TROPICAL GEOMETRY AND MECHANISM DESIGN

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Abstract. We use tropical geometry to analyze finite valued mechanisms. We geometrically characterize all mechanisms on arbitrary type spaces, derive geometric and algebraic criteria for weak monotonicity, incentive compatibility and revenue equivalence. As corollaries we obtain various results known in the literature and shed new light on their nature.

Keywords: Mechanism Design, Tropical Geometry, Tropical Convexity, Incentive Compatibility, Cyclical Monotonicity, Revenue Equivalence

1. Introduction

Mechanisms are games devised to implement a targeted outcome in an informationally constrained economy. A well-designed mechanism elicits agents’ private information truthfully given their strategic behavior. An important class consists of dominant strategy incentive compatible (D-IC) mechanisms. With quasi-linear utilities, the classical theorem of Ro-chet [31] states that a mechanism is (D-IC) if and only if all cycles on a certain weighted graph are nonnegative. This condition, known as cyclical monotonicity [32], is rather difficult to work with in theory. Many papers have been devoted to identifying domains on which simpler conditions, such as weak monotonicity, are sufficient for (D-IC), see [3,7,33], and further references in [37]. In particular, Saks and Yu [33] showed that if the type space is convex, then weak monotonicity implies (D-IC). Ashlagi, Braverman, Hassidim and Monderer [3] showed the converse, if the type space is not convex, then one can construct a mechanism that is weakly monotone but not (D-IC).

Contributions. Our paper contains three main results. For an arbitrary type space $T$, we give a geometric characterization of all possible mechanisms (Theorem 4.2), all possible weakly monotone mechanisms (Theorem 4.11), and all possible (D-IC) mechanisms (Theorem 4.6) that can arise on $T$. These results make checking and visualizing incentive compatibility easy, both theoretically and computationally. Our proofs give a clear understanding of how the allocation rule and geometry of the type space affect (D-IC) and weak monotonicity. We prove a generalization of a Saks-Yu in Theorem 6.3, which is closely related to the results of Kushnir and Galichon [22], discovered independently and using different techniques.

Our characterization also allows results about revenue equivalence, such as those in [9,15,16,37], to be understood in a unified way. The characterization of revenue equivalence derived here (Corollary 4.7 and Theorem 6.6) emphasizes the role of geometry. Despite being less general than that of Heydenreich, Müller, Uetz and Vohra [15], these results clarify the interplay of the allocation rules with the geometry of the type space for revenue equivalence explicitly. More importantly, it shows how assumptions pertaining to revenue equivalence and weak monotonicity can differ, and why. For instance, in Section 5 we construct type
spaces for which there are different implementable rules, which may or may not be revenue equivalent.

**Use of Tropical Geometry.** We state and prove our results using tropical geometry. This is a young and exciting field with connections to algebraic geometry, combinatorics and optimization theory. It has found many applications, amongst them phylogenetics [29], mean pay-off games [1], computational complexity [2], discrete events systems [4], chemical reaction networks [28] and pairwise ranking [13], to name a few; see the monographs [8, 23] and references therein. In particular, under the tropical algebra, deciding (D-IC) becomes a problem of solving a system of linear tropical equations. Objects in (D-IC) mechanism design problems correspond to tropical polytopes and tropical hyperplane arrangements, and economic questions about them can thus be answered using tropical convex geometry.

Many problems in mechanism design are questions of network flows [37], which in turn can be cast as problems in tropical linear algebra, as done in [4, 5, 8]. However, there are three distinct advantages using tropical mathematics. Firstly, tropical geometry provides a convenient language with crisp notation for network flow problems. Secondly, it gives new geometric insights that would be difficult to obtain from network flows alone. Thirdly, tropical geometry is more than just tropical linear algebra. It has tools from combinatorics, integer programming, and algebraic geometry, which do not have immediate network flow interpretations. For instance, the tropical Bezout theorem has been used to characterize general equilibrium with indivisibilities, see [5, 6, 36]. These works are the first to apply tropical geometry in microeconomic theory. The auctions studied there are a special case of mechanisms, although the questions addressed in the above papers are distinct from ours. This indicates that there are more applications of tropical geometry to economics.

The results in our paper do not reflect the limit of tropical methods. Instead, they just provide a starting point for a more systematic development and demonstrate the power of tropical thinking. On the other hand, we certainly do not claim that all problems in mechanism design can be solved tropically. Due to its idempotency, tropical algebra is less rich compared to standard algebra.

**Organization.** Section 2 gives the necessary concepts and results from tropical geometry. Section 3 contains the necessary terminology from mechanism design in tropical language. In Section 4 we present our main results, Theorems 4.2, 4.6 and 4.11. We collect all examples in Section 5. Section 6 elaborates on the economic interpretation of our results, including proofs of a generalization of Saks-Yu (cf. Theorem 6.3) and the geometric intuition of the Ashlagi-Braverman-Hassidim-Monderer Theorem (cf. Theorem 6.5).

**Notation.** For an integer \( n \), define \([n] = \{1, 2, \ldots, n\}\). In general, we use the \(-i\) subscript to denote the set or vector excluding its \(i\)-th component. For instance, for \( T = (T_1, \ldots, T_n) \subseteq (\mathbb{R}^m)^n \), define \( T_{-i} = (T_1, \ldots, T_{j-1}, T_{j+1}, \ldots, T_n) \subseteq (\mathbb{R}^m)^{n-1} \). For \( t \in T \), define \( t_{-i} = (t_j : j \in [n], j \neq i) \in T_{-i} \). For an \( m \times m \) matrix \( L \in \mathbb{R}^{m \times m} \), let \( G(L) \) be the directed graph on \( m \) nodes with edge weights \( L \). Let \( S_m \) denote the symmetric group on \([m]\). For sets \( A, B \subset \mathbb{R}^m \), write \( \text{dist}(A, B) \) for their distances as measured in the Euclidean norm. As a convention, we shall use the underline notation, such as \( \oplus, \odot, \mathcal{H}, \ldots \) to indicate objects defined with arithmetic done in the min-plus tropical algebra, and the overline notation \( \bar{\oplus}, \bar{\odot}, \bar{\mathcal{H}}, \ldots \) to indicate the same objects defined with arithmetic in the max-plus tropical algebra.
2. Elements of Tropical Geometry

We will be concerned with two versions of the tropical algebra on $\mathbb{R}$: max-plus and min-plus. The max-plus algebra $(\mathbb{R}, \oplus, \odot)$ is defined with tropical addition $a \oplus b := \max(a, b)$, and tropical multiplication $a \odot b := a + b$. The min-plus algebra $(\mathbb{R}, \oplus, \odot)$ is defined by $a \oplus b := \min(a, b)$, $a \odot b := a + b$. These algebras are isomorphic via the map $x \mapsto -x$, so theorems that hold for one have obvious analogues in the other.

In short, tropical mathematics is mathematics done in the min-plus or one of its isomorphic algebras. The adjective tropical in this context was coined by French mathematicians in honor of the Hungarian-born Brazilian mathematician Imre Simon. There is no deeper meaning of the word tropical, ‘it simply stands for the French view of Brazil’, as put by Maclagan and Sturmfels [23]. This section is a minimal introduction to tropical linear algebra and tropical convex geometry. For a thorough treatment we refer to [4, 8, 18, 23].

2.1. Basics. Let $L \in \mathbb{R}^{m \times m}$ be a matrix, $x \in \mathbb{R}^m$ a vector and $\lambda \in \mathbb{R}$ a scalar. As usual, scalar-vector multiplication is defined element-wise $\lambda \odot x \in \mathbb{R}^m$, $(\lambda \odot x)_i = \lambda + x_i$ for $i \in [m]$. Matrix-vector multiplication is defined by $L \odot x \in \mathbb{R}^m$, $(L \odot x)_i = \min_{j \in [m]} \{L_{ij} + x_j\}$ for $i \in [m]$. We say that $(x, \lambda)$ is an eigenvector-eigenvalue pair of $L$ if

$$L \odot x = \lambda \odot x,$$

or explicitly,

$$\min_{j \in [m]} \{L_{ij} + x_j\} = \lambda + x_i \text{ for } i \in [m].$$

By Theorem 2.1 of [11], a matrix $L \in \mathbb{R}^{m \times m}$ has a unique tropical eigenvalue. Thus one can speak of the tropical eigenvalue of a matrix $L$, denoted $\lambda(L)$. The tropical eigenspace of $L \in \mathbb{R}^{m \times m}$ is

$$\text{Eig}(L) = \{x \in \mathbb{R}^m : L \odot x = \lambda(L) \odot x\}.$$

Tropical addition is idempotent: $a \odot a = a$ for $a \in \mathbb{R}$. In particular there is no subtraction. This makes tropical linear algebra different from its classical counterpart. For instance, the determinant of $A$ equals its permanent, which is

$$\text{tdet}(L) = \bigoplus_{\sigma \in S_m} L_{1\sigma_1} \odot \ldots \odot L_{m\sigma_m} = \min_{\sigma \in S_m} (L_{1\sigma_1} + \ldots + L_{m\sigma_m}).$$

However, because tropical equations are just a set of classically linear equations and inequalities, many tropical objects can be computed using linear programming. We give three examples relevant for this paper: computing the tropical determinant, tropical eigenvalue and tropical eigenspace.

2.2. Tropical Determinant. Evaluating the tropical determinant means solving the assignment problem. Imagine that there are $m$ jobs and $m$ workers, and each worker needs to be assigned to exactly one job. Let $L_{ij}$ be the cost of paying worker $i$ to do job $j$. The company wants to find a least cost assignment. The minimal cost is thus $\text{tdet}(L)$.

While there are $m!$ possible matchings, to find the optimal matching, there is no need to evaluate them all. The classic assignment problem above is a linear program over the permutahedron, and can be solved efficiently with the Hungarian method [21].
2.3. Tropical Eigenvalues.

Theorem 2.1 ([11]). A matrix $L \in \mathbb{R}^{m \times m}$ has a unique eigenvalue $\lambda(L)$, which equals the minimum mean-weight of all simple directed cycles on $m$ vertices.

