Understanding Preferences: “Demand Types”, and the Existence of Equilibrium with Indivisibilities

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COMMENTS WELCOME!‡

2 September 2016 version—the latest version is at:
elizabeth-baldwin.me.uk/papers/demandtypes.pdf and
www.nuff.ox.ac.uk/users/klemperer/demandtypes.pdf

Abstract

We propose new techniques for understanding agents’ valuations. Our classification into “demand types”, incorporates existing definitions (substitutes, complements, “strong substitutes”, etc.) and permits new ones. Our Unimodularity Theorem generalises previous results about when competitive equilibrium exists for any set of agents whose valuations are all of a “demand type” for indivisible goods. Contrary to popular belief, equilibrium is guaranteed for more classes of purely-complements, than of purely-substitutes, preferences. Our Intersection Count Theorem checks equilibrium existence for combinations of agents with specific valuations by counting the intersection points of geometric objects. Applications include matching and coalition-formation; and the Product-Mix Auction, introduced by the Bank of England in response to the financial crisis.

1 Introduction

This paper introduces a new way to think about preferences for indivisible goods, and obtains new results about the existence of competitive equilibrium.

“Demand types.” Our first key idea is to classify economic agents’ individual and aggregate valuations into “demand types”. A “demand type” is defined by a list of vectors that give the possible ways in which the individual or aggregate demand can change in response to a small generic price change. So the vectors defining a “demand type”

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‡This paper extends and supersedes much of the material in Baldwin and Klemperer (2014). However, large parts of Sections 2.5, 6.2, and 6.3.1 of that paper will be incorporated and developed in Baldwin and Klemperer (in preparation-a), and Sections 4.2, 4.3, and 5, form the main part Baldwin and Klemperer (in preparation-b). This work was supported by ESRC grant ES/L003058/1. Acknowledgements to be completed later.
are analogous to the rows of a Slutsky matrix; they specify the possible comparative
statics of any demand of that “type”.

For example, a purchaser of spectacles who values spare pairs might always buy
lenses and frames in the ratio 2:1, so increase or reduce her demand in this ratio in
response to any price change; her valuation is therefore of “demand type” $\pm\{(2, 1)\}$.  
As another example, you might want to book a large hotel room, or a small room, or
neither, but have no interest in both. So your response, if any, to a small change in
prices would either be to substitute one room for the other, or to increase or decrease
your demand for one of the rooms by 1 without altering your demand for the other.
That is, your valuation is of “demand type” $\pm\{(1, -1), (0, 1), (1, 0)\}$.\footnote{In an auction in which goods’ characteristics suggest natural rates of substitution, bidders might be
asked to express valuations of the corresponding “demand type”; e.g., the Bank of England’s Product-Mix Auction built one-for-one substitution into its design (see Klemperer, 2008, 2010).}

Our classification is parsimonious. For example, the “demand type” that comprises
all possible substitutes preferences is defined by the set of all vectors with at most one
positive integer entry, at most one negative integer entry, and all other entries zero; the
“demand type” that is all complements preferences is defined by the set of all vectors
in which all the non-zero entries (of which there may be any number) are integers of
the same sign; the class of all “strong substitutes” preferences for $n$ goods is a “demand
type” with just $n(n + 1)$ vectors.

Our classification clarifies the relationships between different classes of preferences.
For example, the “demand types” descriptions above show clearly why the conditions for
indivisible goods to all be (ordinary) substitutes are in general far more restrictive than
the conditions for them to all be complements—although they are, of course, symmetric
in the two-good case.

Our classification is very general. It permits multiple units of each good; the agents
can include sellers, buyers, and traders who can both buy and sell; and it can also be
applied to matching models.

Finally, we will see the classification is easy to work with.

**Equilibrium existence.**  Our focus on how agents’ demands change in response to
small price changes yields two new theorems about the existence of competitive equilib-
rium with indivisibilities:

Our “Unimodularity Theorem” characterizes equilibrium existence for “demand types”,
that is, for any valuations in *classes* of preferences. It states that competitive equilib-
rium always exists, whatever is the market supply, if and only if all agents’ valuations are
concave and drawn from a demand type that is defined by a unimodular set of vectors.\footnote{A unimodular set of vectors in $n$ dimensions is one for which every subset of $n$ of them has
determinant 0 or $\pm 1$ (with an additional condition if they are not a spanning set).}

Our characterization immediately yields several earlier existence results, and exten-
sions of them. It also identifies previously-unknown environments in which existence
is assured. Moreover, it disproves the popular perception that existence requires sub-
stitutes valuations (or a “basis change” thereof). Indeed every demand type for which
equilibrium is guaranteed can be obtained as a basis change of a demand type involv-
ing only complements preferences (and for which equilibrium is guaranteed)—and the
 correspon ding result is not true for substitute preferences.
Our “Intersection Count Theorem”, by contrast, concerns whether competitive equilibrium exists for combinations of agents with specific valuations. It relates whether equilibrium always exists whatever is the market supply, for the specific agents, to the number of price vectors at which more than one of these agents is indifferent between more than one bundle.

To illustrate our two Theorems, recall the hotel-room example above. Like you, Elizabeth is interested in either room (the hotel only has two rooms), but not both. Both your and her valuations are therefore of demand type \pm\{(1, -1), (0, 1), (1, 0)\}, which is unimodular (because any matrix formed by two of these vectors has determinant 0 or \pm1). So the Unimodularity Theorem tells us that whatever are your and Elizabeth’s valuations (they will generally be different), there always exist competitive equilibrium prices, that is, prices such that demand exactly equals supply, if you and she are the only potential buyers.

Paul, however, requires two hotel rooms for his family; if they cannot have both, they will go elsewhere. So Paul’s valuation is of demand type \pm\{(1, 1)\}. The (smallest) demand type from which Elizabeth’s and Paul’s valuations are both drawn is therefore \pm\{(1, 1), (1, -1), (0, 1), (1, 0)\}, which is not unimodular (because the determinant of (1, 1) and (1, -1) is -2). So the Unimodularity Theorem tells us that, if Elizabeth and Paul are the potential buyers, then there are some valuation(s) of Elizabeth and Paul for which competitive equilibrium does not exist—but the Theorem does not tell us which those valuation(s) are.

However, our Intersection Count Theorem does tell us whether equilibrium exists for any specific valuations: equilibrium exists for Elizabeth and Paul if and only if either (i) there are exactly two price vectors at which both agents are indifferent between more than one bundle, or (ii) there exists a price at which one agent is indifferent between at least two bundles, and the other is indifferent between at least three (case (ii) is non-generic).

For example, imagine Elizabeth would pay up to £40 for the large room, or £30 for the small. Paul is indifferent between paying £50 for both, and going elsewhere. Then there is only one pair of prices, (£30, £20) for the large and small rooms respectively, such that both agents are indifferent between more than one option. (Paul is indifferent between taking both rooms, and taking neither, while Elizabeth is indifferent between the two rooms so, also, each agent is indifferent between only two bundles.) So the Intersection Count Theorem predicts—and it is not hard to check (and Section 5 will confirm)—that competitive equilibrium will fail: at any prices at which Paul is prepared to take both rooms, Elizabeth will also demand a room.

If, however, Paul is willing to pay up to £100 for the rooms, then there are exactly two price vectors, (£40, £60) and (£70, £30), such that both Elizabeth and Paul are indifferent between more than one option (Paul between taking both rooms and neither, and Elizabeth between taking no room and the one room she considers good value). So the Intersection Count Theorem now predicts that competitive equilibrium does exist. (In fact, any prices that exceed £40 for the large and £30 for the small, and add to less than £100, clear the market.) Section 5 gives full details.

Outline of the paper. Our basic tools are convex geometry and, in Section 5 in particular, the “tropical geometry” recently developed by, among others, Mikhalkin.
(2004, 2005). So we begin, in Section 2, by using the existing mathematics literature to develop an economic understanding of two dual geometric objects:

The first, the “Locus of Indifference Prices” (LIP) comprises the price vectors at which the agent is indifferent between two or more bundles, that is, the prices at which the agent’s demand changes. Since any LIP corresponds to a valuation function, we can develop our understanding of demand by working directly with these geometric objects.

Our dual geometric object, the “Demand Complex”, comprises the convex hulls of the sets of bundles (i.e., quantity vectors) among which the agent is indifferent at some price.

Section 3 then defines a “demand type” by using the set of vectors that describes the ways in which the bundles demanded by the agent can change with prices. These vectors are associated in simple ways with both the LIP and the Demand Complex. So we can easily check whether a demand type is, for example, substitutes, or complements, or “strong substitutes”, or “gross substitutes and complements”, etc.

