C).** for every $z \in Z$, the sets $\{z' \in Z : z' \sim z \}$ and $\{z' \in Z : z \sim z' \}$ are closed.

3. **Finiteness (F).** for every $z \in Z$ and for every $b \in A$, there exists $t$ and $t'$ such that $z \sim (b, t)$ and $(b, t') \sim z$.

Let $\mathcal{R}^c$ be the set of all classical preferences over $Z$.

**Definition 1** The valuation of the agent with preference $R$ for alternative $a \in A$ at bundle $z \in Z$ is defined as $V_R(a; z)$, which uniquely solves

$$(a, V_R(a; z)) \sim z.$$ 

An illustration of the valuation is shown in Figure 1. In the figure, the two horizontal lines correspond to two alternatives. The horizontal lines indicate transfer amounts. Hence,
the two lines are the entire set of consumption bundles of the agent. A preference $R$ can be described by drawing (non-intersecting) indifference curves through these consumption bundles (lines). One such indifference curve passing through $z$ is shown in Figure 1. This indifference curve actually consists of two points - $z \equiv (b, t)$ and $(a, V^R(a, z))$ as shown. Part of the indifference curve which lies between the two consumption bundle lines is useless and has no meaning - it is only displayed for convenience. Since we will use such figures throughout the paper, it is important to understand these issues clearly.

The next claim shows that a valuation always exists.

**Claim 1** For every $a \in A$, for every $z \in \mathcal{Z}$, and for every classical $R$, the valuation $V^R(a; z)$ exists.

**Proof:** By finiteness, there exist transfers $t, t'$ such that $(a, t) R z R (a, t')$. By money monotonicity and continuity, there must exist a unique $t''$ such that $(a, t'') I z$. $\blacksquare$

For any $R$ and for any $z \in \mathcal{Z}$, the valuations at bundle $z$ with preference $R$ is a vector in $\mathbb{R}^{|A|}$ and will be denoted by $V^R(z) \equiv \{V^R(a, z)\}_{a \in A}$. The collection of all such vectors for all $z$ describe the preference ordering $R$.

### 2.1 Incentive Compatibility

We now define the notion of incentive compatibility we use. For the one agent model, we refer to it as simply incentive compatibility, but when this is extended to the multi-agent model, we will be using dominant strategy incentive compatibility.

Let $\mathcal{R} \subseteq \mathcal{R}^c$ be any subset of classical preferences - we will refer to $\mathcal{R}$ as the domain. A mechanism $(x, t)$ is a pair, where $x : \mathcal{R} \to A$ and $t : \mathcal{R} \to \mathbb{R}$. Restriction to such mechanisms is due to the revelation principle.
**Definition 2** A mechanism \((x,t)\) is incentive compatible if for every \(R, \hat{R} \in \mathcal{R}\), we have

\[(x(R), t(R)) \ R (x(\hat{R}), t(\hat{R})).\]

The following monotonicity property will be shown to be necessary for incentive compatibility.

**Definition 3** A mechanism \((x,t)\) is monotone if for every \(R, \hat{R} \in \mathcal{R}\) with \(x(R) \equiv a\) and \((x(\hat{R}), t(\hat{R})) \equiv z\), we have

\[V^R(a,z) \geq V^{\hat{R}}(a,z).\]

An illustration is shown in Figure 2. Take any pair of preferences \(R\) and \(\hat{R}\) and suppose the mechanism chooses consumption bundle \(z\) at \(\hat{R}\). Then, monotonicity requires that valuation for \(x(R)\) at \(z\) is larger for \(R\) than for \(\hat{R}\). There is another natural way to interpret the monotonicity condition. If the mechanism chooses the alternative \(a\) at \(R\), and we consider another preference \(R'\) such that \(V^{R'}(a,z) > V^R(a,z)\), where \(z\) does not involve alternative \(a\). In other words, agent’s liking for the outcome at \(R\) compared \(z\) increases from \(R\) to \(R'\). Then, monotonicity requires \((x(R'), t(R')) \neq z\).

As we show later, this monotonicity condition collapses to the standard weak monotonicity or 2-cycle monotonicity condition if \(\mathcal{R}\) is the quasilinear type space - see Section 3.2 for detailed discussions. We show next that monotonicity along with a simple condition on payments is equivalent to incentive compatibility in a rich type space (which covers many important problems).
3 ORDER CONSISTENT DOMAINS

In this section, we introduce our type space \( \mathcal{R} \) by specifying the richness required. We then establish our main results on this type space. Later, we show how the richness required can be relaxed if \(|A| = 2\).

Fix a vector \( v \in \mathbb{R}^{|A|} \). For any \( a, b \in A \), if \((a, v_a) \triangleq (b, v_b)\), we say \( v \) belongs to an indifference class of preference \( R \). The set of all vectors in \( \mathbb{R}^{|A|} \) which form an indifference class of preference \( R \) is denoted by \( \mathcal{I}(R) \). An equivalent way to describe \( R \) is to describe \( \mathcal{I}(R) \). Note that the elements of \( \mathcal{I}(R) \) are distinct component-wise, i.e., for all \( v, v' \in \mathcal{I}(R) \), \( v_a \neq v'_a \) for all \( a \in A \). Further, the elements of \( \mathcal{I}(R) \) can be ordered (since indifference curves do not intersect), i.e., for every \( v, v' \in \mathcal{I}(R) \), we have \( v_a > v'_a \) if and only if \( v_b > v'_b \) for all \( a, b \in A \).

Let \( \succ \) denote a strict partial ordering of the alternatives in \( A \). A vector \( v \in \mathbb{R}^{|A|} \) respects \( \succ \) if for every \( a, b \in A \), we have

\[
  a \succ b \Rightarrow v_a > v_b.
\]

**Definition 4** A preference relation \( R \) is **consistent** with a strict partial order \( \succ \) if every \( v \in \mathcal{I}(R) \) respects \( \succ \). A domain \( \mathcal{R} \) of preference relations is **order consistent** if there exists \( \succ \) such that every \( R \in \mathcal{R} \) is consistent with \( \succ \).

We will usually denote an order consistent domain as \( \mathcal{R}^\succ \) to indicate that every \( R \in \mathcal{R}^\succ \) is consistent with the partial ordering \( \succ \).

![Figure 3: Indifference curves in order consistent domain](image)

Figure 3: Indifference curves in order consistent domain

An equivalent way to state this domain restriction is the following. Consider \( R \) which is consistent with \( \succ \) and pick \( a \in A \) and \( t \in \mathbb{R} \). Then, for every \( b \in A \), we have \( V^R(b, (a, t)) < t \) if and only if \( a \succ b \). Hence, a domain \( \mathcal{R} \) is order consistent if there exists \( \succ \) such that for
every $R \in \mathcal{R}$, for every $t \in \mathbb{R}$ and every $a, b \in A$, we have

$$a \succ b \text{ implies } V^R(b, (a, t)) < t.$$  

In particular, if we arrange the alternatives according to the specified ordering $\succ$ from bottom to top, the points along the indifference curve shift to the right as we move along $\succ$. This is illustrated in Figure 3 for three alternatives and the relation $\succ$ defined as $c \succ b \succ a$ (though $\succ$ is shown to be a complete relation here, this need not be the case).

Order consistent domains capture many interesting problems. They were introduced in Bikhchandani et al. (2006) for quasilinear domains, and ours is an extension to non-quasilinear domains. We list their applicability below:

- $\succ$ is empty. Then, there are no restrictions in preferences over alternatives. This captures public decision problems, where the set of alternatives are set of public projects. The fact that $\succ$ is empty represents the fact that there is no restriction on the preferences over public projects at a given transfer level.

- $\succ$ is a complete ordering. This can capture a situation where there are a set of homogeneous objects (set of alternatives in this example) and there are no restriction on the number of units that an agent can consume. More units are better for the agents. So, alternatives can be completely ranked by the number of units. The complete ordering can also capture tiered object preferences in multi-object allocation problem with unit demand (Zhou and Serizawa, 2015).

- $\succ$ is a partial ordering. This can capture multiple heterogeneous objects sale (set of alternatives in this example), where agents can have unit demand or multiple demand. If they have unit demand, then the only restriction is that every object is strictly preferred to no object. If agents demand more than one object, then the only restriction is that a subset of a set of objects is less preferred than the original set of objects.

We will require the following richness condition.

**Definition 5** An order consistent domain $\mathcal{R}^\succ$ is **minimally rich** if for every $v$ and $v'$ respecting $\succ$, there exists $R \in \mathcal{R}^\succ$ such that $v, v' \in \mathcal{I}(R)$.

If a domain consists of all possible preferences consistent with $\succ$, then it is minimally rich.

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4There is minor difference between our definition and the definition in Bikhchandani et al. (2006) - they allow for indifferences in $\succ$ but we do not allow indifferences. We make this assumption for technical reasons.
We are now ready to state our first main result, which is a characterization of incentive compatibility in the minimally rich order consistent domain using monotonicity and a simple condition on transfer rule.

**Theorem 1** Suppose $\mathcal{R}^>$ is a minimally rich order consistent domain, where $\succ$ is some strict partial order over $A$. Let $(x, t)$ be a mechanism defined over $\mathcal{R}^>$. Then, the following statements are equivalent.

1. $(x, t)$ is incentive compatible.
2. (a) $(x, t)$ is monotone and (b) there exists a map $\kappa : A \rightarrow \mathbb{R}$ such that for every $R \in \mathcal{R}^>$, we have $t(R) = \kappa(x(R))$.

Condition 2(b) is sometimes referred to as the “taxation principle” in mechanism design. It only requires that if two preferences get the same alternative then their payments should also be the same. Hence, Theorem 1 characterizes incentive compatibility using monotonicity and the taxation principle.

Theorem 1 is the counterpart of monotonicity based characterizations found in the quasi-linear domains - we discuss the precise connection in Section 3.2. Note that Theorem 1 does not imply any kind of revenue equivalence. This is because if we take $x$ and a $\kappa$ map, and then add some constant to the $\kappa$ map to construct another map $\kappa'$, it will define a different transfer rule $t'$. Even though $(x, t)$ is monotone, we may have that $(x, t')$ is no longer monotone. In other words, the monotonicity condition here is a condition on the mechanism, and this hints that revenue equivalence type results found in quasilinear type spaces may not hold. In fact, we show that revenue uniqueness holds.

**Theorem 2** Let $\mathcal{R}^>$ be an order consistent and minimally rich domain and $\tilde{x}$ be onto. If $(\tilde{x}, t)$ and $(\tilde{x}, t')$ are two incentive compatible mechanisms defined on $\mathcal{R}^>$, then $t = t'$.

As we discuss later in Section 3.2, the quasilinear type space is not minimally rich, and hence, Theorems 1 and 2 do not apply to quasilinear type space. However, these theorems help us understand the structure of incentive compatible mechanisms if the designer does not know the precise nature preferences of the agent over consumption bundles. We also point out that our results are not vacuous. In particular, interesting incentive compatible mechanisms exist in many important domains (for various choice of $\succ$) even if all preferences consistent with $\succ$ are considered. As an example, consider the multi-object auction problem where each agent is interested in exactly one object (unit demand). This can be modeled by
a minimally rich consistent domain as follows - $\succ$ is the partial order where every object $a$ is strictly preferred to the null object (alternative that represents no assignment of object), and there is no other relation between other objects. Let $\mathcal{R}^\succ$ represent all classical preferences consistent with $\succ$. Demange and Gale (1985) show the existence of a dominant strategy incentive compatible, Pareto efficient, and individually rational mechanism in this domain - they call this mechanism minimum Walrasian equilibrium price mechanism.  