The mean-weight of a cycle on $L$ is the sum of the edges divided by the number of edges in the cycle. While there are exponentially many simple cycles, to compute $\lambda(L)$, it is not necessary to check them all. Computing the min-plus and max-plus eigenvalue are linear programs over the normalized cycle polytope, for which there is an efficient solution, see [11, 23].

2.4. Tropical Eigenspace. Parallel to classical linear algebra, the tropical eigenspace of an $m \times m$ matrix is generated by at most $m$ extreme tropical eigenvectors $v_1, \ldots, v_m$, in the sense that any $x \in \mathbf{Eig}(L)$ can be written as

$$x = a_1 \odot v_1 \oplus \cdots \oplus a_m \odot v_m$$

for some $a_1, \ldots, a_m \in \mathbb{R}$. As a set, tropical eigenspaces are polytopes [18], so-called because such a set is both a tropical and an ordinary polytope, see Section 2.8 below. To find $\mathbf{Eig}(L)$, one first subtracts $\lambda(L)$ element-wise from $L$ and reduces to the case $\lambda(L) = 0$. In this case, numerically, $v_{ij}$ can be found as the value of the shortest path from $i$ to $j$ on $G(L)$. Abstractly, these vectors are the column vectors of the Kleene star of $L$. In the following definition $I$ denotes the min-plus identity matrix with zeros on its diagonal and $+\infty$ elsewhere.

Definition 2.2. For $L \in \mathbb{R}^{m \times m}$ with $\lambda(L) = 0$, the Kleene star of $L$, denoted $L^*$, is

$$L^* = I \oplus \left( \bigoplus_{i=1}^{m} L^{\odot i} \right).$$

A matrix $M \in \mathbb{R}^{n \times n}$ is called a Kleene star if $M^* = M$.

Theorem 2.3. For $L \in \mathbb{R}^{m \times m}$, let $c_1, \ldots, c_m$ be the column vectors of $(L - \lambda(L))^*$. Then $\{c_1, \ldots, c_m\}$ is the set of tropical generators of $\mathbf{Eig}(L)$.

For this reason, Kleene stars play a fundamental role in tropical spectral theory, and thus have been extensively studied [4, 8, 17, 34]. They are also known as strong transitive closures [8, §1.6.2.1], or distance matrix [26]. Kleene stars, and thus the tropical generators of a tropical eigenspace, can be computed by multiplication and addition of tropical matrices.

2.5. Tropical Projective Torus. In many economic problems only relative valuations or prices matter, not their absolute values. In tropical terms, this means that valuations and prices are points in the tropical projective torus $\mathbb{T}P^{m-1}$. This shall be our ambient space when speaking about geometry in mechanism design problems.

A set $S \subset \mathbb{R}^m$ is closed under tropical scalar multiplication if for all $a \in \mathbb{R}$ we have $a \odot x = (a + x_1, \ldots, a + x_m) \in S$ whenever $x \in S$. Examples include the image of a matrix $L \in \mathbb{R}^{m \times m}$

$$\mathbf{Im}(L) = \{y \in \mathbb{R}^m : L\odot x, x \in \mathbb{R}^m\},$$

or its tropical eigenspace $\mathbf{Eig}(L)$. It is sufficient to consider such a set modulo tropical scalar multiplication. Define an equivalence relation $\sim$ on $\mathbb{R}^m$ via

$$x \sim y \iff x = a \odot y$$

for some $a \in \mathbb{R}^m$. 

(1)
The space $\mathbb{R}^m$ modulo $\sim$ is called the tropical projective torus, or tropical affine space $\text{T}P^{m-1}$. Explicitly, it is $\mathbb{R}^m$ modulo the line spanned by the all-one vector

$$\text{T}P^{m-1} \equiv \mathbb{R}^m / \mathbb{R} \cdot (1, \ldots, 1).$$

Note that the tropical scalar multiplication does not depend on max or min, so on the same space $\text{T}P^{m-1}$ one can speak of max-plus as well as min-plus geometric objects such as tropical polytopes and tropical hyperplane arrangements.

We follow the convention in [18,23] and identify $\text{T}P^{m-1}$ with $\mathbb{R}^{m-1}$ by normalizing the first coordinate via the following homeomorphism.

$$\text{T}P^{m-1} \to \mathbb{R}^{m-1}, [(x_1, \ldots, x_m)] \mapsto (x_2 - x_1, \ldots, x_m - x_1).$$

In particular, we use this map to visualize sets in $\text{T}P^2$. Of course, one could choose other normalizations, such as setting the $i$-th coordinate to zero for some other $i \in [m]$, or require that the sum of the coordinates is a constant.

Often, checking whether a set $S \subseteq \mathbb{R}^m$ is closed under tropical scalar multiplication is straightforward. In such cases, we shall write $S \subseteq \text{T}P^{m-1}$ without taking explicit notice. In particular, we will often write an element $x \in \text{T}P^{m-1}$ as a vector in $\mathbb{R}^m$.

### 2.6. Tropical Polytopes and Hyperplanes.

The central objects in tropical convex geometry are tropical polytopes and tropical hyperplanes. A tropical polytope contains all tropical line segments (tropical convex hull) between any two of its points. For each tropical polytope there is a finite minimal set of points generating it. Dually a tropical polytope can be represented as intersections of tropical hyperplanes. Let us elaborate.

**Definition 2.4.** The tropical *min-plus polytope* generated by vectors $\{c_1, \ldots, c_m\} \subset \mathbb{R}^m$ is

$$\text{tconv}(c_1, \ldots, c_m) = \{z_1 \circ c_1 + \ldots + z_m \circ c_m : z \in \mathbb{R}^m\}.$$

Let $L$ be the matrix whose $i$-th column is $c_i$. By rewriting, one obtains

$$\text{im}(L) = \text{tconv}(c_1, \ldots, c_m).$$

It is immediate that $\text{tconv}(c_1, \ldots, c_m) \subseteq \text{T}P^{m-1}$, and that for constants $a_1, \ldots, a_m \in \mathbb{R}$,

$$\text{tconv}(a_1 \circ c_1, \ldots, a_m \circ c_m) = \text{tconv}(c_1, \ldots, c_m).$$

Thus, we shall work in $\text{T}P^{m-1}$, viewing both $\{c_1, \ldots, c_m\}$ and their tropical convex hull as a subset of $\text{T}P^{m-1}$. Note that for a particular tropical polytope $P$, the matrix $L$ such that $\text{im}(L) = P$ is only defined up to tropical scalings of its columns. Unless stated otherwise, we shall rescale $L$ as to have zeros on its diagonal. This associates $P$ to a unique matrix $L$.

For the rest of this section, fix a matrix $L \in \mathbb{R}^{m \times m}$. Let $c_i$ denote its $i$-th column vector. The tropical polytope $\text{im}(L)$ can also be thought of in terms of its supporting tropical hyperplanes.

**Definition 2.5.** For a point $p \in \mathbb{R}^m$, $j \in [m]$, the *min-plus tropical hyperplane* with apex $-p$, denoted $\mathcal{H}(p)$, is the set of $z \in \mathbb{R}^m$ such that the minimum in the tropical inner product

$$p^{\top} \circ z = \min\{z_1 + p_1, \ldots, z_m + p_m\}$$

is achieved at least twice. Analogously, the *max-plus tropical hyperplane* with apex $-p$, denoted $\overline{\mathcal{H}}(p)$, is the set of $z \in \mathbb{R}^m$ such that the maximum in

$$p^{\top} \circ z = \max\{z_1 + p_1, \ldots, z_m + p_m\}$$

is achieved at least twice.
In classical linear algebra, the complement of a hyperplane is the union of two open half-spaces. In tropical linear algebra, the complement of a tropical hyperplane in $\mathbb{T}P^{m-1}$ is the union of $m$ open sectors.

**Definition 2.6.** The $j$-th open sector of the max-plus tropical hyperplane with apex $-p$, denoted $\mathcal{H}^o_j(p)$, is the set of $z \in \mathbb{R}^m$ such that the maximum (4) is achieved only at $j$. That is,

$$\mathcal{H}^o_j(p) := \{z \in \mathbb{T}P^{m-1} : z_j + p_j > z_k + p_k \text{ for } k \neq j\}.$$ 

Its closure is the $j$-th closed sector of the max-plus hyperplane with apex $-p$,

$$\overline{\mathcal{H}}_j(p) := \{z \in \mathbb{T}P^{m-1} : z_j + p_j \geq z_k + p_k \text{ for } k \neq j\}.$$ 

This is the set of $z \in \mathbb{R}^m$ such that the maximum (4) is achieved at $j$ and possibly at a second coordinate.

For a matrix $L \in \mathbb{R}^{m \times m}$ with zero diagonal, let $L_1, \ldots, L_m \in \mathbb{T}P^{m-1}$ be the $m$ rows of $L$, viewed as vectors in $\mathbb{T}P^{m-1}$. To simplify notation, write $\mathcal{L}_j$ for $\mathcal{H}_j(L_j)$, that is,

$$\mathcal{L}_j = \mathcal{H}_j(L_j) = \{t \in \mathbb{T}P^{m-1} : L_{jk} + t_k \leq t_j \text{ for } k \neq j\}.$$ 

Write $\mathcal{L}^o_j$ for the corresponding $j$-th open sector $\mathcal{H}^o_j(L_j)$

$$\mathcal{L}^o_j = \mathcal{H}^o_j(L_j) = \{t \in \mathbb{T}P^{m-1} : L_{jk} + t_k < t_j \text{ for } k \neq j\}.$$ 

Write $\partial \mathcal{L}_j$ for its boundary $\mathcal{L}_j \setminus \mathcal{L}^o_j$. Note that the boundary is the union of $m - 1$ half-hyperplanes

$$\partial \mathcal{L}_j = \bigcup_{k \in [m], k \neq j} \mathcal{L}_{jk},$$

where the $k$-th piece $\mathcal{L}_{jk}$ is the set

$$\mathcal{L}_{jk} = \mathcal{H}_{jk}(L_j) = \{t \in \mathbb{T}P^{m-1} : L_{jk} + t_k = t_j, L_{jk'} + t_{k'} < t_j \text{ for } k' \neq j, k\}.$$ 

As before, we use the underline notation to mean the analogous quantity in min-plus. For instance, $\mathcal{L}_j$, the $j$-th sector of the min-plus hyperplane with apex $-L_j$ in $\mathbb{T}P^{m-1}$, is

$$\mathcal{L}_j = \{t \in \mathbb{T}P^{m-1} : L_{jk} + t_k \geq t_j \text{ for } k \neq j\}.$$ 

**Definition 2.7** (Tropical hyperplane arrangement). Let $L \in \mathbb{R}^{m \times m}$, $L_1, \ldots, L_m \in \mathbb{R}^m$ be its row vectors. Consider the set of tropical hyperplanes $\{\mathcal{H}(L_i) : i \in [m]\}$ in $\mathbb{T}P^{m-1}$. The intersections of their various sectors partitions $\mathbb{T}P^{m-1}$ into a polyhedral complex called the **tropical hyperplane arrangement** $\mathcal{H}(L)$.