Section 4 proves our Unimodularity Theorem. Danilov et al. (2001) provide a very similar sufficient condition for equilibrium, but our use of tropical ideas allows for a simpler proof. We also show the necessity of the same condition, so that our theorem is a full characterization of when equilibrium exists. Perhaps more important, our concept of “demand types” also shows how this condition can be applied. For example, equilibrium existence results such as those in Sun and Yang (2006), Milgrom and Strulovici (2009), and Hatfield et al. (2013), are obvious corollaries of the Unimodularity Theorem, but none of these papers present their results as applications of Danilov et al.’s earlier work, since the latter’s relevance was unclear.

Section 5 develops our Intersection Count Theorem by applying a version of Bézout’s classic theorem that the number of intersection points of two curves, taking into account “multiplicities” such as tangencies, is equal to the product of the degrees of their defining polynomials. (LIPs can be obtained as “tropical” transformations of “ordinary” geometric objects, and their intersection properties are preserved under these transformations.) We thereby obtain an algorithm for determining whether equilibrium exists, for every market supply, for a specific set of individual agents’ valuations.

Section 6 discusses applications: Sections 6.1 shows that our model encompasses classic models, and clarifies the relationships between them. Sections 6.2-6.3 show how to find new “demand types” for which equilibrium always exists. These include purely-complements “demand types” which are unrelated to any substitutes preferences (including via any basis change), and Section 6.4 shows that there are many such “demand types”. Sections 6.5-6.7 explain that our geometric techniques yield new results in other contexts, including matching models and developing extensions of the Product-

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4We believe the first version of this paper, Baldwin and Klemperer (2012), was the first to apply tropical geometry to economics. Matveenko (2014), Shiozawa (2015), Crowell and Tran (2016) and Weymark (2016) are other applications.

4We especially thank Gleb Koshevoy for very helpful discussions. Analysing “demand types” in price space (as well as, like Danilov et al., in quantity space) also allows us to develop additional implications. We discuss the relationships to Danilov and Koshevoy and their co-authors’ work in detail in Section 4.3. Our Intersection Count Theorem addresses similar issues as Bikchandani and Mamer (1997) and Ma (1998), but our methodology is completely different and, we believe, gives more insight.

5For example, two lines intersect once (possibly at infinity). A quadratic and a line intersect at two points (possibly including points with complex coordinates and points at infinity, and double-counting tangencies). Two quadratics intersect four times (correctly counted), etc.
Mix Auctions introduced by the Bank of England during the financial crisis. Section 7 concludes. The Appendix contains additional examples, and proofs of all results not proved in the text.

This paper has been written for economists. Some of our ideas have been translated for a mathematical audience by Tran and Yu (2015).

2 Representing Indivisible Demand Geometrically

2.1 Assumptions

An agent has a valuation \( u : A \to \mathbb{R} \) on bundles \( x \in A \subseteq \mathbb{Z}^n \). That is, the bundles are formed of \( n \) distinct goods, which come in indivisible units. Note that a bundle may be negative or mixed-sign. So our model allows for sellers with non-trivial supply functions, and more general traders, as well as buyers.

The domain \( A \) of bundles that the agent considers possible, can be any finite set in \( \mathbb{Z}^n \). Note that \( A \) need not contain every integer bundle in its convex hull. Nor need \( A \) include every bundle that is available in the economy. In particular, if a bundle is completely unacceptable to the agent, it is simply not in \( A \). This is equivalent to (and mathematically more convenient than) allowing the agent to value some bundles at “\(-\infty\)”. 

The agent has quasilinear utility, so maximises \( u(x) - p \cdot x \), where \( p \in \mathbb{R}^n \) is the price vector. Thus different units of the same good all have the same price. (If they did not we could treat them as different goods.) We do not specify that valuations are weakly increasing, or that valuations or prices are non-negative, so our model covers “bads” as well as goods.

We will extend our model to multiple agents in Section 3.3. The remainder of this section interprets existing mathematics literature in the context of our model.

2.2 The Locus of Indifference Prices (LIP)

We will be particularly interested in the prices at which the agent’s demand set, \( D_u(p) = \arg \max_{x \in A} \{ u(x) - p \cdot x \} \), contains more than one bundle, that is, those prices at which the agent is indifferent among more than one bundle.

Definition 2.1. The Locus of Indifference Prices (LIP) is \( L_u := \{ p \in \mathbb{R}^n : |D_u(p)| > 1 \} \).

This set is known as a “tropical hypersurface” in the mathematics literature (see Mikhalkin, 2004, and others), but we are introducing new terminology to facilitate understanding among economists. We analyse the structure of this set in more detail because (by continuity of quasilinear utility) it comprises the only prices at which demand can change in response to a price change.

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\(^6\)Bidders in these auctions make sets of “either/or” bids for alternative objects. These bids can be represented geometrically as sets of points in multi-dimensional price space. The then-Governor of the Bank of England (Mervyn King) told the Economist that the Product-Mix Auction “is a marvellous application of theoretical economics to a practical problem of vital importance”; current-Governor Mark Carney announced plans for its greater use; and an updated version has been introduced—see Bank of England (2010, 2011), Milnes (2010), Fisher (2011), Frost et al (2015) and the Economist (2012).
Figure 1: A simple LIP, for valuation $u(0, 0) = 0$, $u(1, 0) = 5$ and $u(0, 1) = 4$. The bundle demanded on each side of the LIP is labelled.

**Definition 2.2.**

1. A cell of $L_u$ is a non-empty set of the form $\{p \in L_u : x^1, \ldots, x^k \in D_u(p)\}$ where $|\{x^1, \ldots, x^k\}| > 1$.
2. A facet is an $(n - 1)$-dimensional cell of $L_u$.7

Thus a cell of $L_u$ is the subset of its prices at which the bundles that the agent demands include a particular collection of at least two bundles. The cell therefore specifies the prices at which the agent’s demand can change between the bundles in this particular collection. By continuity, cells are closed.

At prices not in $L_u$, the agent demands a unique bundle:

**Definition 2.3.** A unique demand region (UDR) of $u$ is a connected component of the complement of $L_u$ in $\mathbb{R}^n$.

Demand is generically unique, so UDRs are $n$-dimensional and open in $\mathbb{R}^n$. The bundle defining the UDR is, again by continuity, also demanded in its closure. It is straightforward that a UDR is convex and each UDR corresponds to a different bundle. Thus the closure of a UDR gives the only points at which this particular bundle is demanded (that is, it has the form $\{p \in L_u : x \in D_u(p)\}$), while a cell of $L_u$ gives the only points at which some collection of multiple bundles is demanded.

Fig. 1 shows a simple example of a LIP. The agent uniquely demands one of the bundles $(0, 0)$, $(0, 1)$, and $(1, 0)$, in the correspondingly-labelled 2-dimensional region, so these regions are the UDRs. The agent demands both bundle $(0, 0)$ and bundle $(0, 1)$ on the line segment $\{(p_1, 4) \in \mathbb{R}^2 : p_1 \geq 5\}$ so this is a facet (1-dimensional cell). There are two other facets. Meanwhile, the price $(5, 4)$ is a 0-dimensional cell (or “0-cell”)—it is the only price at which the agent is indifferent between all three bundles.

If, instead, bundles were formed from three distinct goods, i.e. $n = 3$, the facets would be the plane-segments, separating 3-dimensional UDRs; the facets would then meet in line segments, i.e. 1-cells, which would themselves meet in 0-cells.

We give the facets a specific name because of the economic information they contain. At any price $p$ in a given facet, $F$, the agent is indifferent between the bundles $x$ and $x'$ demanded in the UDRs on either side of $F$. That is, $u(x) - p \cdot x = u(x') - p \cdot x'$, $\forall p \in F$. So $p : (x' - x)$ is constant across all $p \in F$. Therefore $F$ is normal to the vector that gives the change in demand, $x' - x$, between the UDRs on either side of $F$. For example,

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7We always use the natural dimensions. Thus the dimension of a cell is the dimension of its affine span, i.e. the dimension of the smallest linear subspace $U \subseteq \mathbb{R}^n$ such that the cell is contained in $\{c\} + U$ for some fixed vector $c$. Here, and throughout the text, we use Minkowski (set-wise) addition on sets.
in Fig. 1, the facet \( \{ (p_1, 4) \in \mathbb{R}^2 : p_1 \geq 5 \} \) contains prices at which demand can change by \( (0, 0) - (0, 1) = (0, -1) \), which vector is normal to this facet.