3.1 Convex Domains: Two Alternatives

In this section, we focus on the case where $|A| = 2$ - all our applications will be based on this case. Besides covering a wider variety of applications, the two alternatives setting is important because it corresponds to a canonical class of models in mechanism design studied under quasilinearity assumption. Under quasilinearity, if there are two alternatives, valuation of one of the alternatives can always be scaled to zero and the only private information of an agent is his valuation for the other alternative. As a result, the type of an agent becomes one-dimensional. One-dimensional type spaces are ubiquitous in mechanism design literature, and their study parallels the single object auction analysis in Myerson (1981). Hence, analysis of the two alternatives case can be interpreted as an extension of the one-dimensional mechanism design to non-quasilinear preferences.

For simplicity, we represent the set of alternatives as $A = \{0, 1\}$. Given a classical preference relation $R$, we define the willingness to pay of an agent at transfer $t$ as

$$WP(R, t) := V^R(1; (0, t)) - t.$$ 

Notice that the reference consumption bundle is $(0, t)$ - hence, willingness to pay $WP(R, t)$ is the extra transfer needed to make the agent indifferent between $(0, t)$ and $(1, t + WP(R, t))$. Since $V^R$ exists, $WP(R, \cdot)$ exists.

**Definition 6** A domain of preferences $\mathcal{R}$ is convex if for every $R, R' \in \mathcal{R}$, for every $t \in \mathbb{R}$, and for every $v$ such that $v = \lambda WP(R, t) + (1 - \lambda) WP(R', t)$ for some $\lambda \in [0, 1]$, there exists $R'' \in \mathcal{R}$ such that $WP(R'', t) = v$.

A domain of preferences $\mathcal{R}$ is convex rich if it is convex and

$$\inf\{WP(R, t) : R \in \mathcal{R}\} = \inf\{WP(R, t') : R \in \mathcal{R}\} \quad \forall \ t, t' \in \mathbb{R}.$$ 

5 This mechanism corresponds to the minimum Walrasian equilibrium price of the complete information economy - see a characterization of this mechanism in Morimoto and Serizawa (2015).
Convexity is equivalent to requiring that for every $t \in \mathbb{R}$, the set $\{WP(R,t) : R \in \mathcal{R}\}$ is convex. Convex richness requires that the set of willingness to pay at every transfer forms an interval, and the lower support of that interval is the same for all transfers. Convex richness is a much weaker requirement than the richness condition used in Theorem 1. For instance, the set of all quasilinear preferences form a convex rich set, but it is not rich in the sense of Definition 5. In the convex domain of preferences with two alternatives, we get a characterization which is closer to the standard results in quasilinear domains.

**Theorem 3** Suppose $A = \{0,1\}$ and $\mathcal{R}$ is a convex rich domain of preferences. If $(x,t)$ is a mechanism defined on $\mathcal{R}$, then the following statements are equivalent.

1. $(x,t)$ is incentive compatible.

2. There exists a real number $\tau^0$ such that

   (a) **WP-monotonicity.** for every $R, R' \in \mathcal{R}$ with $x(R) = 1$, we have
   
   $$ WP(R', \tau^0) > WP(R, \tau^0) \Rightarrow x(R') = 1. $$

   (b) **Payment formula.** for every $R \in \mathcal{R}$,
   
   $$ t(R) = \tau^0 + x(R) \cdot \tau^1, $$

   where $\tau^1 := \inf\{WP(R', \tau^0) : R' \in \mathcal{R}, x(R') = 1\}$.

We make some remarks about Theorem 3.

1. Theorem 3 continues to hold if we just have a convex preference domain (and drop richness), but use ontoness of the allocation rule (as in Theorem 2). This is obvious from the steps of the proof.

2. The richness requirement in Theorem 3 is milder than the richness required in Theorems 1 and 2. As a result, quasilinear type spaces are covered in Theorem 3. That explains why we need a stronger Condition 2(b) in Theorem 3 than in Theorem 1.

3. Though Condition 2(b) in Theorem 3 is stronger than Condition 2(b) in Theorem 1, WP-monotonicity is weaker than monotonicity. We discuss this issue in detail in Section 3.2, where we also emphasize the fact that WP-monotonicity and monotonicity are equivalent in quasilinear type space.
4. Theorem 3 hints at dramatic reduction in the informational requirement of mechanisms in the two-alternatives model. Every mechanism must announce a pair of numbers \( \tau_0 \) and \( \tau_1 \). Then, it only elicits \( WP(R, \tau_0) \) information from the agent. If \( WP(R, \tau_0) > \tau_1 \), then the object is allocated at price \( \tau_0 + \tau_1 \). If \( WP(R, \tau_0) < \tau_1 \), then the object is not allocated and the agent pays \( \tau_0 \).

5. Condition 2(b) should not be confused as a revenue equivalence result. This is because the monotonicity condition depends on the choice of \( \tau_0 \). To be more precise, suppose we have an incentive compatible mechanism \((x, t)\) defined by \( x, \tau_0 - \tau_1 \) and \( t \) is determined by these two parameters. If we fix \( x \) and consider \( \bar{\tau}_0 := \tau_0 + \Delta \) for some \( \Delta \in \mathbb{R} \), does that define a new incentive compatible mechanism? The answer is not clear. Though this defines a new \( \bar{\tau}_1 := \inf \{ WP(R', \bar{\tau}_0) : x(R') = 1 \} \), and hence, a new payment rule \( \bar{t} \), monotonicity may be violated at \( \bar{\tau}_0 \). This is not an issue in the quasilinear type space because WP values do not depend on transfer amount, and hence, monotonicity is satisfied at \( \bar{\tau}_0 \) also.

To highlight the last remark further, we show below that in the two-alternative model, the revenue uniqueness result can be restored under weaker richness requirements than Theorem 2. To do so, we introduce the notion of positive income effect. We define it using the notion of willingness to pay. As transfer increases, positive income effect requires that willingness to pay must decrease.

**Definition 7** A classical preference \( R \) exhibits **positive income effect** if for all \( t, t' \in \mathbb{R} \) with \( t > t' \), we have \( WP(R, t') > WP(R, t) \).

The set of all classical types with positive income effect is denoted as \( R^{++} \).

We now state the revenue uniqueness result under positive income effect.

**Theorem 4** Suppose \( \mathcal{R} \) is a convex type space such that \( \mathcal{R}^{++} \subseteq \mathcal{R} \). Let \( x : \mathcal{R} \to \{0, 1\} \) be an onto allocation rule. If \((x, t)\) and \((x, t')\) are incentive compatible mechanisms then \( t = t' \).

We also note that a counterpart of Theorem 1 can be proved in the two-alternatives model if the type space contains \( \mathcal{R}^{++} \).

### 3.2 Comparison of Results to Quasilinear Type Space

We now formally compare our results to some of the important results in the quasilinear type space. We start by formally defining a quasilinear preference.
Definition 8. A preference $R$ is quasilinear if there exists $v \in \mathbb{R}^{|A|}$ such that for any $(a, p)$ and $(b, p')$ we have

$$(a, p) \ R \ (b, p') \iff v_a - p \geq v_b - p'.$$

A quasilinear preference can be represented by a valuation vector in $\mathbb{R}^{|A|}$. Denote the valuation attached to quasilinear preferences $R, R', R'', \ldots$ as $v, v', v'', \ldots$ respectively. Let $V \subseteq \mathbb{R}^{|A|}$ be the domain of valuations in the quasilinear preference domain. Notice that two valuation vectors which differ by the same constant in each component, represent the same preference over consumption bundles.

We provide some comparisons of our results to the quasilinear domain literature. We state these results under the assumption that $A$ is finite and mechanisms are deterministic. But some of the results stated below also hold for non-finite $A$ and randomized mechanisms.

**Monotonicity.** Our monotonicity condition reduces to the following condition in quasilinear domains. Take two valuation vectors $v, v' \in \mathbb{R}^{|A|}$ and denote the underlying quasilinear preferences corresponding to them as $R, R'$ respectively. Consider a mechanism $(x, t)$ and denote $x(R) = a$ and $x(R') = b$. Note that

$$V^R(a, (b, t(R'))) = v_a - v_b + t(R') \quad \text{and} \quad V^{R'}(a, (b, t(R'))) = v'_a - v'_b + t(R').$$

Hence, $V^R(a, (b, t(R'))) \geq V^{R'}(a, (b, t(R'))) \iff v_a - v_b \geq v'_a - v'_b$. This is the familiar weak monotonicity or 2-cycle monotonicity condition from the quasilinear domain literature (Bikhchandani et al., 2006; Saks and Yu, 2005; Ashlagi et al., 2010).

**Monotonicity and WP-monotonicity.** The monotonicity condition is somewhat symmetric in quasilinear domain. Consider the same example as above. For $R$ and $R'$ with $x(R) = a, x(R') = b$, we have a pair of monotonicity conditions:

$$V^R(a, (b, t(R')))) \geq V^{R'}(a, (b, t(R')))$$

$$V^{R'}(b, (a, t(R')))) \geq V^R(b, (a, t(R'))).$$

In quasilinear domain, both these conditions are equivalent to:

$$v_a - v_b \geq v'_a - v'_b.$$
only uses one-side of monotonicity. Hence, WP-monotonicity in Theorem 3 is significantly weaker than the monotonicity condition in Theorem 1. In quasilinear domain, monotonicity is equivalent to WP-monotonicity.

**Monotonicity is just monotonicity of** \( x \). In quasilinear domain, monotonicity of the mechanism \((x,t)\) reduces to the monotonicity of the allocation rule \( x \). This follows from the observation above. In particular, we consider the following definition of monotonicity.

**Definition 9** Let \( x \) be an allocation rule defined on a quasilinear domain \( \mathcal{V} \). We say \( x \) is monotone if for every \( v, v' \in \mathcal{V} \), we have

\[
v_{x(v)} - v_{x(v')} \geq v'_{x(v)} - v'_{x(v')}.
\]

Such a condition on allocation rule cannot be a necessary condition in the non-quasilinear type space since the allocation rule and transfer rule are not separable.

**Mechanism characterization.** We report two results from quasilinear domains to compare it with our results. The first result is due to Jehiel et al. (1999) - we report a version of this result given in Ashlagi et al. (2010), which can be shown using subgradient techniques of convex functions (Rockafellar, 1970).

**Theorem 5 (Jehiel et al. (1999); Ashlagi et al. (2010))** Suppose \((x, t)\) is a mechanism defined on a convex domain of valuations \( \mathcal{V} \subseteq \mathbb{R}^{|A|} \). Then, the following statements are equivalent.

1. \((x, t)\) is incentive compatible.
2. (a) \( x \) is monotone.
   (b) for every \( v, v' \in \mathcal{V} \), we have
   \[
t(v) = t(v') + \int_{0}^{1} \psi_{v',v}(z)dz,
   \]
   where \( \psi_{v',v}(z) = (v - v') \cdot I_{x(v' + z(v - v'))} \) for all \( z \in [0, 1] \) and for every \( v'' \in \mathcal{V} \), \( I_{x(v'')} \) is the indicator vector in \( \mathbb{R}^{|A|} \), where the component corresponding to \( x(v'') = 1 \) and all other components are zero.