**Definition 2.8** (Dominant arrangement). Let $L \in \mathbb{R}^{m \times m}$, $L_1, \ldots, L_m \in \mathbb{R}^m$ be its row vectors. The set of closed sectors $\{\mathcal{L}_i : i \in [m]\}$ in $\mathbb{T}P^{m-1}$ is called the **max-plus dominant arrangement** of $L$, denoted $\overline{\mathcal{D}}(L)$. Similarly, $\{\mathcal{L}_i : i \in [m]\}$ is the **min-plus dominant arrangement** of $L$, denoted $\mathcal{D}(L)$.

**Proposition 2.9** (23). Let $L \in \mathbb{R}^{m \times m}$. The tropical hyperplane arrangement $\mathcal{H}(L)$ is a polyhedral complex. Furthermore, the union of the bounded cells of $\mathcal{H}(L)$ equals $\text{Im}(L)$. 

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2.7. Covectors. Let $L \in \mathbb{R}^{m \times m}$. Each hyperplane in $\mathcal{H}(L)$ partitions $\mathbb{TP}^{m-1}$ into $m$ sectors. The numbering of the sectors correspond to the coordinate at which the minimum is achieved once. For a given point $p \in \mathbb{TP}^{m-1}$ we can ask for its position relative to the tropical hyperplanes associated to $L$. This combinatorial information is encoded in the covectors.

Definition 2.10. The min-plus covector (or combinatorial type) of $p \in \mathbb{TP}^{m-1}$ with respect to $\mathcal{H}(L)$, denoted $\text{coVec}_L(p)$, is a matrix in $\{0, 1\}^{m \times m}$ with
\[
\text{coVec}_L(p)_{ki} = 1 \iff p \in H_k(L_i).
\]

One can think of a covector as the adjacency matrix of a $(m, m)$ bipartite graph. It has edge $(k, i)$ iff $p$ is in sector $k$ of the hyperplane with apex at $-L_i$, the $i$-th row of $L$. Covectors are a central concept in tropical convex geometry. This idea was put forward by Develin and Sturmfels as combinatorial types [12] and was subsequently further developed by Joswig and Loho [19].

Points in the same open cell of $\mathcal{H}(L)$ have the same covector. Thus, we can speak of the covector of a cell $\nu$ of $\mathcal{H}(L)$. Denote this $\text{coVec}_L(\nu)$. Say that $\text{coVec}_L(\nu)$ is invertible if for each $i \in [m]$, there exists some $j \in [m]$ such that $\text{coVec}_L(\nu)_{ij} = 1$. The following is a characterization of bounded cells in a tropical hyperplane arrangement by their covectors.

Lemma 2.11 ([12]). Let $\nu \subset \mathbb{TP}^{m-1}$ be a cell in $\mathcal{H}(L)$. Then $\nu$ is bounded (as a subset of $\mathbb{TP}^{m-1}$) if and only if $\text{coVec}_L(\nu)$ is invertible.

2.8. Polytopes and Tropical Eigenspaces.

Definition 2.12. A polytrope $P \subset \mathbb{TP}^{m-1}$ is a tropical polytope that is also an ordinary polytope.

The term polytrope was coined by Joswig and Kulas [18]. There are many equivalent characterizations of polytopes. We collect the results relevant for this paper here. The first three results are classical results, see [8, 18, 23]. The last two statements are characterizations by Murota [26], who calls polytopes $L$-convex sets.

Proposition 2.13. Let $P \subset \mathbb{TP}^{m-1}$ be a non-empty set. The following are equivalent.

1. $P$ is a polytrope.
2. There exists a matrix $M \in \mathbb{R}^{m \times m}$ such that $P = \text{Eig}(M)$.
3. There exists a matrix $M \in \mathbb{R}^{m \times m}$ such that $P = \{y \in \mathbb{TP}^{m-1} : y_i - y_j \leq M_{ij} \text{ for all } i, j \in [m]\}$.
4. $P = \text{Im}(M^*)$ for a unique Kleene star $M^*$.
5. $P$ is both a min-plus tropical polytope and a max-plus tropical polytope
6. $P$ is both a min-plus convex set and max-plus convex set.

As a set in $\mathbb{TP}^{m-1}$, a polytrope $P$ is a compact convex polytope of dimension $k \in \{0, 1, \ldots, m - 1\}$. Say that $P$ is full-dimensional if it has dimension $m - 1$. A polytrope of dimension $k < m - 1$ in $\mathbb{TP}^{m-1}$ is obtained by embedding a $k$-dimensional polytrope into a $k$-dimensional subspace of $\mathbb{TP}^{m-1}$ defined by intersections of hyperplanes of the form
\[x_i - x_j = 0\]
for some $i, j \in [m]$. In matrix terms, this means that if we write a polytrope $P$ as $P = \text{Im}(L)$, then as a set in $\mathbb{TP}^{m-1}$, precisely $k + 1$ of its columns (or rows) are unique. Thus, it is often sufficient to consider the theory for full-dimensional polytopes.
2.9. **Row-Column Duality and Minimal Polytopes.** There is a trivial duality between the min and max algebra stemming from the fact that for \(a, b \in \mathbb{R}\),

\[
\min\{a, b\} = -\max\{-a, -b\}.
\]

This translates to a trivial duality between row and column space of a matrix \(L \in \mathbb{R}^{m \times m}\) as follows.

**Lemma 2.14.** Let \(A \in \mathbb{R}^{m \times m}\). The map \(x \mapsto -x\) sends \(\mathcal{H}(L)\) to \(\mathcal{H}(-L^\top)\).

There is a more astonishing row and column duality, due to Develin and Sturmfels [12]

**Theorem 2.15** ([12]). Let \(L \in \mathbb{R}^{m \times m}\). There is an isomorphism between the polyhedral complexes \(\text{Im}(L)\) and \(\text{Im}(L^\top)\) obtained by restricting the piecewise linear maps \(\mathbb{R}^m \to \mathbb{R}^m\), \(z \mapsto y := L \odot (-z)\), and \(y \mapsto z := (-z)^\top \odot L\), to \(\text{Im}(L)\) and \(\text{Im}(L^\top)\), respectively.

By part (5) of Proposition 2.13, the isomorphism in Theorem 2.15 reduces to the trivial map \(y \mapsto -y\) induced by the map \(L \mapsto -L^\top\) if and only if \(L\) is a Kleene star, or equivalently, if and only if \(\text{Im}(L)\) is a polytrope.

When \(L = -L^\top\), then not only \(\text{Im}(L)\) is both a max-plus and min-plus tropical polytope, it has the same set of min-plus and max-plus generators. An example is the standard minimal polytrope

\[
\Delta_{m-1} = \text{conv}(0, e_1, e_1 + e_2, e_1 + e_2 + e_3, \ldots, e_1 + \ldots + e_m) + \mathbb{R} \cdot (1, \ldots, 1),
\]

where \(\text{conv}\) denotes the classical convex hull, and \(e_i\) is the \(i\)-th standard basis vector in \(\mathbb{R}^m\), that is, the vector with 1 in the \(i\)-th coordinate and zero elsewhere. Note that for this polytrope, its set of max-plus tropical generators, min-plus tropical generators and vertices as ordinary polytope all coincide. In a sense, this is the only polytrope with this property. The following theorem is essentially a restatement of [26, Theorem 7.24]. It is key to the characterization of weak monotonicity in mechanism design, see Section 6.1.

**Definition 2.16.** Say that a polytrope \(P \subset \mathbb{T}P^{m-1}\) of dimension \(k \in \{0, 1\ldots, m - 1\}\) is minimal if as a classical polytope, it has \(k + 1\) vertices.