So the geometry of the LIP tells us the directions of demand changes between pairs of prices. To know how much demand changes in any direction that the LIP specifies, we need one more piece of information:

**Definition 2.4.** Let \( x, x' \) be the bundles demanded in the UDRs on either side of facet \( F \). The weight of \( F \), \( w_u(F) \), is the greatest common divisor of the entries of \( x' - x \).

Now \( \frac{1}{w_u(F)}(x' - x) \) is a primitive integer vector (the greatest common divisor is 1). It points from the UDR where \( x' \) is demanded, to the UDR where \( x \) is demanded, and is in the opposite direction to the change denoted by \( x' - x \). But since \( F \) is \((n-1)\) dimensional, there is a unique primitive integer vector normal to \( F \) and pointing in this direction. So we have shown:

**Proposition 2.5.**

(1) If \( x, x' \) are uniquely demanded on either side of facet \( F \), then \( p \cdot (x' - x) \) is constant for all \( p \in F \).

(2) The change in demand as price changes between the UDRs on either side of \( F \), is \( w_u(F) \) times the primitive integer vector that is normal to \( F \), and points in the opposite direction to the change in price.

That is, the LIP and its weights, taken together, provide full information about how demand changes between UDRs.

**2.2.1 The correspondence between LIPs and valuations**

The previous subsection showed that, starting with a valuation, we could derive a geometric object, encoding economic information. We now show, conversely, that we can start with a purely geometric object and associate economic meaning.

We will use some standard terminology from convex geometry:

**Definition 2.6.**

(1) A *rational polyhedron* is the intersection of a finite collection of half-spaces \( \{ p \in \mathbb{R}^n : p \cdot v^j \leq \alpha^j \} \) for some \( v^j \in \mathbb{Z}^n \) and \( \alpha^j \in \mathbb{R} \).

(2) A *face* of a polyhedron \( C \) maximises \( p \cdot v \) over \( p \in C \), for some fixed \( v \in \mathbb{R}^n \).

(3) The *interior* of polyhedron \( C \) is \( C^\circ := \{ p \in C : p \not\in C' \text{ for any face } C' \subseteq C \} \).

(4) A *rational polyhedral complex* \( \Pi \) is a finite collection of sets \( C \subseteq \mathbb{R}^n \) such that:

- (i) if \( C \in \Pi \) then \( C \) is a rational polyhedron and any face of \( C \) is also in \( \Pi \);
- (ii) if \( C, C' \in \Pi \) then either \( C \cap C' = \emptyset \) or \( C \cap C' \) is a face of both \( C \) and \( C' \).

(5) A *k-cell* is a cell of dimension \( k \).

(6) A polyhedral complex is *k-dimensional* if all its cells are contained in its \( k \)-cells.

(7) A *weighted* polyhedral complex is a pair \( (\Pi, w) \) where \( \Pi \) is a polyhedral complex and \( w \) is a vector assigning a weight \( w(F) \in \mathbb{Z}_{\geq 0} \) to each facet \( F \in \Pi \).

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\(^8\)We follow the convention of Mikhalkin (2004) in not restricting \( \alpha^j \) to be rational for rationality of the complex.
Since the cells of \( L_u \), and the closures of its UDRs, are defined by collections of linear equalities and weak inequalities, it is straightforward that they are all polyhedra and fit together as a ‘complex’ (details in Appendix A.1). In particular:

**Proposition 2.7.** The set of all cells of \( L_u \) is an \((n-1)\)-dimensional rational polyhedral complex.

So if \( C \) is a cell of \( L_u \), then every face \( C' \) of \( C \) satisfying \( C' \subseteq C \) is also a cell of \( L_u \). It follows that at prices in such \( C' \), the agent demands additional bundles to those that she demands in \( C \). But the agent’s demand set is constant in the interior of the cell. That is:

**Lemma 2.8.** \( D_u(p^o) \) is constant across all \( p^o \) in the interior \( C^o \) of a cell \( C \). Moreover \( D_u(p^o) \) defines the cell: \( C = \{ p \in \mathbb{R}^n : D_u(p^o) \subseteq D_u(p) \} \).

Fig. 1 illustrates all these points.

Prop. 2.5 tells us that once we know the demand in one particular UDR, and we know the weights of the LIP, we can infer the demand in every UDR, by stepping across a series of facets. But if we follow an agent along a price path that ends where it started, the demand at the end must be the same as that at the beginning. So the weights on the facets must satisfy the balancing condition:

**Definition 2.9** (Mikhalkin, 2004, Defn. 3). An \((n-1)\)-dimensional weighted polyhedral complex \( \Pi \) is balanced if for every \((n-2)\)-cell \( G \in \Pi \), the weights \( w(F_j) \) on the facets \( F_1, \ldots, F_l \) that contain \( G \), and primitive integer normal vectors \( v_{F_j} \) for these facets that are defined by a fixed rotational direction about \( G \), satisfy \( \sum_{j=1}^{l} w(F_j)v_{F_j} = 0.\)

This balancing condition is, in fact, the only condition that a weighted rational polyhedral complex has to satisfy to be the LIP of some valuation function.\(^{10}\) That is:

**Theorem 2.10** (Mikhalkin, 2004, Prop. 2.4; also Mikhalkin, 2005, Thm. 3.15). Suppose that \((\Pi, w)\) is an \((n-1)\)-dimensional balanced weighted rational polyhedral complex in \( \mathbb{R}^n \), and let \( L \) be the union of its cells. Then there exists a finite set \( A \subseteq \mathbb{Z}^n \) and a function \( u : A \to \mathbb{R} \) such that \( L_u = L \) and \( w_u = w \).

By contrast with Afriat’s theorem (see e.g. Vohra, 2011, Thm. 7.2.1), which starts with a (finite) set of prices paired with demands, this theorem uses only information about the geometrical divisions in price space.

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\(^{9}\)This is just the \( n \)-dimensional generalisation of the requirement in 2 dimensions that, when moving in a sufficiently small circle around any point, the vectors \( v_F \) all point in the direction of travel. To choose a rotational direction around \( G \), pick a 2-dimensional affine subspace \( H \) of \( \mathbb{R}^n \) orthogonal to \( G \), such that the intersection of each \( F_j \) with \( H \) is 1-dimensional. The intersection of \( H \) with the LIP is then a collection of 1-cells meeting at the 0-cell which is \( G \cap H \). An ordinary choice of rotational direction in this two-dimensional picture gives a rotational direction around \( G \) in \( \mathbb{R}^n \).

\(^{10}\)There do not necessarily exist weights to balance a general rational polyhedral complex. For example, in two dimensions, consider three points (0-cells), each contained in three facets, such that each pair of points are both contained in a common facet. There are six weights, which must satisfy six equations (three balancing conditions in each of the two dimensions). But since the conditions are trivially satisfied by setting all weights equal to zero, the conditions can only be satisfied by positive integer weights if the conditions are not linearly independent—which is non-generic.
Thm. 2.10 is not mathematically novel, but its economic implications are important (and, we believe, novel). It shows that a set in $\mathbb{R}^n$ is the LIP of a quasilinear valuation if and only if it has some easily-checked geometric properties. It is also easy to identify the cells of this LIP, in particular its facets, and so understand the economics of the valuation, since:

**Lemma 2.11.** $C \subseteq L_u$ is a cell iff it is the intersection of the closures of a set of UDRs of $u$.

In practice it is often much easier to develop ideas and intuitions by working with these geometric objects, than by working out explicit examples of valuations. Subsequent sections will show in more detail how simply describing the geometry of the LIP, and of related objects, gives insight into the economics.

The next section explores the uniqueness of valuations identified in this way.

### 2.2.2 The correspondence between LIPs and concave valuations

We define concavity of the valuation $u$ in the standard “concave-extensible” sense, but with an extra property since we allow the domain to be any finite subset of $\mathbb{Z}^n$:

**Definition 2.12.**

1. A set $A \subseteq \mathbb{Z}^n$ is **discrete convex** if it contains every integer point within its convex hull, that is, $\text{conv}(A) \cap \mathbb{Z}^n = A$.
2. We write $\text{conv}(u) : \text{conv}(A) \to \mathbb{R}$ for the minimal weakly-concave function everywhere weakly greater than $u$ (sometimes called the “concave majorant” of $u$).
3. $u : A \to \mathbb{R}$ is **concave** if $A$ is discrete-convex and $u(x) = \text{conv}(u)(x)$ for all $x \in A$.