We can contrast our result in Theorem 1 with Theorem 5. Theorem 1 has monotonicity of mechanism and a simple condition on payments. On the other hand, Theorem 5 has
monotonicity of allocation rule and a complicated condition on payments - usually referred to as the revenue equivalence formula or envelope formula (Milgrom and Segal, 2002; Krishna and Maenner, 2001). The richness of our non-quasilinear domain allows us to use the monotonicity of the mechanism along with a simple condition on payments in Theorem 1 to characterize incentive compatibility.

**Allocation rule characterization.** In the quasilinear domain, due to the separability (and linearity) of the payment from the allocation rule, one can focus attention on the implementability question.

**Definition 10** An allocation rule $x$ is implementable if there exists a payment rule $t$ such that $(x, t)$ is incentive compatible.

In quasilinear domain, if we know that $x$ is implementable, then due to Theorem 5, there is an explicit way to compute payments using $x$ (up to an additive constant). The natural question is: when is an allocation rule implementable?

**Theorem 6 (Saks and Yu (2005); Ashlagi et al. (2010))** Suppose closure of $\mathcal{V}$ is convex and $x$ is an onto allocation rule defined on $\mathcal{V}$. Then, $x$ is implementable if and only if it is monotone.

Theorem 6 was first shown for order consistent quasilinear domains in Bikhchandani et al. (2006). Though Theorem 1 is an extension of their result to non-quasilinear domains, it should not be confused as an analogue of Theorem 6. There are two important differences: (1) Theorem 1 is a characterization of incentive compatible mechanisms using monotonicity of the mechanism and Theorem 6 is a characterization of implementable allocation rules in quasilinear domains using monotonicity of allocation rule; (2) Theorem 1 uses richness of non-quasilinear domain and it does not apply to quasilinear domains.

An analogue of Theorem 6 is probably the following straightforward modification of Theorem 1: Suppose $\mathcal{R}$ is a convex domain of (non-quasilinear) preferences and $(x, t)$ is a mechanism defined on $\mathcal{R}$. Then, $(x, t)$ is incentive compatible if and only if $(x, t)$ is monotone and there exists a map $\kappa : A \to \mathbb{R}$ such that $t(R) = \kappa(x(R))$ for all $R \in \mathcal{R}$. Such a conclusion is not true - a simple example in the two alternatives case can be produced. For instance, if this conjecture was true, then Theorem 1 will also hold in quasilinear domains, but we know that we require much stronger payment formula (as indicated in Theorem 5) for quasilinear domain.

In summary, a generalized version of Theorem 3 for arbitrary number of alternatives and convex domain is probably true. But Condition 2(b) in Theorem 3 may change for more than
two alternatives. Further, Condition 2(a) may have to be strengthened to monotonicity. We do not have an answer to this question at this point.

4 Applications

We discuss three applications of our results. All the applications are based on Theorem 3 for the two-alternatives case. This is consistent with the literature on quasilinear type space, where most of the applications are in one-dimensional mechanism design (two alternatives). Our first application is a simple model of screening or monopoly pricing.

4.1 Screening with Non-quasilinearity

In this model, a seller is selling a single indivisible object to a single buyer. Since the outcome is deterministic (either the buyer gets the object or not), the set of alternatives can be written as \( A = \{0, 1\} \), where 0 represents the alternative where the buyer does not get the object and 1 represents the alternative where he gets the object. The set of outcomes is \( Z = A \times \mathbb{R} \), and the buyer (agent) has classical preferences over \( Z \). For convenience, we represent these classical preferences by a utility function \( u : Z \to \mathbb{R} \), where we scale \( u(0, 0) = 0 \) - note that classical preferences allow for such a representation. For any classical preference \( R \) and for any \( u \) representing \( R \), as before we let

\[
WP(u, t) := V^R(1; (0, t)) - t \quad \forall t \in \mathbb{R}.
\]

We refer to \( WP(u, t) \) as the willingness to pay of buyer type \( u \) at transfer \( t \). In particular, \( WP(u, t) \) is the amount that makes a buyer of type \( u \) indifferent between not getting the object at transfer \( t \) and getting the object at transfer \( t + WP(u, t) \).

Rich Type Space. We now define the type space we work with in this model. Type here will be represented by the utility function \( u \) (with \( u(0, 0) = 0 \)). We denote by \( \mathcal{U}^c \) the set of all classical utility functions (i.e., utility functions representing classical preferences).

**Definition 11** A type space \( \mathcal{U} \subseteq \mathcal{U}^c \) is a rich classical type space if for every \( t \in \mathbb{R} \), there exists \( \beta_t > 0 \) such that

\[
\{WP(u, t) : u \in \mathcal{U}\} = (0, \beta_t).
\]

Note that a rich classical type space in this problem is convex rich, and hence, results in Theorem 3 can be applied. In quasilinear domain, this richness will boil down to requiring
that the valuation of the agent for the object lies in an open interval with lower support at zero. The definition above generalizes this richness to an arbitrary subset of classical type space.

Suppose $\mathcal{U}$ is the domain of utility functions (type space). A mechanism $(x, t)$ in this case consists of an allocation rule $x : \mathcal{U} \to \{0, 1\}$ and $t : \mathcal{U} \to \mathbb{R}$. In this section, we investigate the issue of expected revenue maximizing mechanism design. We search over all incentive compatible and individually rational mechanisms.

**Definition 12** A mechanism $(x, t)$ is **individually rational** if for every $u \in \mathcal{U}$, $u(x(u), t(u)) \geq 0$.

To compute expected payment from a mechanism, we need to have a prior over the types of the agents. By Theorem 3, a mechanism only needs information about the willingness to pay of the agent at one point (which will be determined as a part of the optimization exercise). Hence, the expected payment can be computed by using prior information on willingness to pay (WP). For every transfer $\tau \in \mathbb{R}$, we assume that WP values are drawn independently from an identical distribution. We denote it by $G_\tau$ at transfer $\tau$. Hence, $G_\tau(y)$ will denote the probability that the agent has a type $u$ such that $WP(u, \tau) \leq y$. We also assume that at every $\tau \in \mathbb{R}$, $G_\tau$ admits a density function, which we denote by $g_\tau$.

The expected revenue from a mechanism $M \equiv (x, t)$ is denoted as $R(M)$. This will be computed using the prior. If we are given a type space $\mathcal{U}$, we will denote by $\mathcal{M}(\mathcal{U})$ the set of all incentive compatible and individually rational mechanisms. We will say that a mechanism $M \in \mathcal{M}(\mathcal{U})$ is **optimal** in type space $\mathcal{U}$ if

$$R(M) \geq R(M') \text{ for all } M' \in \mathcal{M}(\mathcal{U}).$$

To describe the optimal mechanism, we require $G_0$ to satisfy the following condition.

**Assumption MHR:** $G_0$ satisfies **monotone hazard rate** (MHR) condition if $\frac{g_0(y)}{1-G_0(y)}$ is non-decreasing in $y$.

If $G_0$ satisfies MHR condition, then $y = \frac{1-G_0(y)}{g_0(y)}$ has a unique solution. Denote this unique solution as $r^*$.

**Definition 13** A mechanism $(x, t)$ is a **$r^*$-reserve price** mechanism if for all $u \in \mathcal{U}$,

$$\begin{align*}
WP(u, 0) > r^* \Rightarrow x(u) &= 1 \\
WP(u, 0) < r^* \Rightarrow x(u) &= 0 \\
t(u) &= x(u) \cdot r^*
\end{align*}$$
In a $r^*$-reserve price mechanism, whenever the object is sold, the agent pays $r^*$. When the object is not sold, he pays 0. The object is sold to the agent at price $r^*$ if his willingness to pay at 0 is larger than $r^*$. If the willingness to pay at 0 is less than $r^*$, then the object is not sold to the agent. Hence, an $r^*$-reserve price mechanism only requires WP information of agents at zero transfer. We show here that such a mechanism is optimal.

**Theorem 7** Suppose $\mathcal{U}$ is a rich classical type space and $G_0$ satisfies the MHR condition. Then, an $r^*$-reserve price mechanism is an optimal mechanism in $\mathcal{M}(\mathcal{U})$.

Theorem 7 shows that the optimal mechanism for selling an indivisible object to one buyer is still very similar to the quasilinear domain. As we observed in remarks following Theorem 3, revenue equivalence may not hold in these models. In that sense, it is interesting that we are able to provide a solution of the optimal mechanism.

Unfortunately, this is no longer true in the multiple buyer case, which we illustrate next.

### 4.2 Single Object Auction

We now discuss some applications of Theorem 3 for the many agents case. Let $N = \{1, \ldots, n\}$ be the set of agents. There is a single indivisible object to be allocated among the $n$ agents.

In the first application, the designer is a seller who is interested in maximizing his expected revenue from allocating the object. Though giving a precise description of the expected revenue maximizing mechanism seems analytically challenging, we give some qualitative properties of the optimal mechanism.

In the second application, the designer is a social planner who is interested in allocating the object fairly. We use a minimal notion of fairness, which we call anonymity - outcomes of agents should not depend on their identities. We characterize the class of anonymous and DSIC mechanisms satisfying an additional property called the loser payment independence (LPI) - LPI requires that whenever an agent does not receive an object, his transfer amount is the same. This class of mechanisms can be thought to be a generalization of the class of Groves mechanisms to this setting.

#### 4.2.1 The Model and Extension of Theorem 3

The type of each agent $i \in N$ is a classical utility function $u_i \in \mathcal{U}^c$. As before, we will denote by $\mathcal{U}$ an arbitrary type space which is a subset of $\mathcal{U}^c$. An allocation rule is a map $x : \mathcal{U}^n \to N \cup \{0\}$, where choosing 0 indicates not giving the object to any of the agents. Notationally, at every type profile $\mathbf{u} \equiv (u_1, \ldots, u_n) \in \mathcal{U}^n$, we will write $x_i(\mathbf{u}) \in \{0, 1\}$ to
denote if agent $i$ gets the object at type profile $u$. We use the standard notation $u_{-i}$ to denote a profile of types that does not include the type of agent $i$.

The payment rule of agent $i$ is a map $t_i : \mathcal{U}^n \to \mathbb{R}$. We will denote by $t \equiv (t_1, \ldots, t_n)$ a collection of payment rules. A mechanism is a tuple $(x, t)$. The standard notion of dominant strategy incentive compatibility is defined as follows.

**Definition 14** A mechanism $(x, t)$ is **dominant strategy incentive compatible (DSIC)** if for every $i \in N$, for every $u_{-i} \in \mathcal{U}^{n-1}$, and for every $u_i, u'_i \in \mathcal{U}_i$, we have

\[
u_i(x_i(u_i, u_{-i}), t_i(u_i, u_{-i})) \geq \nu_i(t_i(u'_i, u_{-i}), t_i(u'_i, u_{-i})).\]

With this modification, Theorem 3 extends straightforwardly to the $n$-agent case. For completeness, we state the result without a proof.