**Theorem 2.17.** Let \(P\) be a full-dimensional tropical polytope in \(\mathbb{T}P^{m-1}\). Then the following are equivalent

1. \(P\) is a full-dimensional minimal polytrope.
2. \(P\) is both a min-plus and max-plus full-dimensional tropical polytope, with the same set of \(m\) generators.
3. Up to permutations, there exists a unique matrix \(A \in \mathbb{R}^{m \times m}\) such that \(A = -A^\top\), \(P = \text{Im}(A)\), and as a subset of \(\mathbb{T}P^{m-1}\), the columns of \(A\) are unique.
4. The Kleene star \(A^*\) of \(P\) has the form

   \[
   A^*_{ij} = p_i - p_j + a(A_{\Delta_{m-1}})_{ij},
   \]

   for some vector \(p \in \mathbb{T}P^{m-1}\) and some scalar \(a \in \mathbb{R}\), where \(A_{\Delta_{m-1}}^*\) is the Kleene star of \(\Delta_{m-1}\).
5. Up to permutations, there exists a vector \(p \in \mathbb{T}P^{m-1}\) and a scalar \(a \in \mathbb{R}\) such that

   \[
   P = p + a \cdot \Delta_{m-1}
   \]
Proof. Proposition 2.13 implies (2) \(\Leftrightarrow\) (3) and (4) \(\Leftrightarrow\) (5). We first prove (1) \(\Leftrightarrow\) (5). By Theorem 7.24, any polytrope can be regularly decomposed as unions of smaller polytropes. It follows from this proof, using the Lovász extension of the corresponding \(\text{L}\)-convex functions over \(P\), that the minimal polytropes are precisely the convex hull of maximal chains on \(\{0,1\}^m\). Up to permutation on \([m]\), such a chain is lexicographic. The convex hull of the lexicographic chain is \(\Delta_{m-1}\). So up to permutation, scaling and translation, a minimal polytrope \(P\) equals \(\Delta_{m-1}\), as needed. Now we prove (5) \(\Leftrightarrow\) (2). Suppose \(P\) is given by (5). Then it is straight-forward to check that it satisfies (2). Conversely, suppose (2). Due to Theorem 7.24, any \(\text{L}\)-convex function over \(P\) must also be \(\text{L}\)-concave. Thus, it must be a classical hyperplane. So \(P\) has a unique Lovász extension, which implies that up to scaling and translation, it must be a simplex supported on chains of \(\{0,1\}^m\). Since \(P\) is full-dimensional, the chain is maximal, and thus we have (5).

2.10. An Example. We illustrate all of the concepts above in the following example. Let \(m = 3, \delta \in [0,1]\) and consider the following points in \(\mathbb{T}\mathbb{P}^2\): \(c_1 = (0, -1, -4), c_2 = (0, \delta, 0), c_3 = (0, -4, -1)\). Putting them together as column vectors, we obtain the matrix

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
1 & -\delta & 4 \\
4 & 0 & 1
\end{pmatrix}
\]

View \(c_1, c_2, c_3\) are vectors in \(\mathbb{T}\mathbb{P}^2\), and renormalize so that the \(i\)-th coordinate of the \(i\)-th vector is zero, we obtain the new matrix

\[
L = \begin{pmatrix}
0 & \delta & -1 \\
1 & 0 & 3 \\
4 & \delta & 0
\end{pmatrix}
\]

Let \(P = \text{Im}(L)\). One can check that \(P = \text{Im}(A)\). The matrix \(L\) is the unique matrix with zero diagonal associated with \(P\).

On the graph \(G(L)\), the mean-cycle lengths of two-cycles are \(\frac{1}{2}(1+\delta), \frac{1}{2}(3+\delta)\), and \(3/2\). The mean-cycle lengths of three cycles are \(\frac{1}{3}(\delta+3+4)\) and \(\frac{1}{3}\delta\). The self-loops have mean-cycle lengths zero. So for any \(\delta \in [0,1]\), we have \(\Lambda(L) = 0\).

Assume \(\delta \in [0,1]\). Let us compute the Kleene star of \(L\). Note that

\[
L^{\otimes 2} = \begin{pmatrix}
0 & \delta \oplus (1-\delta) & -1 \\
1 & 0 & 0 \\
1+\delta & \delta & 0
\end{pmatrix},
\]

\[
L^{\otimes 3} = \begin{pmatrix}
0 & -1+\delta & -1 \\
1 & 0 & 0 \\
1+\delta & \delta & 0
\end{pmatrix}
\]

So the Kleene star \(L^* = L^{\otimes 2} \oplus L^{\otimes 3}\) equals \(L^{\otimes 3}\). In this case, the tropical eigenspace \(\text{Eig}(L)\) has facet representation

\[
\text{Eig}(L) = \{ x \in \mathbb{R}^3 : -1 \leq x_1 - x_2 \leq -1+\delta, -1-\delta \leq x_1 - x_3 \leq -1-\delta \leq x_2 - x_3 \leq 0 \}.
\]

Figure 1 shows the tropical hyperplane arrangement \(\mathcal{H}(L^+)\) for \(\delta \in (0,1)\) in panel (A), and \(\delta = 0\) in panel (B). This arrangement consists of min-plus tropical hyperplanes whose apices are the column vectors \(-L_1^+, -L_2^+\) and \(-L_3^+\) of \(L\). In the first case, the green triangle in the middle is the tropical eigenspace \(\text{Eig}(L)\). In the second case, the tropical eigenspace \(\text{Eig}(L)\) is the green point. The tropical polytope \(\text{Im}(L)\) consists of the bounded line segments from the vertices of \(\text{Eig}(L)\) to the points \(c_1, c_2, c_3\) labelled with their coordinates.
The covectors of the points labelled $a$ through $c$ are given by the following matrices.

$$
\text{coVec}_L(a) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{coVec}_L(b) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{coVec}_L(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Note that $a$ and $b$ lie in unbounded cells of the tropical hyperplane arrangement, so their covectors are not invertible, while $c$ lies in a bounded cell, so its covector is invertible.

3. Mechanism Design in Tropical Notation

We consider an economy with $m \in \mathbb{N}$ outcomes which is populated by $n \in \mathbb{N}$ agents with quasi-linear utilities, as in [37, §4]. For $i \in [n]$, agent $i$ has a true type $t^i \in \mathbb{R}^m$, drawn by nature and privately known only to him. The tuple of all possible true types forms the type space $T \subseteq \mathbb{R}^{m \times n}$. A direct mechanism $(g, p)$ consists of an outcome function $g : T \to [m]$ and a payment function $p : T \to \mathbb{R}^n$.

We focus on dominant strategy equilibria in the game induced by the mechanism. A strategy for agent $i$ is a declared type $s^i \in T_i$ for each possible true type. After all agents have submitted their reports, the game’s outcome is $g(s^1, \ldots, s^n) \in [m]$ with payment-vector $p(s^1, \ldots, s^n) \in \mathbb{R}^n$. Agent $i$ chooses his report to maximize his utility, which is

$$
u_i((j, p), t^i) = t^i_j - p_i$$

for game outcome $j \in [m]$ with payment vector $p \in \mathbb{R}^n$.

We say that an agent tells the truth if his declared type is his true type. A central goal of mechanism design is to find mechanisms $(g, p)$ under which all agents report truthfully and the desired outcome is implemented in equilibrium.

**Definition 3.1.** For a mechanism $(g, p)$, say that truth-telling is dominant strategy incentive compatible, abbreviated (D-IC), if for each agent $i \in [n]$, each $t^i \in T_i$ and each $t, s \in T_i(t^i)$,

$$(\text{D-IC}) \quad t^i g((t, t^i)) - p_i((t, t^i)) \geq t^i g((s, t^i)) - p_i((s, t^i)).$$

That is, no matter what the other agents declare, each agent always maximizes his utility by reporting his true type.
By rewriting (D-IC), we obtain the analogous tropical formulation
\[ L_{jk}^g = \inf_{t \in g^{-1}(j)} \{ t_j - t_k \} \]
For a given outcome function \( g \), the classical theorem of Rochet \cite{31} gives a necessary and sufficient condition for (D-IC) in terms of cycles on the allocation graph \( G(L^g_{t-i}) \), \cite{14,37}.

**Theorem 3.2** (Rochet). A mechanism \( g \) is (D-IC) if and only if for each \( i \in [n] \) and each \( t_{-i} \in \mathcal{T}_{-i} \), all cycles in \( G(L^g_{t-i}) \) have nonnegative weights.

We can restrict to the single agent case. Fix \( i \in [n] \) and \( t_{-i} \). Let \( T = \mathcal{T}_i \), define \( g : T \to [m] \) via \( g(t) := g(t,t_{-i}) \) and similarly \( p : T \to \mathbb{R} \) via \( p(t) := p_i([t,t_{-i}]) \). Without loss, assume
\[
g^{-1}(k) = \{ t \in T : g(t) = k \} \neq \emptyset \text{ for all } k \in [m].
\]
By rewriting (D-IC), we obtain the analogous tropical formulation
\[
L^g_{t-i} \odot p = p.
\]
That is, deciding (D-IC) is solving a tropical linear equation. With this view, we restate essential definitions of mechanism design in tropical notation as follows.

**Definition 3.3.** Say that \( g \) is dominant strategy incentive compatible (D-IC) if for each \( i \in [n] \) and each \( t_{-i} \in \mathcal{T}_{-i} \) we have that \( \lambda(L^g_{t-i}) = 0 \). In this case, call the tropical eigenspace \( \text{ Eig}(L^g_{t-i}) \) the set of incentive compatible payments.

**Definition 3.4.** Suppose \( g \) is (D-IC). Say that \( g \) is revenue equivalent (RE) if for each \( i \in [n] \) and each \( t_{-i} \in \mathcal{T}_{-i} \), \( \text{ Eig}(L^g_{t-i}) \) in \( \mathbb{T}^{m-1} \) consists of exactly one point.

For ease of reference, we also collect the following central definitions here.

**Definition 3.5.** Fix a type space \( T \subset \mathbb{R}^m \). Say that a given matrix \( L \in \mathbb{R}^{m \times m} \) is realizable if there exists some outcome function \( g : T \to [m] \) such that \( L = L^g_{t-i} \).

**Definition 3.6.** Let \( L \in \mathbb{R}^{m \times m} \) be a realizable matrix. Say that \( L \) is weakly monotone if \( L + L^\top \) is element-wise nonnegative.

4. **The Geometry of Mechanism Design - Main Results**

4.1. **Geometric Characterization of all Mechanisms.** We fix an agent \( i \) and the types of other agents \( t_{-i} \in \mathcal{T}_{-i} \). An allocation rule \( g \) defines the matrix \( L^g_{t-i} \). Our characterization rests on inverting this map. This allows us to study mechanisms on the space of matrices.

It is without loss to assume that \( T \subseteq \mathbb{T}^{m-1} \). In terms of economics, this means that only relative valuation of the agents matter for truthfulness. Though it is common to normalize in equilibrium theory, this is rather uncommon in mechanism design. In our setting it turns out to be natural and very useful.