It is a standard result that concave valuations are precisely those for which every possible bundle is demanded at some price, and for which the demand set at any price is discrete-convex, just as for divisible, weakly-concave valuations, and for essentially the same reasons:

**Lemma 2.13.** $u : A \to \mathbb{R}$ is concave iff for all $x \in \text{conv}(A) \cap \mathbb{Z}^n$ there exists $p$ such that $x \in D_u(p)$.

For a simple example of failure of concavity, consider the 1-dimensional valuation $u(0) = u(1) = 0$; $u(2) = 10$. Then $D_u(5) = \{0, 2\}$ is not discrete-convex, and there exists no price $p$ such that $1 \in D_u(p)$.

If we weakly increase a valuation until it becomes concave, it is easy to see that the only values we need change are those for bundles which were previously never demanded. And increasing any never-demanded bundle’s value has no effect on the agent’s behaviour until the bundle is just marginally demanded, when the value function becomes locally affine. The marginally defined bundle is then added to the demand at some prices, but is never demanded uniquely, and all other bundles are demanded exactly as they were previously, so the LIP is unchanged. That is:

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11See Appendix A.1; these results are also clear from the example in the next subsection (2.3). For the divisible case see, e.g., Mas-Colell et al. (1995) pp. 135-8, especially Prop. 5.C.1(v), since a quasilinear valuation is equivalent to a standard profit function with a single-output technology.
Lemma 2.14. Let $u : A \rightarrow \mathbb{R}$. Then:

1. for each $x \in A$, $u(x) = \text{conv}(u)(x)$ iff there exists $p$ such that $x \in D_u(p)$;
2. $\mathcal{L}_u = \mathcal{L}_{u'}$, where $u'$ is the restriction of $\text{conv}(u)$ to $\text{conv}(A) \cap \mathbb{Z}^n$.

However, restricting to concave valuations does not resolve the only ambiguity in associating valuations to weighted rational polyhedral complexes. Adding a constant to $u(x)$ leaves the LIP unchanged, as does increasing every available bundle by a fixed bundle and making a corresponding shift in the valuation. So to give a full equivalence between weighted LIPs and concave valuation functions we must specify the exact demand set at some price, and the value of one bundle.

Theorem 2.15 (Mikhalkin, 2004, Remark 2.3). Let $(\Pi, w)$ be an $(n-1)$-dimensional balanced weighted rational polyhedral complex in $\mathbb{R}^n$, let $\mathcal{L}$ be the union of the cells of $\Pi$, and let $p$ be any price not contained in $\mathcal{L}$. Then there exists a unique concave valuation $u$ such that $D_u(p) = \{0\}$, $u(0) = 0$, $\mathcal{L} = \mathcal{L}_u$ and $w = w_u$.

In sum, Thms. 2.10 and 2.15 tell us that we can develop our understanding of valuations by working directly with geometric pictures of unions of cells which form balanced weighted rational polyhedral complexes. Any such geometric picture corresponds to a concave valuation, which is unique up to the ambiguity described. However, we will not restrict attention to concave valuations.

2.3 The Demand Complex

We constructed the LIP in price space. It is now useful to construct a dual geometric object—the demand complex—in quantity space.

We saw that the LIP consists of a complex of cells, each cell being a set of price vectors at which a given set of bundles is demanded. Conversely, the demand complex is a collection of cells, each cell being the convex hull of a set of bundles (quantity vectors) which are demanded at a given set of price vectors. (We will see that working with the convex hulls of demand sets, rather than with the sets themselves, yields a valuable duality with LIPs.)

Definition 2.16.

1. The demand complex $\Sigma_u$ is the set of all cells $\sigma := \text{conv}(D_u(p))$ where $p \in \mathbb{R}^n$.
2. The vertices of the demand complex are its 0-cells.
3. The edges of the demand complex are its 1-cells.
4. The length of an edge is the number of primitive integer vectors, in its direction, of which it is formed (i.e., its Euclidean length divided by the Euclidean length of the primitive integer vector in its direction).

It is easy to see that every cell in $\Sigma_u$ is a rational polyhedron. Furthermore,

Proposition 2.17. The demand complex is a rational polyhedral complex, with dimension equal to that of $\text{conv}(A)$.

Of course, the bundle demanded at any price is then increased by the fixed bundle.
We will understand this proposition via an alternative description of the demand complex, which aids intuition and also makes it easy to quickly develop examples.

First, note it is clear that:

**Lemma 2.18.** \( D_{\text{conv}(u)}(p) = \text{conv}(D_u(p)) \) for all \( p \in \mathbb{R}^n \).

Now, \( \text{conv}(u) \) can be understood as a valuation function on divisible goods. So we can use the standard construction for a concave valuation: any price vector defines a hyperplane, tangent to the graph of the agent’s valuation, which meets this graph at the agent’s demand set for that price. But because \( \text{conv}(u) \) is only weakly-concave, some tangent hyperplanes meet the graph at more than one point, and some demand sets are multi-valued.

For example, Fig. 2a shows a valuation function, \( u \), and Fig. 2b illustrates, using bars to represent the valuations, \( u(x) \), of bundles \( x \). We will always present the feasible bundles increasing to the left, and down. This will reveal the duality between the demand complex and the LIP most clearly.

![Figure 2: A valuation and its roof.](image)

Fig. 2c shows the graph of \( \text{conv}(u) \). We call this the “roof” of the valuation. At any price \( p \), the bundles, \( x \), demanded under the valuation \( \text{conv}(u) \), are those that maximise \( \text{conv}(u)(x) - p \cdot x = (-p, 1) \cdot (x, \text{conv}(u)(x)) \). That is, \( x \) is demanded at \( p \) if the point \( (x, \text{conv}(u)(x)) \) is “farthest out” from the origin in the “direction of that price” (i.e., in the direction \((-p, 1))\). So an intersection between the roof and a supporting hyperplane is a set of the form \( \hat{\sigma} = \{(x, \text{conv}(u)(x)) \in \mathbb{R}^{n+1} : x \in D_{\text{conv}(u)}(p)\} \), where \( p \) is such that \((-p, 1)\) is normal to the hyperplane. And projecting this set from \( \mathbb{R}^{n+1} \) to its first \( n \) coordinates (in \( \mathbb{R}^n \)) just yields the set \( D_{\text{conv}(u)}(p) = \text{conv}(D_u(p)) \) for that \( p \). So:

**Lemma 2.19.** \( \hat{\sigma} \in \mathbb{R}^{n+1} \) is a face of the roof iff the projection of \( \hat{\sigma} \) to its first \( n \) coordinates is a cell \( \sigma \in \mathbb{R}^n \) of the demand complex.

So projecting the faces of the roof into \( \mathbb{R}^n \) yields the collection of all demand complex cells \( \text{conv}(D_u(p)) \). This is illustrated by the projection beneath the roof in Fig. 2c, and
the demand complex in Fig. 3a. Moreover, it is clear that the faces of the roof are faces of a polyhedron, namely, the convex hull of the points \((x, u(x))\). So these faces form a polyhedral complex. Prop. 2.17 follows from the fact that the projection of this complex to its first \(n\) coordinates is one-to-one. (Details are given in Appendix A.2).

Fig. 3a shows the three 2-cells (areas), shaded to match the corresponding pieces of planes of the roof in Fig. 2c. The 2-cells are separated by nine edges (line-segments that are 1-cells), that themselves meet in the seven vertices (0-cells) of the demand complex.

Note that only the “white” circles represent vertices. The grey and black circles represent bundles that are not at vertices of the demand complex, since they are not uniquely demanded at any price. Indeed the demand complex cannot tell us whether non-vertex bundles such as these are ever demanded. However, it does tell us that if a non-vertex bundle is demanded at any price, then it is demanded at the price(s) corresponding to those cells in which it lies. This follows straightforwardly from Lemma 2.14(1)’s result that if a bundle, \(x\), is demanded at any price, then \(u(x) = \text{conv}(u(x))\), together with the observation that \(D_u(p) = \{x : u(x) = \text{conv}(u(x))\} \cap D_{\text{conv}(u)}(p)\), and Lemma 2.18. So we have proved:

**Lemma 2.20.** If there is any price \(p\) at which \(x\) is demanded, and if \(x \in \text{conv} (D_u(p))\), then \(x \in D_u(p)\).

We discuss the example in more detail in the next section.