**Theorem 8 (n-agent Theorem 3)** Suppose $\mathcal{U}^n$ is a rich classical type space and $(x, t)$ is a mechanism defined on $\mathcal{U}$. Then, $(x, t)$ is DSIC if and only if for every agent $i \in N$, there exists a map $\tau_0^i : \mathcal{U}^{n-1} \to \mathbb{R}$ such that for every $u_{-i} \in \mathcal{U}^{n-1},$

- **Monotonicity.** for every $u_i, u'_i \in \mathcal{U}$ with $x(u_i, u_{-i}) = 1,$

\[
\left[WP(u'_i, \tau_0^i(u_{-i})) > WP(u_i, \tau_0^i(u_{-i}))\right] \Rightarrow x(u'_i, u_{-i}) = 1,
\]

- **Payment formula.** for every $u_i \in \mathcal{U},$

\[
t_i(u_i, u_{-i}) = \tau_0^i(u_{-i}) + x_i(u_i, u_{-i}) \cdot \tau_1^i(u_{-i}),
\]

where $\tau_1^i(u_{-i}) = \inf\{WP(u'_i, \tau_0^i(u_{-i})) : u'_i \in \mathcal{U}, x(u'_i, u_{-i}) = 1\}.$

Though Theorem 8 is a simple extension of Theorem 3 to the multi-agent case, it shows the informational complexity of an incentive compatible mechanism in the multi-agent model.

Now, $\tau_0^i$ and $\tau_1^i$ are maps that depend on the preferences of other agents. As a result, the mechanism must elicit infinite number of willingness to pay of each agent. In contrast, a one-agent incentive compatible mechanism only elicits one value of willingness to pay.

In the remainder of the section, we present two applications that apply Theorem 8 to get explicit characterization of classes of mechanisms.

### 4.2.2 Optimal WP-mechanisms

In this section, we explore how to extend the one-agent optimal auction analysis of Section 4.1 to the multiple agent case. There are two difficulties in extending the analysis of the one agent model.
1. First, the two parameters $\tau_{0i}^0$ and $\tau_{1i}^1$ will now depend on the preferences of agents other than $i$ - note that these are infinite dimensional objects. As a result, the one-dimensionality nature of the problem in the one-agent case (Section 4.1) is lost. This makes optimization difficult. To partially overcome this intractability issue, we focus attention on a simpler class of mechanisms that we call WP-mechanisms and make some observations about optimality inside this class.

2. The second problem with the multi-agent case is the payment to losing agents. To remind, the analysis in Section 4.1 became tractable because the payment of the losing agent (i.e., if an agent does not get the object) could be fixed to zero - this is achieved by Lemma 3 (see proof of Theorem 7 in the Appendix). The technique of Lemma 3 does not extend to the multiple agents case because when there are more than one agent, the new mechanism constructed in Lemma 3 may not necessarily be feasible. Since the losing agents’ payment cannot be fixed at zero, it adds new problems to the optimization.

Below, we focus on a restricted class of mechanisms and show that standard Myersonian techniques can be extended inside this class. This allows us to make some observations about the nature of the expected welfare maximizing mechanism.

We say two type profiles $u$ and $u'$ are **WP-equivalent** at $t$ if $WP(u_i, t) = WP(u'_i, t)$ for all $i \in N$. The following class of mechanisms fixes the payment of losing agents at some value and then elicits only the WP values of agents at that payment.

**Definition 15** A mechanism $(x, t)$ is a **WP-mechanism** if there exists a real number $\tau^0$ such that

1. $x_i(u) = 0$ implies $t_i(u) = \tau^0$ for all $u$ and for all $i \in N$,
2. $x_i(u) = x_i(u')$ and $t_i(u) = t_i(u')$ for all $i \in N$ if $u$ and $u'$ are WP-equivalent at $\tau^0$.

By Theorem 8, every WP-mechanism must announce a $\tau^0$ (the amount a losing agent pays), and determines the allocation and transfer amounts based on the WP values of agents at $\tau^0$. Given a value of $\tau^0$ and a type profile $u$, we denote the profile of $WP(u_1, \tau^0), \ldots, WP(u_n, \tau^0)$ as $w(\tau^0)$. For simplicity, we denote by $x_i(w(\tau^0))$ the allocation $x_i(u)$ and by $t_i(w(\tau^0))$ the transfer $t_i(u)$.

By Theorem 8, the payment of agent $i$ at this type profile is

$$t_i(w(\tau^0)) = \tau^0 + \tau_{1i}^1(w_{-i}(\tau^0))x_i(w(\tau^0)).$$
We need to compute the expected payment of each agent from a mechanism. For this, we associate priors on the WP values at $\tau^0$. We assume that for every $t \in \mathbb{R}$, the WP values at $t$ are drawn from some absolutely continuous distribution $G(\cdot; t)$ with density $g(\cdot; t)$ from a support $(0, \beta)$, where $\beta \in \mathbb{R} \cup \{\infty\}$ - note that the support of WP values is independent of $t$.\(^6\) We also assume that $G(\cdot; t)$ is regular in the sense that for all $w > w'$ and for all $t \leq 0$, we have
\[
\frac{g(w; t)}{1 - G(w; t)} \geq \frac{g(w'; t)}{1 - G(w; t)}.
\]

Using such a distribution, we compute the expected payment of agent from a WP-mechanism $(x, t)$ and denote it as $R_i(x, t)$.

A mechanism $(x, t)$ is individually rational if for every $i \in N$ and every $u$, we have $u_i(x_i(u), t_i(u)) \geq 0$. The proof of Lemma 2 can be replicated to show that a mechanism $(x, t)$ is individually rational if and only if the payment of losing agents is non-positive, i.e., for a WP-mechanism at $\tau^0$, we must have $\tau^0 \leq 0$.

A WP-mechanism $(x, t)$ is an optimal WP-mechanism if it is DSIC and individually rational, and for every DSIC and individually rational WP-mechanism $(x', t')$ we have $R(x, t) \geq R(x', t')$.

We provide a partial description of the optimal WP-mechanism below.

**Theorem 9** An optimal WP-mechanism satisfies the following. There exists a $\tau^0 \leq 0$ such that

- each agent is paid $-\tau^0$ before he announces his type,
- a reserve price $r = \phi^{-1}(0)$ is announced, where $\phi$ is the map $\phi(w) = w - \frac{1 - G(w; \tau^0)}{g(w; \tau^0)}$ for all $w$,
- for all $u$,
  - the object is not allocated and no transfers are made if $WP(u_i, \tau^0) < r \ \forall \ i \in N$;
  - else it is allocated to an agent with highest $WP(\cdot; \tau^0)$ value.

In the latter case, the winning agent $i$ pays an amount equal to $\max(r, \max_{j \neq i} WP(u_j, \tau^0))$.

Essentially, Theorem 9 says that the optimal WP-mechanism is the standard Vickrey auction with the appropriate reserve price, but with two significant differences: (1) some payment $-\tau^0$ may have to be made to all the agents before the start of the auction; and (2)\(^6\) This assumption and the symmetry assumption can be changed without changing the qualitative nature of the results below.
the Vickrey auction with reserve price is conducted with respect to the willingness to pay at \( WP(\cdot, \tau^0) \).

In the Appendix, we provide a proof of Theorem 9 - once we use Theorem 8, it follows standard Myersonian techniques. The computation of optimal value of \( \tau^0 \) will depend on the particulars of the model. It seems unlikely that a particular recommendation of \( \tau^0 \) can be made for a general enough class of models.

\[ WP(\cdot, \tau^0) \]

4.3 A Characterization of Anonymous Auctions

In this application, we are interested in allocating the object using a fair mechanism. The notion of fairness we use imposes a minimal notion of fairness.

**Definition 16** A mechanism \((x, t)\) is anonymous if for every \( i, j \in N \), for every \( u_{-ij} \in U^{n-2} \), and for every \( u_i, u_j \in U \), we have

\[
u_i(x_i(u_i, u_j, u_{-ij}), t_i(u_i, u_j, u_{-ij})) = u_j(x_j(u_i', u_j', u_{-ij}), t_j(u_i', u_j', u_{-ij}))
\]

where \( u_i' = u_j \) and \( u_j' = u_i \).

We now define the adjusted Vickrey auction with a variable reserve price. This is an extension of the Vickrey auction defined on this problem by Saitoh and Serizawa (2008). We can think of our mechanism as a class of Clarke-Groves mechanisms (Clarke, 1971; Groves, 1973) for allocating a single object with non-quasilinear preferences.

These mechanisms are defined by an anonymous but variable reserve price map \( \tau : U^{n-1} \rightarrow \mathbb{R} \). For every agent \( i \), the reserve price of agent \( i \) at a profile \((u_i, u_{-i})\) is \( r(u_{-i}) \). These mechanisms also specify \( \tau^0 \in \mathbb{R} \), which is the payment of any losing agent. To define the mechanism, we define the following notations. Given a reserve price map \( r \), at every profile \( \mathbf{u} \in U^n \), define

\[
W(\mathbf{u}; r) := \{ i \in N : WP(u_i, \tau^0) \geq \max_{j \neq i} r(u_{-i}) \}
\]

\[
W^*(\mathbf{u}; r) := \{ i \in N : WP(u_i, \tau^0) > \max_{j \neq i} r(u_{-i}) \}
\]

Note that if \( W^*(\mathbf{u}; r) \) is non-empty, then it is a singleton. \( W(\mathbf{u}; r) \) are the set of agents who have the highest willingness to pay (at \( \tau^0 \)) and whose willingness to pay (at \( \tau^0 \)) is higher than their respective reserve prices. Now, we describe the mechanism. The mechanism is a little complicated to describe because of tie-breaking issues. Informally, it fixes a reserve price for each agent for every type profile of other agents. At every type profile, it considers the set of agents who have the highest WP at \( \tau^0 \) and allocates the object to one of these agents if his willingness to pay is larger than the reserve price.

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Definition 17 The mechanism \((x^*, t^*)\) is an adjusted Vickrey auction with a variable reserve price (AVVR) mechanism if there exists a real number \(\tau^0 \in \mathbb{R}\) and a map \(r : \mathcal{U}^{n-1} \to \mathbb{R}\) such that at every type profile \(u \in \mathcal{U}^n\), we have

- if \(W(u; r) = \emptyset\), then \(x^*(u) = 0\) for all \(i \in N\),
- if \(W^*(u; r) \neq \emptyset\), then \(x^*(u) \in W^*(u; r)\), and
- if \(x^*(u) \neq 0\), then \(x^*(u) \in W(u; r)\),

and for every \(i \in N\),

\[
t^*_i(u) = \tau^0 + x^*_i(u) \cdot \max \left( r(u_{-i}), \max_{j \neq i} WP(u_j, \tau^0) \right).
\]

Payments in the AVVR mechanism is like a Vickrey auction with personalized reserve price - winning agent pays the maximum of his reserve price and the second highest willingness to pay (plus the \(\tau^0\) amount). Usually, the \(\tau^0\) in an arbitrary incentive compatible mechanism could depend on the agent and the reports of other agents. But the AVVR mechanism uses simpler payments for losing agents. In general, we restrict our analysis below to such mechanisms.

Definition 18 A mechanism \((x, t)\) satisfies loser payment independence (LPI) if there exists a real number \(\tau^0\) such that for every \(i \in N\) and for every \(u \in \mathcal{U}^n\) with \(x_i(u) = 0\) we have \(t_i(u) = \tau^0\).

We are now ready to state the characterization result for AVVR mechanism - it is proved in the Appendix using Theorem 8.

Theorem 10 Suppose \(\mathcal{U}\) is a rich classical type space. Let \((x, t)\) be a mechanism defined on \(\mathcal{U}^n\). Then, the following statements are equivalent.