**Definition 4.1.** For \( j, k \in [m], j \neq k \), define \( \mathcal{I}_{jk} = \mathcal{L}_{jk} \cap T \). A \((j, k)\)-witness is a sequence \( \{ s^j(r) : r \geq 1 \} \subset T \cap \mathcal{L}_{j}^\top \) such that
\[
\lim_{r \to \infty} \text{ dist}(s^j(r), \mathcal{L}_{jk}) = 0.
\]
We say that \( L \) separates \( T \) at \((j, k)\) if
\[
\text{ dist}(T \cap \mathcal{L}_{j}, \mathcal{L}_{jk}) = 0,
\]
and in addition, whenever \( \mathcal{I}_{ik} = \mathcal{I}_{kj} = \{ s \} \) for some \( s \in \mathbb{T}^{m-1} \), then there exists a \((j, k)\)-witness or a \((k, j)\)-witness. Say that \( L \) separates \( T \) if \( L \) separates \( T \) for all \( j, k \in [m], j \neq k \).
The witnessing sequence condition is to exclude strange cases in which two sectors are held in place by one point. In such a case, the dominant arrangement will move if the shared point on the boundary is assigned to different outcomes. Thus such a matrix cannot be realizable, see Example 5.2 below. We now state our first main result, which geometrically characterizes all possible mechanisms on arbitrary type-spaces.

**Theorem 4.2.** Let $L \in \mathbb{R}^{m \times m}$ with zero diagonal. Then $L$ is realizable if and only if

$$T \subseteq \bigcup_{k=1}^{m} \mathcal{L}_k$$

and $L$ separates $T$.

For $L$ to be realizable, $L$ needs to separate $T$, as for each $i, j \in [m]$, we want

$$L_{ij} = \inf_{t \in g^{-1}(i)} (t_i - t_j),$$

so this infimum needs to be achievable by some sequence in $g^{-1}(i)$. The other condition comes from the following observation.

**Lemma 4.3.** Suppose $L \in \mathbb{R}^{m \times m}$ is realizable with outcome function $g : T \subset \mathbb{P}^{m-1} \to [m]$. Then $g^{-1}(j) \subseteq \mathcal{L}_j$ for each $j \in [m]$.

*Proof.* For any $t \in g^{-1}(j)$ we have that $L_{jk} = \inf_{s \in g^{-1}(j)} (s_j - s_k) \leq t_j - t_k$, so $L_{jk} + t_k \leq t_j$. Thus $\max_{k \in [m]}(L_{jk} + t_k) \leq t_j$. But by definition for $j = k$, $L_{jj} = 0 = t_j - t_j$, so $L_{jj} + t_j = t_j$ and hence $\max_{k \in [m]}(L_{jk} + t_k) = L_{jj} + t_j = t_j$. It follows that $t \in \mathcal{L}_j$.

*Proof of Theorem 4.2* Suppose $L$ is realizable with outcome function $g$. By Lemma 4.3 for $k = 1, 2, \ldots, m$, $g^{-1}(k) \subseteq \mathcal{L}_k$. Since $g$ partitions $T$, we have that,

$$T = \bigcup_{k=1}^{m} g^{-1}(k) \subseteq \bigcup_{k=1}^{m} \mathcal{L}_k.$$

It remains to show that $L$ separates $T$. Fix $j, k \in [m], j \neq k$. Since $L_{jk} = \inf_{t \in g^{-1}(j)} \{ t_j - t_k \}$, this infimum is achieved either by a $(j, k)$-witness, or by a point $t \in L_{jk}$. So $\text{dist}(T \cap \mathcal{L}_j, \mathcal{L}_{jk}) = 0$. Furthermore, if $\mathcal{I}_{jk} = \mathcal{I}_{kj} = \{s\}$, then if $g(s) = j$, the infimum in the definition of $L_{kj}$ must be achieved by a $(j, k)$-witness, so there must exist a $(k, j)$-witness. Conversely, if $g(s) = k$, then there must exist a $(j, k)$-witness. This shows that $L$ separates $T$ at $\{j, k\}$. Since the pair was chosen arbitrarily, $L$ separates $T$.

For the converse direction, suppose $L$ is a matrix that separates $T$. Define $g : T \to [m]$ as follows. For a point $t \in T \cap \mathcal{L}_j$, let

$$g(t) = j.$$

The remaining points lie on the boundary $T \cap \bigcup_{j, k \in [m], j \neq k} \mathcal{L}_{jk} = \bigcup_{j, k \in [m], j \neq k} \mathcal{L}_{jk}$. Assign these points such that points on $\mathcal{I}_{jk}$ have either outcome $j$ or $k$, and such that on every non-empty boundary $\mathcal{I}_{jk}$ there exists a point with outcome $j$. The only case where this cannot be done is if $\mathcal{I}_{jk} = \mathcal{I}_{kj} = \{s\}$. In this case, if there is a $(j, k)$-witness, set $g(s) = k$, else, since $L$ separates $T$, there must exist a $(k, j)$-witness, so set $g(s) = j$. We claim that $L$ is realized by $g$. Fix $j, k \in [m], j \neq k$. By definition of $\mathcal{L}_j$

$$\inf_{s \in g^{-1}(j)} (s_j - s_k) \geq L_{jk}.$$
By the above assignment, there must be a point in $I_{jk}$ or a $(j, k)$-witness in $g^{-1}(j)$. So the infimum is achieved, thus $g$ realizes $L$ as claimed. 

It is immediate that when two rows of $L$ coincide in $\mathbb{TP}^{m-1}$, say, $L_m \equiv L_{m-1}$, then $L_{mi} = L_{(m-1)i}$ for all $i \in [m], i \neq m-1, m$. In that case, one can reduce checking realizability by projecting onto the subspace $x_{m-1} - x_m = 0$, and deleting the last coordinate. In economic terms, this means that under any incentive compatible payment, outcomes $m$ and $m-1$ have the same transfers. We can strengthen this argument to an equivalence by introducing liftable type spaces. Essentially, this condition requires that the type space has enough points, so that upon embedding into a space of one dimension higher, there are enough witnesses.

**Definition 4.4.** Suppose $L$ separates $T$. For a pair $j, k \in [m], k \neq j$, let $w_{jk}$ be the number of $(j, k)$-witnesses. Let $q_{jk}$ be the number of points in $\mathcal{I}_{jk} \cap T$. Now fix $r \in [m]$. Say that $T$ is liftable at coordinate $r$ with respect to $L$ if for each $k \in [m], k \neq r$, one of the following cases hold

- $w_{rk} + q_{rk} \geq 3$
- $w_{rk} + q_{rk} = 2$, and $\mathcal{I}_{rk} \cap \mathcal{I}_{kr} \cap T$ is empty
- $w_{rk} + q_{rk} = 2, \mathcal{I}_{rk} \cap \mathcal{I}_{kr} \cap T$ is non-empty, and $q_{kr} > 0$.

We now obtain the following dimension reduction result, which follows straight from Theorem 4.2 by a definition chase.

**Corollary 4.5.** Let $L \in \mathbb{R}^{m \times m}$ be a zero diagonal matrix. Suppose that two rows of $L$ are the same in $\mathbb{TP}^{m-1}$, say $L_{m-1} \equiv L_m$. Let $\Pi : \mathbb{R}^{m} \to \mathbb{R}^{m-1}$

$$\Pi(x) = (x_1, \ldots, x_{m-2}, \min(x_{m-1}, x_m))$$

be the map given by projecting orthogonally onto the subspace $x_{m-1} - x_m = 0$ and deleting the $m$-th entry. View this as a map from $\mathbb{TP}^{m-1}$ to $\mathbb{TP}^{m-2}$. Write $\Pi(L) \in \mathbb{R}^{(m-1) \times (m-1)}$ for the matrix obtained by removing the $m$-th row and column of $L$. Then $L$ is realizable on type space $T \subset \mathbb{TP}^{m-1}$ if and only if $\Pi(L)$ is realizable on type space $\Pi(T) \subset \mathbb{TP}^{m-2}$ and $T$ is liftable at coordinate $m-1$ with respect to $L$.

4.2. **Geometric Characterization of Incentive Compatibility.** Consider an allocation matrix $L = L^{g,t-1}$. Note that it has zero diagonal, so necessarily $\Lambda(L) \leq 0$. To decide whether it is incentive compatible, we want to know whether $\Lambda(L) < 0$ or $\Lambda(L) = 0$.

**Theorem 4.6.** Let $L \in \mathbb{R}^{m \times m}$ be a matrix with zero diagonal. Define

$$\text{Eig}_0(L) := \bigcap_{j=1}^{m} \mathcal{L}_j.$$ 

Then $\Lambda(L) = 0$ if and only if $\text{Eig}_0(L) \neq \emptyset$. In that case, $\text{Eig}(L) = \text{Eig}_0(L)$.

**Proof.** By definition,

$$\text{Eig}_0(L) = \{p \in \mathbb{TP}^{m-1} : L \odot p = p\}.$$ 

The theorem follows from uniqueness of the tropical eigenvalue. If $\Lambda(L) = 0$, then $\text{Eig}_0(L)$ is the tropical eigenspace of $L$, which is non-empty. Conversely, suppose $\text{Eig}_0(L) \neq \emptyset$. This means that there exists a tropical eigenvector of $L$ with tropical eigenvalue zero. But the tropical eigenvalue of $L$ is unique, thus, $\Lambda(L) = 0$. 

□
This geometric result allows us to visualize and prove incentive compatibility, or its failure, on arbitrary type spaces without calculating cycle lengths. When $\lambda(L) = 0$, it gives a geometric description for the tropical eigenspace of $L$, i.e. the tropical semi-module of incentive compatible payments. We defer the discussion of the economic interpretation to Section 6.3.

**Corollary 4.7.** Let $L \in \mathbb{R}^{m \times m}$ be a matrix with zero diagonal. Then $\text{Ei}g_0(L)$ is a singleton if and only if any mechanism $g$ that realizes $L$ is revenue equivalent.

4.3. **Verifying Incentive Compatibility Algorithmically.** Determining if $\lambda(L) = 0$ can be reduced to computing the tropical determinant. Thus verifying (D-IC) can be solved by the Hungarian method.