### 2.4 Duality

We can now see an instructive (and beautiful) duality between the demand complex and the LIP.\(^{14}\)

Since the vertices of the demand complex are at bundles which are uniquely demanded for some price, they correspond to UDRs. And an edge of the demand complex between vertices \(x\) and \(x'\) indicates the existence of prices, \(p\), for which the demand set contains both these bundles. Moreover, such \(p\) form an \((n-1)\)-dimensional facet of the LIP, as they are defined by only one equality constraint \(u(x) - p \cdot x = u(x') - p \cdot x'\).\(^{15}\)

And as we saw in Prop. 2.5, \(p \cdot (x' - x) = \text{constant}\), for all these price vectors, \(p\). So each edge of the demand complex is normal to the facet that corresponds to it in the LIP. And more generally:

**Proposition 2.21.** There is a bijective correspondence between: vertices of the demand complex and closures of UDRs; between edges of the demand complex and facets of the LIP; and, for \(k \geq 1\), between \(k\)-cells \(\sigma\) of the demand complex and \((n-k)\)-cells \(C_{\sigma}\) of the LIP; such that:

\[
(1) \quad \sigma = \text{conv} (D_u(p)) \iff p \in C_{\sigma}^{\circ};
\]

\(^{13}\)We depict the demand complex by drawing its top-dimensional cells, on a grid of integer bundles. The remaining cells are easily identified as faces of the top-dimensional cells, while the grid allows us to identify the ‘lengths’ of edges and the bundles in any cell. We omit axes, since replacing \(A\) with \(A + x\) for some \(x \in \mathbb{Z}^n\), and re-defining \(u\) correspondingly, yields the same demand complex (and the same weighted LIP).

\(^{14}\)The construction uses Legendre-Fenchel duality; e.g. see Murota (2003). For more on these ‘regular subdivisions’ and on polytopes in general see Thomas (2006) and De Loera et al. 2010.

\(^{15}\)If there are additional points in \(A\) lying on the edge, they do not impose additional linearly independent constraints on such \(p\); see the discussion following relating to the ‘dark-grey edge’.

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Figure 3: (a) The demand complex and (b) LIP of the valuation $u$ given in Fig. 2a, with dual geometric objects drawn with the same style and shading; (c) a LIP of a different valuation from that of Fig. 2a, but also dual to the demand complex of Fig. 3a.

(2) $C_{\sigma} = \{ p \in \mathbb{R}^n : \sigma \subseteq \text{conv} (D_u(p)) \}$;

(3) inclusion relationships reverse: $\sigma \subsetneq \sigma' \iff C_{\sigma'} \subsetneq C_{\sigma}$;

(4) corresponding cells are orthogonal: $(p' - p) \cdot (x' - x) = 0$ for all $p, p' \in C_{\sigma}, x, x' \in \sigma$;

(5) facets $F_{\sigma}$ correspond to edges $\sigma$ of length $w_u(F_{\sigma})$.

The demand complex and weighted LIP of the valuation of Fig. 2a are pictured in Figs. 3a and 3b respectively; cells which are dual are depicted in the same style.

Thus the 0-cells of the LIP at the prices $(4,8)$, $(1,2)$, and $(0,1)$ correspond to the dotted-, wavy-, and light-grey-, shaded 2-cells of the demand complex, respectively; the nine facets of the LIP correspond to the nine correspondingly-styled edges of the demand complex; and each of the seven UDRs, around the LIP, corresponds to one of the seven bundles at the white circles that are the seven vertices of the demand complex.\(^{16}\)

Notice that the dark-grey horizontal edge at the top of the demand complex passes through a bundle, and has length 2 (in the sense of Defn. 2.16(4)). It is dual to the dark-grey vertical facet of the LIP, which correspondingly has weight 2, and is so labelled. Recall from Prop. 2.5 that a facet’s “weight” times its primitive integer normal vector is the change in demand between the UDRs it separates. All other edges of this demand complex have length 1; all other facets of the LIP correspondingly have weight 1.

As we noted in the previous subsection, neither the grey bundle, nor the black bundle, is a vertex of the demand complex, since neither is ever uniquely demanded for any price, so they have no corresponding UDRs.

Furthermore, neither the LIP nor the demand complex can tell us whether a non-vertex bundle such as one of these is ever demanded. However we do know, from Lemma 2.20, that because the central wavy-shaded (five-sided) demand complex cell is the only

\(^{16}\)Clockwise from the top right of the LIP, the bundles demanded in the UDRs are $(0,0)$, $(0,1)$, $(0,2)$, $(1,2)$, $(2,2)$, $(2,1)$, and $(2,0)$. 
demand complex cell that the black bundle lies in, the corresponding wavy-shaded 0-cell of the LIP in which that bundle is “hidden” indicates the only price ((1,2)) at which that bundle might be demanded. Similarly, because the dark-grey horizontal edge at the top of the demand complex is the lowest-dimensional demand complex cell that the grey bundle lies in, the corresponding dark-grey vertical facet of the LIP in which that bundle is “hidden” indicates the only prices ((4, p_2) for p_2 ≥ 8–see Fig. 3b) at which that bundle might be demanded.

In fact, (x, u(x)) is in the roof for a non-vertex bundle, x—and so the bundle is demanded— if and only if the valuation, u, is affine in the relevant range. The grey bundle is an example of this. It is at (1,0), and its valuation, 4, is the average of the valuations, 0 and 8, of the bundles (0,0) and (2,0), so it is demanded at the prices \{(4, p_2) : p_2 ≥ 8\}.

However, if u is non-concave at a non-vertex bundle, the bundle’s value lies strictly below the roof, so it is never demanded—it is “jumped over” as we cross between UDRs. The black bundle in the centre of demand complex illustrates this. Its value under u is strictly below its value under conv(u), so it lies strictly under the “roof” (see Fig. 2c) and is never demanded at any price. (See Appendix A.2 for more discussion.)

Prop. 2.21, and the remark above Prop. 2.7, allow us to characterise the set of prices at which a bundle x is demanded, if it is demanded at any prices:

**Corollary 2.22.** If σ is the minimal cell of the demand complex such that x ∈ σ, and if x is demanded for any price, then x ∈ D_u(p) iff p ∈ C_σ. In particular, the set of prices at which x is demanded forms a polyhedron.

Finally, note that, for any single demand complex, there are multiple LIPs which satisfy the correspondences and orthogonality relationships of Prop. 2.21. For example, Figs. 3b and 3c give two different LIPs—and therefore two different valuations—that both correspond to the demand complex of Fig. 3a. So it is natural to group together all valuations whose demand complexes are either the same, or differ only by a constant shift by some bundle, x:

**Definition 2.23.** Two valuations u, u’ have the same combinatorial type if they have the same demand complex, or if there exists x ∈ Z^n such that σ ∈ Σ_u iff \{x\} + σ ∈ Σ_{u’}.

It is easy to list all the possible demand complexes, and examples of their corresponding LIPs, when the set of feasible bundles is not too large–see Figs. 11 and 12 in Appendix A.2.

### 2.5 Representation in Price Space vs. Quantity Space

Although the LIP and demand complex are “dual”, there is an important distinction. In price space, any rational polyhedral complex satisfying the simple “balancing condition” of Defn. 2.9 corresponds to some valuation (see Thm. 2.10). But, in quantity space, it is not true that every way of subdividing conv(A) into a rational polyhedral complex yields a demand complex. (See Maclagan and Sturmfels, 2015, Fig. 2.3.9 for an example of a subdivision which corresponds to no LIP, and therefore to no valuation.) Nor does there seem to be any simple check of which polyhedral complexes in quantity space correspond to any valuation function.
So while we can develop examples to, e.g., test conjectures, by working with geometric objects in price space, and be certain that the corresponding valuations will exist, it is hard to do this in quantity space. Furthermore, a demand complex shows only collections of bundles among which the agent is indifferent for some prices, while LIPs show which prices bundles are demanded at. So we have found in practice that it is usually easier to develop ideas by working with our geometric objects in price space, than by working either in quantity space, or directly with valuation functions.

It is also much easier to aggregate agents’ valuations in price space (see Section 3.3). So we will mostly work in price space.

However, some information that is only implicit in the LIP becomes obvious in the demand complex. For example, we will see in Sections 4.1 and 5 that a 0-dimensional, or low-dimensional, cell of the LIP sometimes “hides” important detail that is much more easily seen in the higher-dimensional dual object in the demand complex, in quantity space.