1. \((x, t)\) is DSIC, anonymous, and satisfies LPI.
2. \((x, t)\) is an AVVR mechanism.

The proof in Theorem 10 constructs one reserve price map given a DSIC, anonymous mechanism satisfying LPI. But there can be other choices for the reserve price map also. Any such choice leads to an AVVR mechanism.
The literature on mechanism design with quasilinearity is long, and almost impossible to describe exhaustively. We discussed some relevant papers in detail in Section 3.2, but refer to the reader to two excellent books on this topic (Vohra, 2011; Borgers et al., 2015). After Myerson (1981), a long literature has focused on extending his monotonicity and revenue equivalence characterizations to various multidimensional models, where more than two alternatives are allocated and agents have values for each of those alternatives. Because of quasilinearity, the allocation rule and payment rule of a mechanism appear in separable form in the utility function of the agent. This allows one to provide separate characterization of implementable allocation rules (i.e., allocation rules for which payment rules can be found to make the tuple incentive compatible) and the class of payment rules that implement an implementable allocation rule. The former is characterized by a simple monotonicity property - results in that spirit appear in Jehiel et al. (1999); Bikhchandani et al. (2006); Saks and Yu (2005); Ashlagi et al. (2010); Cuff et al. (2012); Muller et al. (2007); Mishra and Roy (2013); Mishra et al. (2014); Carbajal and Müller (2015). The latter is characterized by a revenue equivalence (or envelope theorem) formula - results in that spirit are in Krishna and Maenner (2001); Milgrom and Segal (2002); Chung and Olszewski (2007); Heydenreich et al. (2009). These results exploit the geometric structure induced by quasilinearity, and make use of convex analysis machinery (Rockafellar, 1970).

In the absence of quasilinearity, it is not possible to provide separate characterization of monotonicity and revenue equivalence since the allocation and payment are no longer separable. Our monotonicity condition is indeed a condition on the mechanism, but reduces to the monotonicity condition on allocation rule used in quasilinear type spaces. Unlike the quasilinear type space, we find revenue uniqueness of incentive compatible mechanisms in our benchmark non-quasilinear type space. Further, we do not (and cannot) rely on convex analysis techniques.

There is a short but important literature on mechanism design with non-quasilinear preferences. The closest paper to ours is Kos and Messner (2013). They derive a necessary condition for incentive compatibility in a model with non-quasilinearity. Their condition is a generalization of the cycle monotonicity condition in Rochet (1987) for quasilinear preferences. They show that their condition is not sufficient for incentive compatibility. In contrast our monotonicity condition is significantly weaker than their condition and is a necessary and sufficient condition for incentive compatibility. However, we focus on deter-
ministic mechanisms in rich type spaces, and Kos and Messner (2013) do not make such assumptions.

Baisa (2016) considers the single object auction model and allows for randomization in a model with non-quasilinear preferences. He introduces a novel mechanism in his setting and studies its optimality properties (in terms of revenue maximization). Garratt and Pycia (2016) study the bilateral trading model with non-quasilinear preferences and allows for randomization. Their main finding is that for generic set of non-quasilinear preferences the Myerson-Satterthwaite impossibility result (Myerson and Satterthwaite, 1983) on bilateral trading disappears. Unlike both these papers, we do not consider randomization. More importantly, we have a more general model of non-quasilinear preferences, albeit with deterministic mechanisms. In a recent paper, Noldeke and Samuelson (2015) analyze principal-agent problem and matching problem with non-quasilinear preferences. They establish a certain implementation duality using abstract convex analysis techniques, which allows them to extend results from quasilinear type spaces. Their topological assumptions do not apply to our model, and hence, their results are different from ours. Moreover, our results are specific to deterministic mechanisms, which is not assumed in Noldeke and Samuelson (2015).

There is a literature on axiomatic treatment of mechanisms in non-quasilinear models. This literature identifies specific mechanisms in different problems with non-quasilinear preferences and axiomatically characterizes these mechanisms. The central axiom in this literature is dominant strategy incentive compatibility, but it is often accompanied by Pareto efficiency, individual rationality, and other axioms. A seminal paper in this literature is Demange and Gale (1985), who consider the matching model without quasilinearity and define the minimum price Walrasian equilibrium (MPWE) mechanism. They show that the MPWE mechanism is stable and dominant strategy incentive compatible. Morimoto and Serizawa (2015) characterize the MPWE mechanism using Pareto efficiency, individual rationality, incentive compatibility, and other axioms on payments - see an extension of this characterization in a smaller type space in Zhou and Serizawa (2015). These characterizations require a rich type space in the matching model (unit-demand preferences of agents). Saitoh and Serizawa (2008); Sakai (2008) define the generalization of Vickrey auction in the single object auction model with non-quasilinear preferences. They characterize the Vickrey auction in a rich non-quasilinear type space using incentive compatibility, Pareto efficiency, individual rationality, and no-subsidy for losers. Sakai (2013b) and Adachi (2014) characterize the same generalized Vickrey auction mechanism using incentive compatibility, continuity, and some fairness axioms. Sakai (2013a) considers an extension of this generalized Vickrey auction with reserve price, and characterizes it incentive compatibility, weak
Pareto efficiency, and zero-payment for losers. The mechanism we propose for allocating an object is more general than the auction in Sakai (2013a) since it has personalized reserve prices for every agent which vary with preferences of other agents. Hashimoto and Saitoh (2010) define the generalization of the Pivotal mechanism in the single public good provision model with non-quasilinear preferences and characterize it using Pareto efficiency, incentive compatibility, and an equity axiom. When the set of preferences include all or a very rich class of non-quasilinear preferences, impossibility results exist - no incentive compatible and Pareto efficient mechanism exists that satisfy other mild axioms (Kazumura and Serizawa, 2015; Ma et al., 2016).

The main differences between this literature and our paper is that (a) we have general results for a very general class of problems whereas each paper in this literature focuses on a particular mechanism defined on a specific problem; (b) our focus on axiomatic characterization of a class of mechanisms for single object allocation in Section 4.3 illustrates the use of a particular general result, and that class of mechanisms is more general than any specific mechanism discussed in the literature.

There is a literature in auction theory and algorithmic game theory on single object auctions with budget-constrained bidders - see Che and Gale (2000); Pai and Vohra (2014); Ashlagi et al. (2010); Lavi and May (2012). Mualem (2015) considers the budget-constrained single object auction problem. She extends Rochet’s cycle monotonicity (Rochet, 1987) characterization to her problem, and finds conditions under which revenue equivalence holds. The budget-constraint in these papers introduces a particular form of non-quasilinearity in preferences of agents. Further, the budget-constraint in these models is hard, i.e., the utility from any payment above the budget is minus infinity. This assumption is not satisfied by the preferences considered in our model. Hence, we cannot apply our results to these models directly. However, if we consider soft budget-constraints (where agents can borrow from banks at an interest rate), then our results on the two-alternatives case will apply to some of these models.

6 Conclusion

We have presented a systematic analysis of the structure of incentive compatible mechanisms when agents have non-quasilinear preferences. Our characterization results give novel insights into the complex nature of incentive constraints in this model. It also explores the tractability of performing optimization with non-quasilinear preferences. In particular, we apply our results to some of the models where similar results with quasilinear preferences have been
successfully applied. While we were successful in some of them, we only have partial answers in others. This shows the delicate nature of incentive compatible mechanisms as we move from quasilinear to non-quasilinear preferences.

A troubling feature of mechanisms in non-quasilinear type spaces is the informational requirement - preferences over infinite number of consumption bundles have to be elicited. Mechanisms that only elicit finite amount of information must be studied and its performance must be compared with a general mechanism. This will be crucial to apply mechanism design theory to non-quasilinear type spaces. We leave this important issue as an area of future research. Another area of future research will be to find specific models of non-quasilinear type spaces and apply our results in such models.
APPENDIX: OMITTED PROOFS

Proof of Theorem 1

Proof: Suppose \((x, t)\) is incentive compatible. Consider any \(R, R'\) with \(x(R) = x(R') = a\). By incentive compatibility

\[(a, t(R)) \sim (a, t(R')) \text{ and } (a, t(R')) \sim (a, t(R)).\]

Money monotonicity of classical preferences implies that \(t(R) = t(R')\). Hence, define for every \(a \in A\), \(\kappa(a) := t(R)\) for some \(R \in \mathcal{R}\) with \(x(R) = a\). Hence, 2(b) holds.

Pick \(R, R' \in \mathcal{R}\) such that \(x(R) = a, x(R') = b\). By incentive compatibility

\[(a, \kappa(a)) \sim (b, \kappa(b)) \sim (a, V^R(a, (b, \kappa(b)))).\]

Hence, \(\kappa(a) \leq V^R(a, (b, \kappa(b)))\). But incentive compatibility also implies

\[(a, V^{R'}(a, (b, \kappa(b)))) \sim (b, \kappa(b)) \sim (a, \kappa(a)).\]

Hence, \(\kappa(a) \geq V^{R'}(a, (b, \kappa(b)))\). Combining these two observations, we get

\[V^R(a, (b, \kappa(b))) \geq V^{R'}(a, (b, \kappa(b))).\]

Hence, \((x, t)\) is monotone.

We now establish the converse implication. Suppose \((x, t)\) is monotone and there exists \(\kappa : A \rightarrow \mathbb{R}\) such that for every \(R\), \(t(R) = \kappa(x(R))\).

Assume for contradiction that there is a preference \(R\) such that \(x(R) = a\) and an another preference \(\hat{R}\) such that \(x(\hat{R}) = b \neq a\) but \((b, \kappa(b)) \sim (a, \kappa(a))\). This implies that \(\kappa(a) > V^R(a, (b, \kappa(b)))\). By monotonicity,

\[V^R(a, (b, \kappa(b))) \geq V^{\hat{R}}(a, (b, \kappa(b))) \text{ and } V^{\hat{R}}(b, (a, \kappa(a))) \geq V^R(b, (a, \kappa(a))).\]

Since \(\kappa(a) > V^R(a, (b, \kappa(b)))\), we get \(\kappa(b) < V^R(b, (a, \kappa(a)))\) - see Figure 4 for an illustration. This implies that we can choose \(\epsilon > 0\) but arbitrarily close to zero such that

\[\kappa(b) < V^R(b, (a, \kappa(a))) - \epsilon = \min(V^R(b, (a, \kappa(a))), V^{\hat{R}}(b, (a, \kappa(a)))) - \epsilon. \quad (1)\]

Since \(\kappa(a) > V^R(a, (b, \kappa(b)))\), we have

\[V^R(c, (b, \kappa(b))) < V^R(c, (a, \kappa(a))) \text{ and } V^{\hat{R}}(c, (b, \kappa(b))) < V^{\hat{R}}(c, (a, \kappa(a))) \text{ for all } c \notin \{b, a\}.\]
Hence, for any $c \notin \{b, a\}$, we have
\[
\min(V^R(c, (b, \kappa(b))), V^\hat{R}(c, (b, \kappa(b)))) < \min(V^R(c, (a, \kappa(a))), V^\hat{R}(c, (a, \kappa(a)))). \tag{2}
\]

Choose such an $\epsilon > 0$ but arbitrarily close to zero to satisfy Inequality 1 and define $v$ and $v'$ as follows.
\[
v_c := \begin{cases} 
\kappa(c) & \text{if } c = b \\
\min(V^R(c, (b, \kappa(b))), V^\hat{R}(c, (b, \kappa(b)))) - \epsilon & \text{if } c \in A \setminus \{b\}.
\end{cases}
\]
\[
v'_c := \begin{cases} 
\kappa(c) & \text{if } c = a \\
\min(V^R(c, (a, \kappa(a))), V^\hat{R}(c, (a, \kappa(a)))) - \epsilon & \text{if } c \in A \setminus \{a\}.
\end{cases}
\]

We first show that $v < v'$. To see this, note that $v_a < V^R(a, (b, \kappa(b))) < \kappa(a) = v'_a$, where the first inequality follows from the definition of $v$ and the second inequality follows because $\kappa(a) > V^R(a, (b, \kappa(b)))$. By Inequality 1, we have
\[
v_b = \kappa(b) < \min(V^R(b, (a, \kappa(a))), V^\hat{R}(b, (a, \kappa(a)))) - \epsilon = v'_b.
\]
For any $c \notin \{a, b\}$, Inequality (2) implies that $v_c < v'_c$.