**Proposition 4.8.** Let $L \in \mathbb{R}^{m \times m}$ have zero diagonal. The following are equivalent.

1. $\lambda(L) = 0$.
2. Any cycle of $G(L)$ has nonnegative length.
3. $\text{tdet}(L) = 0$
4. The identity assignment $e : 1 \mapsto 1, \ldots, m \mapsto m$ is a minimal matching in the classic perfect matching problem on a bipartite graph with weight $L$.

**Proof.** The assertion (1) $\Leftrightarrow$ (2) is the Rochet theorem, see [37] for proof using network flows. Since $L$ has zero diagonal, the identity assignment has value 0. So (3) $\Leftrightarrow$ (4). Let us prove (2) $\Leftrightarrow$ (3). Since $L$ has zero diagonal, the identity assignment has value 0. Thus $\text{tdet}(L) \leq 0$. Suppose (2). Any assignment $\sigma \in \mathbb{S}_m$ can be decomposed as a union of simple directed cycles on a disjoint set of vertices. Thus (2) implies $\text{tdet}(L) \geq 0$, and so we have (3). Now instead, suppose (3). Any simple cycle $C$ can be extended to a permutation by adding self-loops at vertices which are not in $C$. Thus we have (2). \qed

**Corollary 4.9.** For each $i \in [n]$ and $t_{-i} \in \mathcal{T}_{-i}$ checking whether all cycles of $G(L^{g_{t-i}})$ are nonnegative can be done algorithmically using the Hungarian algorithm for perfect matching.

4.4. **Characterization of Weak Monotonicity.** Weak monotonicity has received wide attention in the literature, it is known to be sufficient on some domains, cf. Theorem 6.4. The following characterizations of weak monotonicity can be traced back to [10].

**Lemma 4.10 ([10]).** Suppose $L$ is realizable. For $j, k \in [m],$

$$L_{jk} + L_{kj} > 0 \Leftrightarrow \mathcal{L}_j \cap \mathcal{L}_k = \emptyset,$$

$$L_{jk} + L_{kj} = 0 \Leftrightarrow \mathcal{L}_j \cap \mathcal{L}_k \text{ on their boundaries, i.e. } \mathcal{L}_j \cap \mathcal{L}_k = \partial \mathcal{L}_j \cap \partial \mathcal{L}_k$$

$$L_{jk} + L_{kj} < 0 \Leftrightarrow \mathcal{L}_j \cap \mathcal{L}_k \text{ in their interiors, i.e. } \mathcal{L}_j^o \cap \mathcal{L}_k^o \neq \emptyset.$$

**Proof.** Note that $\mathcal{L}_j$ and $\mathcal{L}_k$ are closed polyhedra, so they either do not intersect, intersect on their boundaries, or intersect in their interiors. So it is sufficient to prove the last two equivalences in the statements. Suppose there exists $t \in \mathcal{L}_j \cap \mathcal{L}_k$. Then by definition of the sectors,

$$L_{jk} + L_{kj} \leq (t_j - t_k) + (t_k - t_j) = 0.$$

In particular, strict equality holds if and only if $t$ lies on the boundary of $\mathcal{L}_j \cap \mathcal{L}_k$, while strict inequality holds if and only if $t$ lies in either of their interiors. In that case, one can pick a $t' \in \mathcal{L}_j^o \cap \mathcal{L}_k^o$, then $L_{jk} + L_{kj} \leq (t'_j - t'_k) + (t'_k - t'_j) = 0$. This proves the desired statement. \qed

**Theorem 4.11.** $L$ is weakly monotone if and only if it is realizable, and the open sets $\mathcal{L}_1^o, \ldots, \mathcal{L}_m^o$ are pairwise disjoint.

**Proof.** This follows from Lemma 4.10 and Definition 3.6. \qed


5. Examples

In this section we illustrate our results with various examples of mechanisms with three outcomes \((m = 3)\). In particular, these examples show that it is simple to construct mechanisms and verify realizability, incentive compatibility (D-IC) and revenue equivalence (RE).

In all figures, for \(j \in \{1, 2, 3\}\), the max-plus sector \(\mathcal{L}_j\) has apex \(-L_j\), where \(L_j\) (up to tropical scaling) is the \(j\)-th row of \(L\). The boundary of the max-plus sector \(\mathcal{L}_j\) is shown in red. The boundary of the min-plus sector \(\mathcal{L}_j\) is shown in red. Type spaces are shaded in gray. The tropical eigenspace \(\text{Eig}_0(L)\) is shaded green.

**Example 5.1** (Geometric representation of mechanism and verification of D-IC). Figure 2 (A) is the geometric representation of a mechanism \(g\) via the dominant arrangement \(\mathcal{D}(L)\) of its allocation matrix \(L = L^{g,t_1} \). Note that \(T\) lies in the union of the sectors \(\mathcal{L}_1\), \(\mathcal{L}_2\), and \(\mathcal{L}_3\). Furthermore, the sectors are disjoint and \(T\) has points on all of the boundaries \(\mathcal{L}_{12}, \mathcal{L}_{13}, \mathcal{L}_{23}, \mathcal{L}_{31}\) and \(\mathcal{L}_{32}\). So \(L\) separates \(T\), as expected from Theorem 4.2. Figure 2 (B) verifies that the mechanism defined by Figure 2 (A) is D-IC via the min-plus dominant arrangement \(\mathcal{D}(L)\) using Theorem 4.6. Indeed, the intersection of the min-plus sectors \(\mathcal{L}_1, \mathcal{L}_2\), and \(\mathcal{L}_3\) is the green region, which is non-empty. This region is the tropical eigenspace \(\text{Eig}_0(L)\) with eigenvalue zero, which corresponds to incentive compatible payments.

![Figure 2](image-url)

**Figure 2.** Geometric representation of a mechanism \(g\) (left) and verification that it is D-IC (right). Figures accompanying Example 5.1.

**Example 5.2** (Realizable and non-realizable matrices). On a common type space, Figure 3 shows the dominant arrangement \(\mathcal{D}(L)\) of a realizable matrix \(L\) in panel (A), and that of a non-realizable matrix in panel (B). The type space \(T = \{t_1, t_2, t_3, t_4\}\) consists of four points in \(\mathbb{T}^2\). In Figure 3 (A), the mechanism \(g\) is defined by \(g(t_2) = g(t_3) = 3, g(t_1) = 1\) and \(g(t_4) = 2\). One can readily verify that \(L^{g,t_1} = L\).

Now consider the matrix \(L\) defined via its dominant arrangement \(\mathcal{D}(L)\) in Figure 3 (B). By Theorem 4.2 for a mechanism \(g\) to realize \(L\) as \(L = L^{g,t_1}\), we must have \(g(t_4) = 2, g(t_1) = 1, g(t_3) = 3\), and \(g(t_2)\) either 1 or 3. Suppose \(g(t_2) = 1\). Then \(L_{31}(t_3) = \inf_{s \in g^{-1}(1)} \{s^3 - s^1\} = t_3^3 - t_3^1, L_{32}^g(t_3) = \inf_{s \in g^{-1}(2)} \{s^3 - s^2\} = t_3^3 - t_3^2\) and \(L_{33}^g = 0\). Hence \(-L_3^g = [(0, -t_3^1, t_3^2), ((-L_{31}, -L_{32}, 0)] = [(0, -t_3^1, t_3^2, t_3^3)]\) in \(\mathbb{T}^2\) and thus \(L^g \neq L\). So we cannot have \(g(t_2) = 1\). Now suppose
instead that $g(t_2) = 3$. Then $\overline{D}(L^{g,t-1})$ must equal the arrangement in Figure 3 (A), so $L \neq L^g$. Thus $L$ is not realizable. Realizability fails since $L$ does not separate $T$ at $\{1,3\}$. Here, one has $\mathcal{I}_{13} = \overline{L}_{13} \cap \overline{L}_{31} \cap T = \{t_2\}$, but there is no $(1,3)$ nor a $(3,1)$ witnesses.

(A) The dominant arrangement $\overline{D}(L)$ of a realizable matrix $L$.

(B) The dominant arrangement $\overline{D}(L)$ of a non-realizable matrix $L$. Here $L$ does not separate $T$.

**Figure 3.** Realizable and non-realizable matrices, defined by their dominant arrangements $\overline{D}(L)$. Figure accompanies Example 5.2

**Example 5.3** (Realizable matrices with unusual domains). In Figure 4 (A), the domain consists of the open gray-shaded trapezium and the four-point set $\{t_1, t_2, t_3, t_4\}$. Indeed, $T \subseteq \bigcup_{k=1}^3 \overline{L}_k$, and $L$ separates $T$. One can define a mechanism that realizes $L$ as follows. Set $g(t_1) = g(t_2) = 1$, $g(t_4) = 2$, $g(t_3) = 2$, and $g(t) = 3$ for all remaining points $t \in T$. By Theorem 4.11 in any mechanism $g$ that realizes Figure 4 (A), the two-cycle $(2 \to 3 \to 2)$ is negative. Thus $g$ is not D-IC.

(A) The dominant arrangement $\overline{D}(L)$ of a realizable matrix $L$. Any corresponding mechanism $g$ is not D-IC.

(B) The max-plus arrangement $\overline{D}(L)$ in blue, and the min-plus arrangement $\overline{L}(L)$ in red, of a realizable matrix $L$. The unique corresponding mechanism is RE.

**Figure 4.** Dominant arrangements $\overline{D}(L)$ of two realizable matrices $L$ with unusual type spaces $T$. Using Theorem 4.12 one can write down all the mechanisms $g : T \to \{1,2,3\}$ with allocation matrix $L$. Figure to Example 5.3
In Figure 4 (b), the domain consists of three connected pieces: two open circles and a large open set cut out by a curve. There is a unique mechanism that realizes $L$ which is RE and D-IC since the three sectors in the min-plus dominant arrangement $D(L)$ intersect at $-L_3$.