Moreover, the easiest way to compute the LIP of a specific valuation is often by first finding the demand complex—it is easy to go from Fig. 2a to Fig. 3a in our example, and then also easy to use the duality to find a LIP of the correct combinatorial type, and from that to find the exact LIP (that is, Fig. 3b is easily found from Fig. 3a, see Appendix A.2).17 It is generally much harder to construct the LIP directly from the valuation, especially for more complicated examples than ours.

The fact that the different representations are useful in different contexts makes the ability to move easily between them, using duality, especially valuable.

3 “Demand Types”

3.1 Definition of Demand Types, and Comparative Statics

We saw in the previous section that the LIP’s facet normals describe how demand changes between UDRs (Prop. 2.5). They therefore give all the possible changes in demands that can generically result from a small change in prices. So it is natural to classify valuations into “demand types” according to these facet normals. A valuation’s demand type then gives us comparative statics information that is analogous to the information that the Slutsky matrix provides for a valuation on divisible goods. (The dimensionality is low enough with indivisibilities that we can characterise a class of valuations globally in this way, by contrast with the divisible case for which the Slutsky matrix provides information only at a point.)

Definition 3.1. Let \( \mathcal{D} \subseteq \mathbb{Z}^n \) be a set of non-zero primitive integer vectors such that if \( \mathbf{v} \in \mathcal{D} \) then \(-\mathbf{v} \in \mathcal{D}\). The demand type defined by \( \mathcal{D} \) comprises valuations \( \mathbf{u} \) such that every facet of \( \mathcal{L}_u \) has normal vector in \( \mathcal{D} \).

(We will slightly abuse notation by also writing “\( \mathbf{u} \) is of demand type \( \mathcal{D} \).”)

For example, the valuation of Fig. 1 is of demand type \( \pm\{(1, 0), (0, 1), (-1, 1)\} \), as are many other valuations, for example, all those shown in Figs. 9a-c. Note that a

\[ ^{17} \text{Constructing the LIP via the demand complex separates the question “in what directions are there line segments?” from the question “where in space are they?”}, and clarifies which bundle values have to be compared to fix the precise locations of the LIP’s cells. \]
valuation is of any demand type which contains the facet normals of its LIP; we do not restrict to the minimal such set.\footnote{Thus, the valuations of Figs. 1 and 9a-c are also of demand type $\pm\{(1,0),(0,1),(-1,1),(-2,1)\}$ which is the minimal demand type of the valuations of Figs. 2–3. Note our definition does not consider the weights on facets; see Baldwin and Klemperer (2012, note 25, and 2014, note 42).}

By duality (Prop. 2.21), we could equivalently classify valuations according to the directions of their demand complexes’ edges.\footnote{Danilov, Koshevoy and their co-authors’ work (see Section 4.3) examine these vectors in quantity space. However, they do not use them to create a taxonomy of demand or, e.g., interpret them as giving comparative statics information. We, by contrast, develop a general framework to understand them in economic terms (see also Baldwin and Klemperer, 2012, 2014 and in preparation-b).} But our description makes clear that the demand type provides the generic comparative statics.

**Proposition 3.2.** The following are equivalent for a valuation $u$:

1. $u$ is of demand type $D$.
2. For valuation $u$ and generic $p, t$, if $\exists \epsilon > 0$ such that $p$ and $p + \epsilon t$ are in distinct UDRs, and such that $\exists \epsilon' \in (0, \epsilon)$ such that $p + \epsilon' t$ is in a third distinct UDR, then the difference between bundles demanded at $p$ and $p + \epsilon t$ is an integer multiple of some vector in $D$.

That is, the change in demand between a generic starting price, $p$, and the next UDR in any given generic direction of price change, $t$, is described by one of the demand type’s vectors. (The conditions of the proposition ensure there is no third UDR between those containing $p$ and $p + \epsilon t$.) Furthermore, since the domain $A$ is finite, the response to any specific price change can, generically, be broken down into a series of steps of this form.

That (1)$\Rightarrow$(2) is immediate from Prop. 2.5. For (2)$\Rightarrow$(1), assume (1) fails, so $L_u$ has a facet $F$ with primitive integer normal $v / \in D$. Then we can violate (2) by choosing $p, p'$ in the UDRs adjacent to $F$ and positioned close to $F$, and letting $t = p' - p$.

Baldwin and Klemperer (2014 and in preparation-b) give a full discussion of non-generic price changes (which, e.g., pass through a lower-dimensional cell than a facet) and also give other equivalent characterisations of demand types, but Prop. 3.2 will suffice for our purposes.

### 3.2 Substitutes, Complements, and other “Demand Types”

It follows straightforwardly that demand types provide simple characterisations of familiar concepts such as ordinary substitutes, ordinary complements, and “strong substitutes”. These characterisations are easier to generalise than standard ones based on imposing restrictions on $u$ directly. Moreover, they more clearly reveal and explain features such as the lack of symmetry between substitutes and complements. We begin by recalling standard definitions:

**Definition 3.3** (Standard).

1. A valuation $u$ is ordinary substitutes if, for any UDR prices $p' \geq p$ with $D_u(p) = \{x\}$ and $D_u(p') = \{x'\}$, we have $x'_k \geq x_k$ for all $k$ such that $p_k = p'_k$.\footnote{We write, as is standard, $p' \geq p$ when the inequality holds component-wise.}
(2) A valuation \( u \) is ordinary complements if, for any UDR prices \( p' \geq p \) with \( D_u(p) = \{x\} \) and \( D_u(p') = \{x'\} \), we have \( x'_k \leq x_k \) for all \( k \) such that \( p_k = p'_k \).

(3) A valuation \( u \) is strong substitutes if, when we consider every unit of every good to be a separate good, it is a valuation for ordinary substitutes.\(^{21}\)

It is easy to use Prop. 3.2 to provide alternative, equivalent, definitions of these concepts as demand types. For substitutes:

**Definition 3.4.** The (\( n \)-dimensional) ordinary substitutes vectors are the set of non-zero primitive integer vectors \( v \in \mathbb{Z}^n \) with at most one positive coordinate entry, and at most one negative coordinate entry. They define the ordinary substitutes demand type (for \( n \) goods).

**Proposition 3.5.** A valuation is an ordinary substitutes valuation iff it is of the ordinary substitutes demand type.

So a vector that is normal to a facet of a substitutes LIP cannot have two non-zero entries of the same sign. (See, for example, Figs. 1 and 3b-3c, for this property holding, and Fig. 4 for it failing.) To understand the necessity of this, see Fig. 4, which depicts a facet whose primary integer normal vector’s first and third coordinates have the same sign. Increasing the price on either good 1 (as pictured) or good 3, can therefore take us across the facet – decreasing demand for both goods 1 and 3. So any such facet generates complementarities at some prices, and so cannot be part of a substitutes LIP.

To prove that being of the ordinary substitutes demand type is sufficient for a valuation to be ordinary substitutes, we apply Prop. 3.2 as we cross the finite number of facets between prices \( p, t \). The fact that Prop. 3.2 refers only to generic \( p, t \) does not matter, because Defn. 3.3(1) only requires us to examine UDR prices, and UDRs are open and dense in \( \mathbb{R}^n \). So we can always pick “close-by” prices \( \tilde{p} \) and \( \tilde{p}' \) so that Prop. 3.2 does apply for this starting price and direction of price change, and so that demand is the same as at \( p \) and \( p' \), respectively.) Now recall the standard result that \( (x'' - x) \cdot (p'' - p) < 0 \), where \( x'', x \) are the bundles demanded at any prices \( p'', p \), respectively (see e.g. Mas-Colell et al., 1995, Prop. 2.F.1). So, as we cross each facet, demand is strictly reduced for some good whose price has strictly increased (since the remaining prices are constant). Since this good corresponds to a negative entry in the facet’s normal vector, and since there is at most one such entry in an ordinary substitutes vector, demand weakly increases at each facet crossing for all goods whose price does not change, so Defn. 3.3(1) is satisfied.

We call “ordinary substitutes” what most others (e.g., Ausubel and Milgrom, 2002, Hatfield and Milgrom, 2005) simply call “substitutes”. We do this for clarity, since some have defined “substitutes” in other ways. In particular, although Kello and Crawford’s (1982) definition is equivalent in their model, it is not generally equivalent if it is extended to multiple units of three or more goods (which yields Milgrom and Strulovici’s, 2009, definition of “weak substitutes”); see Danilov et al., 2003, Ex. 6 and Thm 1. Our definition (3.3(1)) seems the most natural one in the general case. It is also equivalent to several properties that seem to naturally characterise “substitutes”, and to the indirect utility function (\( \max_{x \in A} \{ u(x) - p \cdot x \} \)) being submodular—see Baldwin, Klemperer and Milgrom (in preparation). See also Baldwin and Klemperer (2014). Hatfield et al. (2013)—see our Section 6.1—and Danilov et al. (2003) use definitions equivalent to 3.3(1), and the latter authors make a similar observation to our Prop. 3.5 when they say “each cell of a valuation’s parquet is a polymatroid”.