Next, we show that for sufficiently small $\epsilon > 0$, $v$ and $v'$ respect $\succ$. We consider various cases.

**CASE 1.** Suppose $a \succ b$. Then, using the fact that $R$ is consistent with $\succ$, we get $V^R(b, (a, \kappa(a))) < \kappa(a)$. Using the fact that $v'_b < V^R(b, (a, \kappa(a)))$ and $v'_a = \kappa(a)$, we get $v'_a > v'_b$. Now,
\[
v_a = \min(V^R(a, (b, \kappa(b))), V^\hat{R}(a, (b, \kappa(b)))) - \epsilon = V^\hat{R}(a, (b, \kappa(b))) - \epsilon > \kappa(b) - \epsilon = v_b - \epsilon,
\]
where the strict inequality follows because $\hat{R}$ is consistent with $\succ$. Since $\epsilon$ is chosen arbitrarily close to zero, we get $v_a > v_b$.

A similar argument can be made if $b \succ a$.

**CASE 2.** Suppose $a \succ c$, where $c \notin \{a, b\}$. Then $v'_a = \kappa(a) > V^R(c, (a, \kappa(a))) > v'_c$, where the first inequality follows from the fact that $R$ is consistent with $\succ$ and the second follows from the definition of $v'_c$. Now,
\[
v_a = \min(V^R(a, (b, \kappa(b))), V^\hat{R}(a, (b, \kappa(b)))) - \epsilon = V^\hat{R}(a, (b, \kappa(b))) - \epsilon > V^\hat{R}(c, (b, \kappa(b))) - \epsilon \geq v_c - \epsilon,
\]
where the second strict inequality follows from the fact that $\hat{R}$ is consistent with $\succ$. Since $\epsilon$ is arbitrarily close to zero, we get that $v_a > v_c$. 

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A similar argument can be made if \( c \succ a \). We also skip the cases when \( b \succ c \) and \( c \succ b \) since they are similar.

**Case 3.** Suppose \( c \succ d \), where \( c, d \notin \{a, b\} \). Then,

\[
v_c = \min(V^R(c, (b, \kappa(b))), V^\hat{R}(c, (b, \kappa(b)))) - \epsilon > \min(V^R(d, (b, \kappa(b))), V^\hat{R}(d, (b, \kappa(b)))) - \epsilon = v_d,
\]

where the strict inequality follows from the fact that \( R \) and \( \hat{R} \) are consistent with \( \succ \). A similar argument can be made to show \( v'_c > v'_d \).

This completes all the cases, and establishes the fact that \( v \) and \( v' \) respects \( \succ \). By minimal richness, there is a preference \( \tilde{R} \) such that \( v, v' \in \mathcal{I}(\tilde{R}) \). We show an illustration of this construction in Figure 4.

![Figure 4: Illustration of construction of \( \tilde{R} \).](image)

Now, if \( x(\tilde{R}) = a \), monotonicity will imply that

\[
v_a = V^\tilde{R}(a, (b, \kappa(b))) \geq V^R(a, (b, \kappa(b))).
\]

But this contradicts the fact that \( v_a < V^R(a, (b, \kappa(b))) \). Next, if \( x(\tilde{R}) = b \), monotonicity will imply that

\[
v'_b = V^\tilde{R}(b, (a, \kappa(a))) \geq V^R(b, (a, \kappa(a))).
\]

But this contradicts the fact that \( v'_b < V^R(b, (a, \kappa(a))) \). Finally, if \( x(\tilde{R}) = c \) for some \( c \notin \{a, b\} \), monotonicity will imply that

\[
v'_c = V^\tilde{R}(c, (a, \kappa(a))) \geq V^R(c, (a, \kappa(a))).
\]

But this contradicts the fact that \( v'_c < V^R(c, (a, \kappa(a))) \). This implies that no outcome can be chosen for \( \tilde{R} \) by the mechanism, which is a contradiction. \( \blacksquare \)
Proof of Theorem 2

Proof: We start by proving an important lemma.

Lemma 1 Suppose $x$ is onto. $(x, t)$ is incentive compatible if and only if for every $R, R' \in R^+$

$$t(R) = \inf_{R: x(R) = x(R')} V^R(x(R), (x(R'), t(R'))) = \sup_{R: x(R) = x(R')} V^R(x(R), (x(R'), t(R'))).$$

Proof: Suppose $(x, t)$ is incentive compatible. By Theorem 1, there exists $\kappa : A \to \mathbb{R}$ such that $t(R) = \kappa(x(R))$ for all $R$. Pick $a, b \in A$ such that $b \succ a$. Since $(x, t)$ is incentive compatible, for any $R$ with $x(R) = a$, we have

$$(b, V^R(b, (a, \kappa(a)))) I (a, \kappa(a)) R (b, \kappa(b)).$$

This implies that $V^R(b, (a, \kappa(a))) \leq \kappa(b)$. But $b \succ a$ implies that $V^R(b, (a, \kappa(a))) > \kappa(a)$. Hence, $\kappa(b) > \kappa(a)$.

A consequence of this observation is that the vector $v$, defined as $v_c = \kappa(c)$ for all $c \in A$, respects $\succ$. Now, pick any $a \in A$. Choose any $\epsilon > 0$ but arbitrarily close to zero. Then, the vector $v'$ defined as $v'_c = v_c$ for all $c \neq a$ and $v'_a = v_a + \epsilon$, also respects $\succ$. Hence, by minimal richness, there is a preference $R$ such that $v' \in I(R)$. Note that for any $b \neq a$, we have $(a, \kappa(a)) P (b, \kappa(b))$. Hence, incentive compatibility implies that $x(R) = a$. By definition of $v$ and $v'$, we get that for all $b \neq a$,

$$V^R(a, (b, \kappa(b))) = \kappa(a) + \epsilon.$$ 

Since $\epsilon$ was arbitrarily close to zero, by letting $\epsilon \to 0$, we get

$$\kappa(a) = \inf_{R: x(R) = a} V^R(a, (b, \kappa(b)))$$

for all $b \in A$.

Similarly, we can define $v''$ as $v''_c = v_c - \epsilon$ for all $c \neq a$ and $v''_a = v_a$. Note that for sufficiently small $\epsilon > 0$, $v''$ respects $\succ$. Hence, by minimal richness, there is a preference $R''$ such that $v'' \in I(R'')$. Note that for any $b \neq a$, we have $(a, \kappa(a)) P'' (b, \kappa(b))$. Hence, incentive compatibility implies that $x(R'') = a$. By definition of $v$ and $v''$, we get that for all $b \neq a$,

$$V^{R''}(b, (a, \kappa(a))) = \kappa(b) - \epsilon.$$ 

Since $\epsilon > 0$ was arbitrarily close to zero, by letting $\epsilon \to 0$, we get

$$\kappa(b) = \sup_{R'': x(R'') = a} V^{R''}(b, (a, \kappa(a))).$$
For the converse, choose any $R, R' \in \mathcal{R}^\tau$ and we get,

$$t(R) \leq V^R(x(R), (x(R'), t(R'))).$$

Hence, $(x(R), t(R)) \not\in R(x(R'), t(R'))$. This shows that $(x, t)$ is incentive compatible. \qed

Now, suppose $(\tilde{x}, t)$ and $(\tilde{x}, t')$ are two incentive compatible mechanisms. Theorem 1 implies that there are two maps $\kappa : A \to \mathbb{R}$ and $\kappa' : A \to \mathbb{R}$ such that for all $R$, $t(R) = \kappa(x(R))$ and $t'(R) = \kappa'(x(R))$. If $\kappa(a) = \kappa'(a)$ for some $a \in A$, then by Lemma 1, $\kappa(b) = \kappa'(b) = \inf_{R:x(R)=b} V^R(b, (a, \kappa(a)))$ for all $b \in A$, and we are done.

Hence, assume for contradiction, $\kappa(a) \neq \kappa'(a)$ for all $a \in A$. Without loss of generality, assume that $\kappa(a) > \kappa'(a)$ for some $a \in A$. We first argue that $\kappa(b) > \kappa'(b)$ for all $b \in A$. To see this, by Lemma 1, for any $b \neq a$,

$$\kappa(b) = \inf_{R:x(R)=b} V^R(b, (a, \kappa(a))).$$

Since $\kappa'(a) < \kappa(a)$, again using Lemma 1, we get

$$\kappa'(b) = \inf_{R:x(R)=b} V^R(b, (a, \kappa'(a))) < \inf_{R:x(R)=b} V^R(b, (a, \kappa(a))) = \kappa(b).$$

Now, define the vectors $v$ and $v'$ as $v_c = \kappa(c)$ and $v'_c = \kappa'(c)$ for all $c \in A$. As shown in the proof of Lemma 1, both $v$ and $v'$ respect $\succ$. Pick any $a \in A$ and choose $\epsilon > 0$ but arbitrarily close to zero. Define two new vectors $\hat{v}$ and $\hat{v}'$ as follows: $\hat{v}_a = v_a + \epsilon$ and $\hat{v}'_a = v'_a - \epsilon$; $\hat{v}_c = v_c$ and $\hat{v}'_c = v'_c$ for all $c \neq a$. Since $\epsilon > 0$ but arbitrarily close to zero, $\hat{v}$ and $\hat{v}'$ respect $\succ$. Also $\hat{v} > \hat{v}'$ since $v > v'$. By minimal richness, there is a preference $R$ such that $\hat{v}, \hat{v}' \in \mathcal{I}(R)$.

Now, note that $(a, \kappa(a)) \not\in R(c, \kappa(c))$ for all $c \neq a$. By incentive compatibility of $(\tilde{x}, t)$, we get that $\tilde{x}(R) = a$. But $(c, \kappa'(c)) \not\in R(a, \kappa'(a))$ for all $c \neq a$. By incentive compatibility of $(\tilde{x}, t')$, we get that $x(R) \neq a$. This is a contradiction. \qed

**Proof of Theorem 3**

**Proof:** Suppose $(x, t)$ is incentive compatible. The fact that $(x, t)$ is WP-monotone and there exists a map $\kappa : A \to \mathbb{R}$ such that $t(R) = \kappa(x(R))$ for all $R \in \mathcal{R}$, follow from identical arguments as in Theorem 1 - note here that monotonicity in Theorem 1 implies WP-monotonicity.