![Diagram](image)

**Figure 5.** A revenue equivalent mechanism on a convex type space. Figure accompanies Example 5.4

**Example 5.4** (Saks-Yu demonstration). In Figure 5 the type space $T$ is the closed half-space shaded gray. Any mechanism $g$ that realizes the arrangement $D(L)$ given in Figure 5 (A) must assign $-L_3$ to outcome 3, $-L_2$ to outcome 2, and assign points in the interior of $\overline{L}_1$ and $\overline{L}_2$ to outcomes 1 and 2, respectively. Points on the boundary of any pair $\overline{L}_j$ and $\overline{L}_k$ can take either outcome $j$ or $k$. By Theorem 4.11 any realizable $g$ has all of its two-cycles equal to 0. Figure 5 (B) shows the tropical eigenspace $\text{Eig}(L^{g,t-1})$ as the intersection of the min-plus sectors $L_j$. In this case the eigenspace consists of a point, so the mechanism is D-IC and RE. This example is an instance where Theorem 6.4 below applies.

![Diagram](image)

**Figure 6.** Two different mechanisms on a non-convex type space. Figure accompanies Example 5.5

(a) A mechanism that is weakly monotone but not D-IC.
(b) The min-plus arrangement $D(L)$ verifies non-D-IC.
(c) On the same type space, another mechanism that is RE.
Example 5.5 (Non-convex, not path-connected domain with revenue equivalence). Figure 6 depicts a non-convex type space (gray shaded) on which any implementable mechanism must be revenue equivalent. Figures 6 (A) and (B) show a weakly monotone mechanism that is not D-IC. On convex type spaces such a situation cannot occur. Figure 6 (C) depicts a D-IC mechanism that is RE. Note that the closure of the type-space is not path-connected. Also the domain is not ‘boundedly grid-wise connected’ cf. [7] Theorem 1 and 4.

6. Economic Interpretation

We present Theorem 6.3, a generalization of a theorem by Saks and Yu [33], on the sufficiency of weak monotonicity on convex domains. The converse of Saks-Yu was proved by Ashlagi, Braverman, Hassidim and Monderer [3]; we give the geometric intuition of their proof here. Using the simple geometric characterization of revenue equivalence in Corollary 4.7, we obtain a special case of a theorem of Heydenreich-Müller-Uetz-Vohra [15]. We conclude with a discussion of the economic interpretation of Theorem 4.6 in terms of the min-max duality of linear programming.

6.1. Convexity and Weak Monotonicity. Consider the type graph \( T^g : T \times T \to \mathbb{R} \), with \( T^g(t, s) := t_g(t) - t_g(s) \). For a path \( \gamma_{st} \subset T \) of finite Euclidean length from \( s \) to \( t \), let \( \ell^g(\gamma_{st}) \) denote the length of \( \gamma_{st} \) with respect to \( T^g \). Note that such a path has length of the form \( \ell^g(\gamma_{st}) = L^g_{i_0i_1} + L^g_{i_1i_2} + \ldots + L^g_{i_{k-1}i_k} \) for some finite sequence \( i_0 \to \ldots \to i_k \subset [m] \), called the length sequence of \( \gamma_{st} \). Note that \( i_0 = g(s), i_k = g(t) \). Call a path \( \gamma_{st} \) simple if its length sequence contains no repeated nodes.

Our assumptions on the domain \( T \) are

(H1) \( T \) is closed
(H2) \( T \) is simply connected
(H3) For any \( i, j \in [m] \), any point \( t \in T \cap \partial L_i \cap \partial L_j \) is the limit of a sequence in the strict interior \( T \cap L_i \) and also the limit of a sequence in \( T \cap L_j \).
(H4) For any \( i, j \in [m] \), there exist \( u \in T \cap \partial L_{ij}, v \in T \cap \partial L_{ji} \), such that the line segment \( [u, v] \subseteq T \).

Hypothesis (H3) allows one to reassign \( t \) to either outcome \( i \) or \( j \) leaving the matrix \( L \) unchanged. Thus, we have a family of mechanisms \( G = \{ g : T \to [m] \} \) such that \( L^g = L^{g'} = L \) for all \( g, g' \in G \), whose values only differ on boundary points, namely, points in the skeleton \( T' := T \cap \bigcup_{i,j \in [m]} (L_{ij} \cap L_{ji}) \). For a point \( p \in T \), write \( G(p) = \{ g(p) : g \in G \} = \{ i_1, i_2, \ldots, i_k \} \subseteq [m] \) to be the set of possible labels at \( p \).

Lemma 6.1. Suppose \( T \) satisfies (H3). If \( s, t \in T \) are two points in the strict interior of sectors \( i \) and \( j \), respectively, then \( \ell^g(\gamma_{st}) = \ell^{g'}(\gamma_{st}) \) for all \( g, g' \in G \).

Proof. Consider \( \gamma_{st} \cap T' \). As the path \( \gamma_{st} \) has finite length and \( T' \) is piecewise linear, this is a union of points and line segments. Let \( p \in \gamma_{st} \cap T' \) be an isolated point. Since \( p \neq s, t \), the point \( p \) is in the strict interior of the path \( \gamma_{st} \). Let \( p^-, p^+ \in \gamma_{st} \) be points immediately to the left and right of \( p \), respectively. Since \( p \) is isolated, \( p^-, p^+ \notin S \), one may assume that \( G(p^+) = \{ i_1 \}, G(p^-) = \{ i_2 \} \). But \( p \in \bigcap_{i \in G} L_{ii} \cap T \), so \( L_{i_1i} + L_{i_2i} = L_{i_1i_2} \) for all \( i \in G \). So we are done. The case of a line segment is similar, as one can define the line segments so that all points in each line segment have the same set of labels. This proves the claim. □

With the above lemma, we can define \( \ell(\gamma_{st}) \) to be \( \ell(\gamma_{st}) := \ell^g(\gamma_{st}) \) for some \( g \in G \).
Proposition 6.2. Suppose \( T \) satisfies (H1)-(H3). If \( s, t \in T \) are two points in the strict interior of sectors \( i \) and \( j \), respectively, then \( \ell(\gamma_{st}) = \ell(\gamma'_{st}) \) for all paths \( \gamma_{st}, \gamma'_{st} \subset T \).

Proof. Note that \( T \) is simply connected. Continuously transform \( \gamma_{st} \) until the first time that it becomes a path \( \gamma'_{st} \) with a different length sequence. We want to show that \( \ell(\gamma_{st}) = \ell(\gamma'_{st}) \).

By choosing intermediate points \( s', t' \in T \) if necessary to break the problem down, one can assume that \( \gamma_{st} \) has length sequence \( i \to k \to j \), and \( \gamma'_{st} \) has length sequence \( i \to i_1 \to \ldots \to i_r \to j \) with \( k \neq i_1, \ldots, i_r \), and that this length sequence cannot be further reduced.

In other words, there exists points \( p_1, \ldots, p_r \in \gamma_{st} \) such that \( G(p_1) \cap \{i, i_1, \ldots, i_r, j\} = i_1 \), \( G(p_2) \cap \{i, i_1, \ldots, i_r, j\} = i_2 \), and so on. Now, each point of \( \gamma'_{st} \) is the limit of a sequence of points in \( T \cap (\mathcal{L}_i \cup \mathcal{L}_j \cup \mathcal{L}_k) \). Since \( T \) is closed, \( \{i, j, k\} \cap G(q) \neq \emptyset \) for each \( q = p_1, \ldots, p_r \).

By assumption, \( i, j \notin G(q) \), thus \( k \in G(q) \) for each \( q = p_1, \ldots, p_r \). Therefore, \( L_{ik} + L_{kj} = L_{i1} + L_{i2} + \ldots + L_{ir,j} \), so \( \ell(\gamma_{st}) = \ell(\gamma'_{st}) \), as needed. □

The above Proposition says that all cycles in \( L^g \) that are realizable as the length of sequence of a loop \( \gamma_{ss} \in T \) have length zero. However, if such a cycle involves the edge \( i \to j \), say, then one must have \( L_{ij} + L_{ji} = 0 \). In general, not all edges satisfy this property: \( L^g \) can have a strictly positive two-cycle. However, with hypothesis (H4), one can be sure that edges involved in a strict positive two-cycle must be large.

Theorem 6.3. Suppose \( T \) satisfies (H1)-(H4). If all two-cycles of \( L^g \) are nonnegative, then \( g \) is D-IC.

Proof. For \( i, j \in [m] \), take points \( u, v \) as in hypothesis (H4). Let \( i \to i_1 \to \ldots \to i_r \to j \) be the length sequence of the path \([u, v]\). We claim a reversed triangle inequality:

\[
    u_i - u_j = L_{ij} \geq L_{i1} + \ldots + L_{ir,j}.
\]

Indeed, let \( w := v - u \). There exists a sequence of points \( p^1, \ldots, p^r \in [u, v] \) corresponding to a sequence of constants \( 0 \leq \lambda_1 \leq \ldots \leq \lambda_r \leq 1 \) such that \( u + \lambda_1 w \in T \cap \mathcal{L}_i \cap \mathcal{L}_{i1} \), \( u + \lambda_2 w \in T \cap \mathcal{L}_{i1} \cap \mathcal{L}_{i2} \), and so on. If any of the inequalities are equalities, say, \( \lambda_1 = 0 \), then \( L_{i1} = L_{i1} + L_{i1i2} \), so one can replace \( u \) by \( p^1 \). Thus, one may assume that all inequalities are strict. Now, consider the line segment \([p^1, p^2]\). Let \( e_i \) be the \( i \)-th standard coordinate vector. Note that

\[
    p^1_{i1} - p^1_{i2} = \langle u, e_{i1} - e_{i2} \rangle + \lambda_1 \langle w, e_{i1} - e_{i2} \rangle = L_{i1i2},
\]

and

\[
    p^2_{i1} - p^2_{i2} = \langle u, e_{i1} - e_{i2} \rangle + \lambda_2 \langle w, e_{i1} - e_{i2} \rangle = -L_{i2i1} \leq L_{i1i2}.
\]