\(^{21}\)This is equivalent to Milgrom and Strulovici’s (2009) definition—see Danilov et al (2003, Cor. 5). There are many other equivalent definitions (see Shioura and Tamura, 2015), the most important being \( M^2 \)-concavity of the valuation (Murota and Shioura, 1999).
Figure 4: A facet (shaded) defined by \( \{ \mathbf{p} \in \mathbb{R}^3 : p_1 + p_3 = p_2; p_1, p_2, p_3 \geq 0 \} \), with its normal \((1,-1,1)\) (the arrow shown in bold). Increasing either \(p_1\) (as shown with a dotted arrow), or \(p_3\), demonstrates complementarities between goods 1 and 3, as the bundle demanded switches from \((1,0,1)\) to \((0,1,0)\).

For complements, a price change that reduces demand for a good can of course reduce (but not increase) demand for other goods. So, applying Prop. 3.2 in the same way as for Prop. 3.5, we define and then prove:

**Definition 3.6.** The \((n\text{-dimensional})\) **ordinary complements vectors** are the set of non-zero primitive integer vectors \(\mathbf{v} \in \mathbb{Z}^n\) whose non-zero coordinate entries are all of the same sign. They define the **ordinary complements demand type** (for \(n\) goods).

**Proposition 3.7.** A valuation is an ordinary complements valuation iff it is of the ordinary complements demand type.

The lack of symmetry between substitutes and complements, and the reason for it are now clear: ordinary complements vectors may have any number of non-zero entries (of the same sign), but any pair of non-zero entries in an ordinary substitutes vector must be of opposite signs (recall Fig. 4), so ordinary substitutes vectors can have at most two non-zero entries (see Ex. A.4 for more discussion).

The characterisation of strong substitutes as a demand type also gives an intuitive description of them:

**Definition 3.8.** The **strong substitute vectors** are those non-zero \(\mathbf{v} \in \mathbb{Z}^n\) which have at most one \(+1\) entry, at most one \(-1\) entry, and no other non-zero entries. They define the **strong substitutes demand type**.

**Proposition 3.9** (See Baldwin and Klemperer, 2014, Cor. 5.20; and Shioura and Tamura, 2015, Thm. 4.1(i)). A valuation is strong substitutes iff it is concave and is of the strong substitutes demand type.

So Figs. 1, 5a, and 9a-c show examples of LIPs of strong substitutes valuations. Note that this characterisation is also parsimonious; with \(n\) goods, the strong substitutes...
demand type is defined by just \( n(n + 1)/2 \) vectors (and their negations).\(^{22}\)

We will see in Sections 6.2 and 6.3 that demand types also allow us to characterise significant new classes of valuations.

In Section 4 we show that analysing the properties of demand types helps us understand when competitive equilibrium exists.

### 3.3 Aggregate Demand, and the “Demand Type” of the Aggregate of Multiple Agents

An important feature of our “demand types” classification—that, in particular, greatly facilitates the study of equilibrium—is that the demand type when we aggregate valuations from multiple agents is just the union of the sets of vectors that form the individual agents’ demand types.

So we now consider a finite set of agents, \( j = 1, \ldots, m \): agent \( j \) has valuation \( u^j \) for integer bundles in a finite set, \( A^j \). Their aggregate demand is, of course, the (Minkowski) sum of the individual demands, but to apply our techniques to this, we want to treat it as the demand of a single “aggregate” agent.

**Definition 3.10.** An aggregate valuation of \( u^1, \ldots, u^m \) is a valuation \( U \) with domain \( A := A^1 + \cdots + A^m \) such that \( D_U(p) = D_{u^1}(p) + \cdots + D_{u^m}(p) \forall p \in \mathbb{R}^n \).

The rest of this subsection proves and discusses:

**Lemma 3.11.** If \( u^1, \ldots, u^m \) are valuations:

1. an aggregate valuation \( U \) exists;
2. \( \mathcal{L}_U = \mathcal{L}_{u^1} \cup \cdots \cup \mathcal{L}_{u^m} \);
3. If \( F \) is a facet of \( \mathcal{L}_U \), then \( w_U(F) = \sum_{F \in \mathcal{F}} w_u(F^j) \), in which \( \mathcal{F} \) is the set of all facets of the individual \( \mathcal{L}_{u_j} \) which contain \( F \).

**Corollary 3.12.** A collection of individual valuations are all of demand type \( \mathcal{D} \) iff every aggregate valuation of every finite subset of them is of demand type \( \mathcal{D} \).

For example, Figs. 5a-b show the LIPs of Elizabeth’s and Paul’s valuations for the hotel rooms of our introductory example. Both valuations have domain \( \{0, 1\}^2 \). Elizabeth regards the rooms as substitutes; her valuation is \( u^s(x_1, x_2) = \max\{40r_1, 30x_2\} \) (Fig. 5a). Paul regards them as complements; his valuation is \( u^c(x_1, x_2) = \min\{50r_1, 50x_2\} \) (Fig. 5b).

It is easy to see that an aggregate demand set consists of a unique bundle iff all the individual demand sets do (and so to prove Lemma 3.11(2)).\(^{23}\) Thus Fig. 5c shows the LIP of the aggregate valuation of \( u^s \) and \( u^c \). It is obvious that any demand type contains the individual valuations iff it contains the aggregate valuation (Cor. 3.12).

From the aggregate LIP we can obtain a polyhedral complex, \( \Pi \), in the usual way (Prop. 2.7 and Lemma 2.11). Its cells are either cells, or subsets of cells, of the individual

\(^{22}\)Our description is closely related to the “step-wise gross substitutes” of Danilov et al. (2003), which they link to the edge vectors of (what we call) the demand complex; these edge vectors are also linked to \( M^\flat \)-concavity by Murota and Tamura (2003).

\(^{23}\)This has been previously observed in pictures of demand correspondences, see for example Murota, 2003, Section 11.2.
LIPs: cells of individual LIPs that intersect in their interiors are split up into new, smaller cells in the aggregate LIP. Thus the price (30, 20) in Fig. 5c is a 0-cell, on the boundary of four distinct 1-cells.

The change in aggregate demand between any pair of prices is just the sum of the changes of the individual demands. So the weight of any facet $F$ of the aggregate LIP is just the sum of the weights of all the facets $F'$ of the individual LIPs for which $F \subseteq F'$ (which proves Lemma 3.11(3)). And since the weighted polyhedral complex $(\Pi, w)$ is derived from balanced complexes, it is itself balanced, and so (using Thm. 2.10) it is the LIP of some valuation (so Lemma 3.11(1) holds).

It is well-known that $\max \left\{ \sum_{j} u^j(x^j) : x^j \in A^j, \sum_{j=1}^{m} x^j = y \right\}$ gives an aggregate valuation. However, we never use this formula for $U(y)$ as it is easier and more intuitive to work directly from Defn. 3.10.

Note, however, as explained above, we can construct the aggregate LIP from the individual LIPs without knowing the form of $U$; the fact that we can avoid using the cumbersome formula for $U$ is an important advantage of aggregation in price space.

Notice also that we cannot find the demand complex of an aggregate valuation using only the individual demand complexes--because a demand complex does not correspond to a unique valuation, and different valuations may aggregate in different ways.

For example, the demand complexes corresponding to the LIPs of Figs. 5a-b are shown in Figs. 6a-b. The demand complex corresponding to their aggregate LIP (Fig. 5c) is shown in Fig. 6c; its domain is $\{0, 1\}^2 + \{0, 1\}^2 = \{0, 1, 2\}^2$. If Paul’s valuation increases to $u^c(x_1, x_2) = \min\{100x_1, 100x_2\}$, then his demand complex remains that of Fig. 5a. However, the LIP, $L_{U^c}$, of the aggregation of $u^s$ with $u^c$, is shown in Fig. 5d, and its demand complex is that of Fig. 6d. So there is no unique aggregate demand complex corresponding to the demand complexes of Fig. 6a and Fig. 6b.