We show the payment formula. If $x(R) = 0$ for all $R \in \mathcal{R}$. Then, the payment formula is trivially satisfied by setting $\tau^0 = \kappa(0)$. If $x(R) = 1$ for all $R \in \mathcal{R}$, then $x$ is not onto but

$$\inf\{WP(R, \tau^0) : R \in \mathcal{R}, x(R) = 1\} = \inf\{WP(R, \tau^0) : R \in \mathcal{R}\},$$

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is independent of any choice of \( \tau^0 \). Suppose we set

\[
\tau^1 := \inf \{ WP(R, t) : R \in \mathcal{R} \} \text{ for some } t \in \mathbb{R},
\]

and \( \tau^0 = \kappa(1) - \tau^1 \), then the payment formula equation is trivially satisfied.

We now establish the payment formula for the case where \( x \) is onto. By incentive compatibility for any \( R \in \mathcal{R} \) with \( x(R) = 1 \), we have

\[
(1, \kappa(1)) \ R (0, \kappa(0)) \ I (1, V^R(1, (0, \kappa(0)))),
\]

which implies that

\[
\kappa(1) \leq \inf_{R \in \mathcal{R} : x(R) = 1} V^R(1, (0, \kappa(0))).
\]

Let

\[
\ell := \inf_{R \in \mathcal{R} : x(R) = 1} V^R(1, (0, \kappa(0))).
\]

We show that \( \kappa(1) = \ell \).

Assume for contradiction \( \kappa(1) < \ell \). By definition, there exists a preference \( R \) such that \( x(R) = 1 \) and \( V^R(1, (0, \kappa(0))) \) is arbitrarily close to \( \ell \). Note that \( V^R(1, (0, \kappa(0))) > \kappa(1) \).

By ontoness, there is a preference \( \hat{R} \) such that \( x(\hat{R}) = 0 \). Incentive compatibility of \((x, t)\) implies that \( V^{\hat{R}}(1, (0, \kappa(0))) \leq \kappa(1) \).

Now, since \( V^{\hat{R}}(1, (0, \kappa(0))) \) is arbitrarily close to \( \ell \) but greater than \( \kappa(1) \), there is a \( \lambda \in (0, 1) \) such that

\[
v := \lambda V^{\hat{R}}(1, (0, \kappa(0))) + (1 - \lambda)V^R(1, (0, \kappa(0))),
\]

and satisfies \( \kappa(1) < v < \ell \). By convexity of the domain, there exists \( \tilde{R} \) such that \( V^{\tilde{R}}(1, (0, \kappa(0))) = v \). An illustration of construction of \( \tilde{R} \) is shown in Figure 5 - we only show the indifference curves passing through \((0, \kappa(0))\).

Since \( V^{\tilde{R}}(1, (0, \kappa(0))) > \kappa(1) \), incentive compatibility of \((x, t)\) implies that \( x(\tilde{R}) = 1 \). But \( V^{\tilde{R}}(1, (0, \kappa(0))) < \ell \) and \( x(\tilde{R}) = 1 \) contradicts the definition of \( \ell \).

This shows that \( \kappa(1) = \ell \). But note that

\[
\ell = \inf_{R \in \mathcal{R} : x(R) = 1} V^R(1, (0, \kappa(0)))
\]

\[
= \inf_{R \in \mathcal{R} : x(R) = 1} [WP(R, \kappa(0)) + \kappa(0)]
\]

\[
= \kappa(0) + \inf_{R \in \mathcal{R} : x(R) = 1} WP(R, \kappa(0)).
\]

Then, setting \( \tau^0 \equiv \kappa^0 \) and \( \tau^1 \equiv \inf_{R \in \mathcal{R} : x(R) = 1} WP(R, \tau^0) \), gives us the desired payment formula property.
Now, for the converse, suppose \((x, t)\) is monotone and satisfies Property (2) of the claim. Pick \(R\) with \(x(R) = 1\) and assume for contradiction \((0, \tau_0)\) \(P(1, \tau^0 + \tau^1)\). Then, \(\tau^0 + \tau^1 > V^R(1, (0, \tau^0)) = WP(R, \tau^0) + \tau^0\), which implies \(\tau^1 > WP(R, \tau^0)\). This violates the definition of \(\tau^1\).

Next, pick \(R\) with \(x(R) = 0\) and assume for contradiction \((1, \tau_1 + \tau_0)\) \(P(0, \tau_0)\). Then, \(\tau_0 > V^R(0, (1, \tau^0 + \tau^1))\). As a result, \(V^R(1, (0, \tau^0)) > \tau^0 + \tau^1\) or \(WP(R, \tau^0) > \tau^1\). By definition of \(\tau^1\), there is \(R'\) with \(x(R') = 1\) and \(WP(R', \tau^0)\) arbitrarily close to \(\tau^1\). In particular, we can choose \(R'\) such that \(WP(R', \tau^0) < WP(R, \tau^0)\) (this can be done because \(WP(R, \tau^0) > \tau^1\)). But \(x(R') = 1\) and \(x(R) = 0\) while \(WP(R', \tau^0) < WP(R, \tau^0)\) violates WP-monotonicity, giving us the desired contradiction.

**Proof of Theorem 4**

**Proof:** Let \(x\) be a non-trivial allocation rule on \(\mathcal{R}\), where \(\mathcal{R}^{++} \subseteq \mathcal{R}\). If \((x, t)\) and \((x, t')\) are incentive compatible mechanisms, then by Theorem 3, there exists real numbers \(\kappa(0)\) and \(\bar{\kappa}(0)\) such that for all \(R \in \mathcal{R}\), \(t(R) = \kappa(0)\) and \(t'(R) = \bar{\kappa}(0)\) if \(x(R) = 0\). \(^8\) We show that \(\kappa(0) = \bar{\kappa}(0)\), and by Theorem 3, we will be done.

Assume without loss of generality \(\kappa(0) < \bar{\kappa}(0)\). For every \(R \in \mathcal{R}\), let \(t(R) = \kappa(1)\) and \(t'(R) = \bar{\kappa}(1)\) if \(x(R) = 1\). Since \(f\) is onto, monotonicity in Theorem 3 implies that \(\kappa(1) > 0\) and \(\bar{\kappa}(1) > 0\). We consider two possible cases.

**Case 1.** Suppose \(\bar{\kappa}(1) \geq \kappa(1)\). Construct a positive income effect classical type \(R\) such that \(WP(R, \bar{\kappa}(0)) = \bar{\kappa}(1) - \epsilon\), \(WP(R, \kappa(0)) = \bar{\kappa}(1) + \epsilon\),

\(^8\)Note that since \(x\) is onto, we only need convexity of preferences for Theorem 3 to hold.
for some arbitrarily small $\epsilon > 0$. Notice that since $\bar{\kappa}(1) > 0$, we have $WP(R, \bar{\kappa}(0)) < WP(R, \kappa(0))$. Since $\mathcal{U}$ consists of all positive income effect classical types, such a $R$ exists. Since $WP(R, \bar{\kappa}(0)) < \bar{\kappa}(1)$, Theorem 3 implies that $x(R) = 0$. But $WP(R, \kappa(0)) > \bar{\kappa}(1) \geq \kappa(1)$, implies $x(R) = 1$. This is a contradiction.

**Case 2.** Suppose $\bar{\kappa}(1) < \kappa(1)$. Construct a positive income effect classical type $R$ such that

$$WP(R, \bar{\kappa}(0)) = \bar{\kappa}(1) + \epsilon, \quad WP(u, \kappa(0)) = \kappa(1) - \epsilon,$$

for some arbitrarily small $\epsilon > 0$. Notice that since $\epsilon > 0$ can be chosen arbitrarily small, $WP(R, \kappa(0)) > WP(R, \bar{\kappa}(0))$, and since the type space consist of all positive income effect types, such a $R$ exists.

Since $WP(R, \bar{\kappa}(0)) > \bar{\kappa}(1)$, Theorem 3 implies that $x(u) = 1$. On the other hand, since $WP(R, \kappa(0)) < \kappa(1)$, we have $x(u) = 0$. This is a contradiction.

Contradictions in all possible cases establish that $\kappa(0) = \bar{\kappa}(0)$. Hence, $t = t'$.

**Proof of Theorem 7**

**Proof:** We begin the proof by establishing two useful lemmas. We use Theorem 3 to first establish that every DSIC mechanism can be described by $\tau^0$ and $\tau^1$.

**Lemma 2** Suppose $(x, t)$ is an incentive compatible mechanism. Then, it is individually rational if and only if $\tau^0 \leq 0$, where $\tau^0$ is as defined in Theorem 3.

**Proof:** If $(x, t)$ is incentive compatible and individually rational, then by Theorem 3, for all $u$ with $x(u) = 0$ we have $u(0, t(u)) = u(0, \tau^0) \geq 0 = u(0, 0)$, where the inequality follows from individual rationality. Hence, $\tau^0 \leq 0$.

For the converse, suppose $\tau^0 \leq 0$. Then, for all $u$ with $x(u) = 0$, we have $u(0, t(u)) = u(0, \tau^0) \geq u(0, 0) = 0$. If $x(u') = 1$ for all $u'$, then $\tau^1$ (as defined in Theorem 3) has a value of 0. As a consequence $t(u') = \tau^0$ for all $u'$. Since $\tau^0 \leq 0$, for all $u'$, we have $u'(1, \tau^0) > u'(0, \tau^0) \geq u'(0, 0) = 0$. Now if $x(u) = 0$ for some $u$ and $x(u') = 1$ for some $u'$, by Theorem 3, $t(u) = \tau^0$ and $t(u') = \tau^0 + \tau^1$. By the definition of $\tau^1$, $WP(u', \tau^0) \geq \tau^1$. Hence,

$$t(u') = \tau^0 + \tau^1 \leq \tau^0 + WP(u', \tau^0).$$

So,

$$u'(1, t(u')) \geq u'(1, \tau^0 + WP(u', \tau^0)) = u'(0, \tau^0) \geq 0,$$
where the equality follows the definition of WP and last inequality follows from the fact that \( \tau^0 \leq 0 \).

We now describe another useful lemma.

**Lemma 3** Suppose \((x, t)\) is an incentive compatible and individually rational mechanism. Then, there exists an incentive compatible and individually rational mechanism \((\tilde{x}, \tilde{t})\) such that for all \(u \in \mathcal{U}\)

- \(\tilde{t}(u) = 0\) if \(\tilde{x}(u) = 0\)
- \(\tilde{t}(u) \geq t(u)\).

**Proof:** Suppose \((x, t)\) is an incentive compatible and individually rational mechanism with corresponding \(\tau^0\) and \(\tau^1\) (as defined in Theorem 3). By Lemma 2, \(\tau^0 \leq 0\). If \(\tau^0 = 0\), then there is nothing to prove. Suppose \(\tau^0 < 0\).

We construct a new mechanism \((\tilde{x}, \tilde{t})\) by specifying the corresponding \(\tilde{\tau}^0\) and \(\tilde{\tau}^1\). If \(x(u') = 1\) for all \(u'\), then \(\tau^1 = 0\) and \(\tilde{t}(u') = \tau^0\) by definition. Similarly, if \(x(u') = 0\) for all \(u'\), then \(t(u') = \tau^0\) by definition. In both the cases, defining \(\tilde{\tau}^0 = 0\) and \(\tilde{\tau}^1\) as

\[
\tilde{\tau}^1 = \inf \{ WP(u', 0) : u' \in \mathcal{U}, x(u') = 1 \}.
\]

Figure 6 shows the construction of \(\tilde{\tau}^1\) - it is fairly obvious that \(\tilde{\tau}^1 \geq \tau^1 + \tau^0\), but we prove this formally below.