Therefore,

\[
    \langle w, e_{i1} - e_{i2} \rangle \leq \frac{1}{\lambda_2 - \lambda_1} (L_{i1i2} + L_{i2i1}) = 0.
\]

So in particular,

\[
    u_i - u_j = \langle p^1, e_{i1} - e_{i2} \rangle - \lambda_1 \langle w, e_{i1} - e_{i2} \rangle \geq L_{i1i2}.
\]

By the same argument with segments \([p^1, p^2], \ldots, [p^r, v]\), one concludes that

\[
    u_{i2} - u_{i3} \geq L_{i2i3}, \ldots, u_{ir} - u_{ij} \geq L_{irij}.
\]

Therefore,

\[
    L_{ij} = u_i - u_j = (u_i - u_{i1}) + (u_{i1} - u_{i2}) + \ldots + (u_{ir} - u_{ij}) \geq L_{i1i} + \ldots + L_{ir,j}.
\]

Since the line \([u, v]\) goes through adjacent sectors,

\[
    L_{i1} + \ldots + L_{ir,j} = -(L_{i1i} + \ldots + L_{j1i}).
\]
therefore we have 
\[-(L_{i_1i} + \ldots + L_{ji_r}) \leq L_{ij} + L_{ji} + \ldots + L_{i_1i} \geq 0.\]

That is, the cycle \(j \to i_r \to \ldots \to i_1 \to j \to i\) has nonnegative length. Reversing the role of \(u\) and \(v\) implies that the reverse cycle also has nonnegative length.

Finally, it remains to show that any simple cycle involving \(j\) and \(i\) can be decomposed this way. Take an arbitrary cycle \(i \to i_1 \to \ldots \to i_r \to j \to i\). Each edge is greater than or equal to the length of a line segment \([u^i, v^i], [u^{i_1}, v^{i_1}], [u^{i_2}, v^{i_2}], \ldots, [u^j, v^j]\), where the superscript indicates the label of the point. Since \(T\) is connected, there exists simple paths \(\gamma_{vi_1i_1}, \gamma_{vi_1i}, \gamma_{vui}\). Thus, the path \(\gamma_{ii} := [u^i, v^i] \cup \gamma_{v_1i_1} \cup [u^{i_1}, v^{i_1}] \cup \ldots \cup \gamma_{vi} \) is a path in \(T\) such that \(\ell(\gamma_{ii}) = 0\), and is less than or equal to the length of the cycle in consideration. This completes the proof.

We now obtain Saks-Yu as a corollary.

**Theorem 6.4** (Saks-Yu). Suppose \(L\) is realizable, and all two-cycles of \(L\) are nonnegative. If \(T\) is convex, then \(\lambda(L) = 0\).

**Proof.** Without loss of generality, one can assume that \(T\) is closed and for each \(i \in [m]\), \(T \cap \mathcal{L}_i^0 \neq \emptyset\). Thus, \(T\) satisfies (H1)-(H4). The result follows by Theorem 6.3.

We now consider the converse of Saks-Yu for finite-valued mechanisms. This result was first proved by Ashlagi, Braverman, Hassidim and Monderer [3].

**Theorem 6.5** (Ashlagi-Braverman-Hassidim-Monderer). Suppose \(T = \mathcal{T}_i\) is closed. If \(T\) is not convex, then there exists a mechanism \(g\) such that all two-cycles of \(L^{g, t_{-i}}\) are nonnegative, but \(\lambda(L) < 0\).

**Proof.** Let us sketch the idea for the case where \(T\) is full-dimensional, and \(\text{conv}(T) \setminus T\) contains an open ball \(B(p, \epsilon)\) around some point \(p\). Inside this open ball, fits a full-dimensional minimal polytrope \(P\). By Theorem 2.17 there exists a matrix \(L = -L^\top\) such that \(P = \text{Im}(L)\). Then with careful analysis, one can choose \(P\) such that \(L\) separates \(T\). By Theorem 12 \(L\) is realizable, that is, \(L = L^{g, t_{-i}}\) for some \(g\). By construction, all the two-cycles of \(L\) are zero, and the polytrope \(P\) has full-dimension, so \(\lambda(L) < 0\).

6.2. **Weak Monotonicity and Revenue Equivalence.** The literature on revenue equivalence is vast [8,15,16,20,25,27,37], see [9,37] for a comprehensive review. Many results give conditions that ensure revenue equivalence for all implementable allocation rules. As was pointed out in [15], this is a very stringent requirement as there are cases in which revenue equivalent and non-revenue equivalent mechanisms can coexist on the same type-space. The following corollary of Proposition 2.13 is a special case of the characterization of revenue equivalence in [15], it is the linear algebraic analogue of the geometric characterization in Corollary 4.7.

**Theorem 6.6** (15, special case for finite outcomes). Let \(L\) be a realizable \(m \times m\) matrix with zeros on its diagonal and with tropical eigenvalue \(\lambda(L) = 0\). The following are equivalent.

1. \(L\) has a unique eigenvector in \(\mathbb{T}^{m-1}\).
2. \(L^*\) is skew-symmetric.
6.3. An Interpretation via Linear Programming Duality. Loosely formulated we have the following duality. The rows of $L^{g,t-i}$ correspond to the agent’s problem (the informed side of the economy), he chooses optimal reports given his type. The columns of $(L^{g,t-i})^*$ correspond to the problem of the mechanism (the uninformed side), it must simultaneously balance all incentives of reporting a given type. This observation can be made precise via linear programming dualty and the geometry of the dominant arrangements. A similar interpretation in terms of zero sum games \[24,35\] was given in \[30\].

Consider the $j$-th row of $-L^{g,t-i}$, its entries

$$-L^{g,t-i}_{jk} = \sup_{s \in g^{-1}(j)} \{s_k - s_j\} \quad k \in [m]$$

correspond to the most any agent whose truthful statement would entail outcome $j$ could gain by misreporting to change the outcome to $k$.

With quasi-linear utilities the set of payments that offset the incentive to misreport is $L_j$. To see this, note that by definition

$$p \in L_j = \{p \in \mathbb{T}^{m-1} : L_{jk} + p_k \geq p_j \text{ for } k \neq j\}.$$ 

Thus $\inf_{g^{-1}(j)} \{t_j - t_k\} + p_k \geq p_j$ and hence $t_j - p_j \geq t_k - p_k$ for all $t \in g^{-1}(j)$. However, there is a consistency requirement. Since the mechanism is informationally constrained, it can only set one payment that must incentivize all possible types simultaneously.

Suppose $L^{g,t-i}$ is realizable and $\Lambda(L^{g,t-i}) = 0$. The min-plus covector of $p \in \text{Eig}_0(L^{g,t-i})$ with respect to $L^{g,t-i}$ systematically lists for which types the incentive constraints hold, i.e.

$$\text{coVec}_{L^{g,t-i}}(p)_{jj} = 1 \iff p \in L_j$$

The intersection of all the sectors $L_j$ for $j \in [m]$ is the set of jointly incentive compatible payments. Theorem $4.6$ identified this set as the the tropical eigenspace, i.e. the tropical semi-module of D-IC payments. It is the min-plus image of $(L^{g,t-i})^*$.

Rewriting the eigenvector condition, we obtain

$$\max_{k \in [m]} \left\{-L^{g,t-i}_{jk} - p_k\right\} = -p_j \forall j \in [m].$$

The terms $-L^{g,t-i}_{jk} - p_k$ are utility gains relative to truth-telling net payments. Thus the mechanism makes any agent pay the most he could gain.

This can be linked to linear programming duality *viz*. A reporting strategy of an agent is to announce a preference for some allocation that maximizes his utility, i.e. a probability distribution $s \in \Delta^{m-1}$. Recall that types in $\mathbb{T}^{m-1}$ are normalized, i.e. $t = (0, t_2-t_1, \ldots, t_m-t_1)$. The mechanism tries to find a vector $p \in \mathbb{T}^{m-1}$ that solves the following family of linear programs simultaneously for all $j \in [m]$.

$$(P) \quad \inf_{\mu} \sup_{s \geq 0} \left\{ \sum_{k=1}^m s_k \left(t_k - t_j\right) - (p_k - p_j) - \mu \left(\sum_{k=1}^m s_k - 1\right) \right\} \quad \forall t \in g^{-1}(j)$$

The inner problem is the agent’s, he maximizes utility relative truth-telling.

Now, suppose $p \in \text{Eig}_0(L^{g,t-i})$ and hence $\text{coVec}_{L^{g,t-i}}(p)$ has ones on its diagonal. It follows that for any $j \in [m]$ and any $t \in g^{-1}(j)$:

$$t_j - p_j \geq t_k - p_k \quad k \in [m], k \neq j$$
Thus, the inner problem of (P) is maximized by setting $s_j(t) = 1$ and $s_k(t) = 0$ for $k \neq j$, i.e. truth-telling. Implicitly this is the content of Theorem 4.6. For each $j \in [m]$, linear programing duality lets us then equivalently solve the dual.

\[
(D) \sup_{s \geq 0} \inf_{\mu} \left\{ \mu + \sum_{r=1}^{m} s_r \left( (t_r - t_j) - (p_r - p_j) - \mu \right) \right\} \forall t \in g^{-1}(j)
\]

Explicitly the problem is to minimize $\mu$ subject to $\mu \geq 0$ and $(t_k - t_j) - (p_k - p_j) \leq \mu$ for $k \neq j$. In economics $\mu(t)$ is the shadow-value of an agent with type $t$ from lying. If $\mu > 0$ then the agent has a strict incentive to misreport. For $p \in \text{Eig}_0(L_{g,-j})$ this program is solved by setting $\mu(t) = 0$. Thus $\{1\}$ is the agent’s problem of maximizing his utility relative to truth-telling for given payments. Problem $\{D\}$ is the mechanism’s problem of minimizing incentives to lie given optimizing behavior. By flipping sectors at their apices and intersecting the regions $L_j$, we geometrically characterize payments that induce truth-telling.

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