Clearly, \(\tilde{\tau}^1 \geq 0\). Hence, \(\tilde{\tau}^1 + \tilde{\tau}^0 = \tilde{\tau}^1 \geq 0\). Assume for contradiction \(\tilde{\tau}^1 < \tau^1 + \tau^0\). Then, there exists \(u' \in \mathcal{U}\) with \(x(u') = 1\) and \(WP(u', 0) < \tau^1 + \tau^0\). Using the fact that \(x(u') = 1\), we get \(WP(u', \tau^0) \geq \tau^1\). Hence, \(WP(u', 0) < WP(u', \tau^0) + \tau^0\). This implies that

\[
u'(0, 0) = u'(1, WP(u', 0)) > u'(1, WP(u', \tau^0) + \tau^0) = u'(0, \tau^0),
\]

which is a contradiction since \(\tau^0 < 0\). This establishes that \(\tilde{\tau}^1 \geq \tau^1 + \tau^0\).

Now, pick any \(u \in \mathcal{U}\). If \(x(u) = 0\), then \(t(u) < 0\) and \(\tilde{t}(u) \geq 0\). Suppose \(x(u) = 1\). Then, by definition \(WP(u, 0) \geq \tilde{\tau}^1\). Hence, by Theorem 3, \(\tilde{x}(u) = 1\) and \(\tilde{t}(u) = \tilde{\tau}^1 \geq \tau^1 + \tau^0 = t(u)\). This completes the proof.

Lemmas 2 and 3 prove that if \((x, t)\) is an optimal mechanism then the corresponding \(\tau^0 = 0\). Hence, \(t(u) = \tau^1\) if \(x(u) = 1\) in an optimal mechanism, where \(\tau^1\) is as defined in Theorem
3 with $\tau^0 = 0$. The expected revenue from such a mechanism is thus given by $\tau^1 (1 - G_0(\tau^1))$. But the definition of $r^*$ ensures that $r^*$ is the optimal solution to $\max_z z (1 - G_0(z))$ under the MHR assumption of $G_0$. Hence, the $r^*$-reserve price mechanism is optimal.

**Proof of Theorem 9**

**Proof:** A consequence of our result in Theorem 8 is that a DSIC and IR WP-mechanism is completely described by a real number $\tau^0 \leq 0$ and the allocation rule $x$ (which only depends on the WP values at $\tau^0$).

Fix any such WP-mechanism that satisfies $\tau^0 \leq 0$. For simplicity, we will denote the WP of an agent $i$ at the specified $\tau^0$ as $w_i$. At a profile of WP $w$, the payment of agent $i$ is given by

$$t_i(w_i) = \tau^0 + \tau_i^1(w_{-i})x_i(w) = \tau^0 + w_i x_i(w) - \int_0^{w_i} x_i(y_i, w_{-i}) dy_i.$$

Hence, expected payment of agent $i$ with WP $w_i$ is given by

$$EP(w_i; \tau^0) = \tau^0 + w_i \int x_i(w_i, w_{-i}) g_{-i}(w_{-i}; \tau^0) dw_{-i} - \int_0^{w_i} x_i(y_i, w_{-i}) g_{-i}(w_{-i}; \tau^0) dw_{-i} dy_i.$$

$$= \tau^0 + w_i EF(w_i; \tau^0) - \int_0^{w_i} EF(y_i; \tau^0) dy_i,$$

where $EF(y_i; \tau^0) := \int x_i(y_i, w_{-i}) g_{-i}(w_{-i}; \tau^0) dw_{-i}$ for all $y_i$. Hence, expected revenue from
Consider the following property.

Step 1. We now do the proof in three steps.

Proof: One direction is easy and skipped. For the other direction, let \((x, t)\) be a DSIC, anonymous mechanism on a rich classical type space \(U\) satisfying LPI. By Theorem 8 and LPI property, there exists \(\tau^0\) such that for every \(i \in N\), for all \(u \in U\) with \(x_i(u) = 0\) implies 
\[ t_i(u) = \tau^0 \text{ and } x_i(u) = 1 \text{ implies } t_i(u) = \tau^0 + t \_i(u) \], where \(\kappa_i(u)\) is as defined in Theorem 8. We now do the proof in three steps.

Step 1. Consider the following property.
Definition 19 An allocation rule $x$ satisfying LPI (at $\tau^0$) is restricted WP-efficient if for all $u \in \mathcal{U}$ with $x_i(u) = 1$ for some $i \in N$, we have $WP(u_i, \tau^0) \geq WP(u_j, \tau^0)$ for all $j \in N$.

We show that $x$ is restricted WP-efficient. Assume for contradiction that there is a type profile $u$ such that $x_i(u) = 1$ and $WP(u_i, \tau^0) < WP(u_j, \tau^0)$ for some $j \neq i$. Consider $u'_i = u_j$. By definition $WP(u'_i, \tau^0) > WP(u_i, \tau^0)$ and monotonicity in Theorem 8 implies that $x_i(u'_i, u_j, u_{-ij}) = 1$. Further, $WP(u'_i, \tau^0) > WP(u_i, \tau^0) \geq \tau^1_i(u_{-i})$. By anonymity,

$$u'_i(1, \tau^0 + \tau^1_i(u_{-i})) = u_j(0, \tau^0) = u_j(1, \tau^0 + WP(u_j, \tau^0)).$$

This implies that $\tau^1_i(u_{-i}) = WP(u_j, \tau^0) = WP(u'_i, \tau^0)$, a contradiction.

Step 2. In this step, we define the reserve price map. Consider two agents $i$ and $j$. Fix a type $u_i \in \mathcal{U}$ and $u_{-ij} \in \mathcal{U}^{n-1}$. Notationally, whenever we write a type profile in this step, we first write the type of agent $i$, followed by the type of agent $j$, and then $u_{-ij}$. We first show that there is $u_i \in \mathcal{U}$ such that $x_i(u_i, u_{0, u_{-ij}}) = 1$ if and only if there is $u_j \in \mathcal{U}$ such that $x_j(u_0, u_j, u_{-ij}) = 1$. To show this, suppose $x_i(u_i, u_0, u_{-ij}) = 1$. By Theorem 8, we can choose $u_i$ such that $WP(u_i, \tau^0) > \tau^1_i(u_{0, u_{-ij}})$. Now, consider the type profile $(u_0, u_j, u_{-ij})$, where $u_j = u_i$. If $x_j(u_0, u_j, u_{-ij}) = 0$, then anonymity implies $u_j(0, \tau^0) = u_i(1, \tau^0 + \tau^1_i(u_0, u_{-ij}))$. But $u_j = u_i$ and $u_j(0, \tau^0) = u_j(1, \tau^0 + WP(u_j, \tau^0))$ implies $\tau^1_i(u_0, u_{-ij}) = WP(u_i, \tau^0)$, a contradiction.

Hence, there is $u_i \in \mathcal{U}$ such that $x_i(u_i, u_0, u_{-ij}) = 1$ if and only if there is $u_j \in \mathcal{U}$ such that $x_j(u_0, u_j, u_{-ij}) = 1$. Now, suppose that there is $u_i \in \mathcal{U}$ such that $x_i(u_i, u_0, u_{-ij}) = 1$. We show that $\tau^1_i(u_0, u_{-ij}) = \tau^1_j(u_0, u_{-ij})$. Suppose $\tau^1_i(u_0, u_{-ij}) < \tau^1_j(u_0, u_{-ij})$. We know that there is $u_j \in \mathcal{U}$ such that $x_j(u_0, u_j, u_{-ij}) = 1$. By Theorem 8,

$$WP(u_j, \tau^0) \geq \tau^1_j(u_0, u_{-ij}) > \tau^1_i(u_0, u_{-ij}).$$

Consider the type profile $(u'_i, u_0, u_{-ij})$, where $u'_i = u_j$. Note that since $WP(u'_i, \tau^0) > \tau^1_i(u_0, u_{-ij})$, by Theorem 8, $x_i(u'_i, u_0, u_{-ij}) = 1$. By anonymity,

$$u'_i(1, \tau^0 + \tau^1_i(u_0, u_{-ij})) = u_j(1, \tau^0 + \tau^1_i(u_0, u_{-ij})).$$

Using $u'_i = u_j$, we get $\tau^1_i(u_0, u_{-ij}) = \tau^1_j(u_0, u_{-ij})$, a contradiction.

Hence, if there is $u_i \in \mathcal{U}$ such that $x_i(u_i, u_0, u_{-ij}) = 1$, then define $r(u_0, u_{-ij}) = \tau^1_i(u_0, u_{-ij}) = \tau^1_j(u_0, u_{-ij})$. If $x_i(u_i, u_0, u_{-ij}) = 0$ for all $u_i \in \mathcal{U}$, then define $r(u_0, u_{-ij}) = \beta$, where we had assumed that the willingness to pay at any transfer level lies in $(0, \beta)$. 

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Step 3. We now verify the three conditions in the definition of AVVR mechanism with respect to the reserve price map $r$ defined in Step 2. Fix a type profile $u \in U^n$.

First, suppose $W(u; r) = \emptyset$. Assume for contradiction for some $i \in N$, $x_i(u) = 1$. Then, by definition, $r(u_{-i}) = \tau^1(u_{-i})$. Hence, by Theorem 8, $WP(u_i, \kappa^0) \geq \tau^1(u_{-i}) = r(u_{-i})$. Step 1 implies $i \in W(u; r)$, a contradiction.

Second, suppose $W^*(u, r) \neq \emptyset$. Then, by definition, there is some $i \in N$ such that $\{i\} = W^*(u, r)$. So, $WP(u_i, \tau^0) > r(u_{-i}) \geq \tau^1(u_{-i})$. By Theorem 8, $x_i(u) = 1$.

Finally, if $x_i(u) = 1$ for some $i \in N$, then Theorem 8 implies $WP(u_i, \tau^0) \geq \tau^1(u_{-i}) = r(u_{-i})$. Step 1 then implies that $i \in W(u; r)$.

We conclude the proof by showing the payment in this mechanism coincides with the payment formula given in the definition of the AVVR mechanism. By Theorem 8 and LPI, if $x_i(u) = 0$ then $t_i(u) = \tau^0$. Further, if $x_i(u) = 1$ then $t_i(u) = \tau^0 + \tau^1(u_{-i}) = \tau^0 + r(u_{-i})$. We show that $\max_{j \neq i} WP(u_j, \tau^0) \leq r(u_{-i})$.

To see this, suppose $r(u_{-i}) < \max_{j \neq i} WP(u_j, \tau^0)$, then we can choose $u'_i$ such that $\tau^1(u_{-i}) \leq r(u_{-i}) < WP(u'_i, \tau^0) < \max_{j \neq i} WP(u_j, \tau^0)$. By Theorem 8, $x_i(u'_i, u_{-i}) = 1$, contradicting restricted WP-efficiency in Step 1. Hence, $\max_{j \neq i} WP(u_j, \tau^0) \leq r(u_{-i})$, and this shows

$$t_i(u) = \tau^0 + \max_{j \neq i} \left( r(u_{-i}), \max WP(u_j, \kappa^0) \right).$$

This concludes the proof.