This is a standard implication of quasi-linear preferences. Mathematically, it says that the aggregate valuation is the tropical product of the tropical polynomials that are the individual valuations. Economically, it says that the aggregate value of a bundle is the maximum sum of agents’ values that can be obtained by apportioning the bundle among the agents.
4 The Unimodularity Theorem—when does Equilibrium always exist for a “Demand Type”?

This section shows that our “demand types” classification yields a powerful theorem about when competitive equilibrium is and is not guaranteed.

This theorem requires much weaker assumptions about agents’ preferences than used in the existing leading economics literature (though we retain the standard assumption of quasilinear preferences). So our condition for equilibrium is correspondingly much more general. It immediately generalises, for example, equilibrium results in Kelso and Crawford (1982), Gul and Stacchetti (1999), Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014). In particular it is not necessary for all agents to have substitutes valuations (or some basis change thereof) for equilibrium to always exist; complements valuations guaranteeing equilibrium are easy to find.

We state the Unimodularity Theorem and some immediate corollaries in the next subsection; give intuition and the proof in Section 4.2; and explain the close connections with Danilov et al. (2001) in Section 4.3.

A variety of applications of the theorem are in Section 6.

4.1 Statement of Results

We are interested in the standard notion of competitive equilibrium:

**Definition 4.1.** An *equilibrium* exists, for a market supply $x \in \mathbb{Z}^n$ and a finite set of valuations, if $x$ is in the valuations’ aggregate demand set for some price.

It is standard (Lemma 2.13) that concavity of an aggregate valuation $U$ is necessary and sufficient for equilibrium to exist for all integer bundles in the convex hull of the domain of $U$. We therefore refer to these bundles as the *relevant supply bundles*: equilibrium will clearly never exist for other bundles, as they are the wrong “size”.

Figure 6: Demand complexes corresponding to the LIPs shown in Fig. 5, with the relevant grids of integer bundles. (We do not distinguish which bundles are never demanded.)
Concavity of individual valuations is therefore necessary even for all one-agent economies to have equilibrium, so our results will also restrict attention to concave valuations.

With indivisible goods (unlike with divisible goods), individual concavity is not sufficient to guarantee aggregate concavity (and so, for example, supporting hyperplanes do not necessarily exist). However, our geometric approach provides a simple additional condition that is sufficient to guarantee equilibrium. First we define:

**Definition 4.2.** A set of vectors in $\mathbb{Z}^n$ is **unimodular** if every linearly independent subset can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant $\pm 1$.

By “the determinant” of $n$ vectors we mean the determinant of the $n \times n$ matrix which has them as its columns. If the set of vectors spans $\mathbb{R}^n$, then there exist sets of $n$ of them that are linearly independent; it is therefore, of course, sufficient to check that all $n$-element sets have determinant $\pm 1$ or 0. Alternative equivalent conditions for unimodularity are given by Facts 4.8 and A.7, so unimodularity of a demand type’s vectors is not too hard to check. We refer to “unimodular demand types” in the obvious way.

We can now state the key result:

**Theorem 4.3 (The Unimodularity Theorem).** An equilibrium exists for every pair of concave valuations of demand type $\mathcal{D}$, for all relevant supply bundles, iff $\mathcal{D}$ is unimodular.

We will prove Thm. 4.3 as the combination of Props. 4.10 and 4.17, below.

It follows that if the demand type is unimodular, then a valuation obtained by aggregating any two concave valuations is also a concave valuation, so we can apply the result repeatedly:

**Corollary 4.4.** An equilibrium exists for every finite set of concave valuations of demand type $\mathcal{D}$, for all relevant supply bundles, iff $\mathcal{D}$ is unimodular.

It is also immediate from the discussion above that:

**Corollary 4.5.** With $n$ goods, if the vectors of $\mathcal{D}$ span $\mathbb{R}^n$, then an equilibrium exists for every finite set of concave valuations of demand type $\mathcal{D}$, for all relevant supply bundles, iff every subset of $n$ vectors from $\mathcal{D}$ has determinant 0 or $\pm 1$.

Many standard results are immediate special cases. For example, it is well-known that “strong substitute” vectors form a unimodular set (Poincaré, 1900), and these valuations are (by definition) concave, so:

**Proposition 4.6** (Danilov et al., 2001, 2003, and Milgrom and Strulovici, 2009). An equilibrium exists for every finite set of strong substitutes valuations, for all relevant supply bundles.

Other familiar equilibrium results, such as those of Kelso and Crawford (1982) and Gul and Staccchetti (1999), of course also immediately follow.

Section 6 gives many other applications.

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25 We ignore the order of the vectors since we are only ever interested in the absolute values of determinants.
4.2 Intuition and Proof for the Unimodularity Theorem

4.2.1 The Role of Intersections

The first critical observation is that we can determine whether equilibrium exists by focusing only on intersections of individual LIPs: we know equilibrium always exists, that is, every relevant bundle is demanded at some price, iff the aggregate valuation is concave iff the aggregate demand set is discrete-convex at every price (Lemma 2.13). But if all but one of the agents have unique demand at some price, the aggregate demand set is simply the shift of the remaining agent’s demand set by the other agents’ (unique) demands. And this set must be discrete-convex, since we assumed that every individual valuation is concave. So we only need to check prices at which two or more agents have non-unique demand. This proves:

Lemma 4.7. For concave valuations \( u^s, \ldots, u^s \), an equilibrium exists for every relevant supply bundle \( \mathcal{L}_{u^j} \cap \mathcal{L}_{u^{j'}} \) for \( j, j' = 1, \ldots, s, j \neq j' \).

So, for example, for the case of our “hotel rooms” example in the introduction which had simple two-goods substitutes and complements valuations \( u^s(x_1, x_2) = \max\{40x_1, 30x_2\} \) (Fig. 5a) and \( u^c(x_1, x_2) = \min\{50x_1, 50x_2\} \) (Fig. 5b), whose aggregate LIP is shown in Fig. 5c, the only price we need to analyse is the intersection \((30, 20)\).

4.2.2 Unimodularity

Our “hotel rooms” example also illustrates the role of unimodularity.

The aggregate demand of \( u^s \) and \( u^c \) at the (only) price we need to check, the intersection price \((30, 20)\), is the sum of the individual demands, \( D_{u^s}(30, 20) = \{(1, 0), (0, 1)\} \) and \( D_{u^c}(30, 20) = \{(0, 0), (1, 1)\} \). So \( D_U(30, 20) \) consists of the bundles at the corners of a square, \{\((1, 0), (0, 1), (2, 1), (1, 2)\)\} (the diamond at the centre of Fig. 6c). However, this square also contains the non-vertex integer bundle \((1, 1)\), and since \((1, 1)\) is not in the demand set at this price, it is not demanded at any price (by Lemma 2.20). So equilibrium fails; although both \( u^s \) and \( u^c \) are concave, their aggregate valuation is not. (Observe that we did not need to actually aggregate the valuations \( u^s \) and \( u^c \) to determine this.)

To see the relevance of unimodularity to the example and also, we will see, more generally, note two equivalent conditions to unimodularity:

Fact 4.8. A set of vectors in \( \mathbb{Z}^n \) is unimodular iff the following equivalent conditions hold for every linearly independent subset \( \{v^1, \ldots, v^s\} \) of this set:

1. the parallelepiped whose edges are these vectors, that is, \( \{ \sum_{j=1}^s \lambda^j v^j : \lambda^j \in [0, 1] \} \), contains no non-vertex integer point;
2. if \( x \in \mathbb{Z}^n \) and \( x = \sum_{j=1}^s \alpha^j v^j \) with \( \alpha^j \in \mathbb{R} \), then \( \alpha^j \in \mathbb{Z} \) for all \( j \).

Fact 4.8(1) says that a parallelepiped (that is, an \( s \)-dimensional parallelogram) whose vertices are integer points contains a non-vertex integer point iff the vectors along its edges do not form a unimodular set. Now recall two geometric facts for the case \( s = n \): that the volume of a parallelepiped is given by the (absolute value of the) determinant of its edge vectors; and that a parallelepiped whose vertices are integer points contains
a non-vertex integer point if its volume strictly exceeds 1. In our example, the edges of
the demand complex cell (the diamond in Fig. 6c) are in directions (1,1) and (−1,1).
Their determinant is 2, so the area of the cell is 2, and it therefore contains a non-vertex
integer point at which equilibrium may fail (as Thm. 4.3 warned, since the determinant
of 2 directly reveals a failure of unimodularity).

So equivalent condition 4.8(1) will help to demonstrate the necessity of Thm. 4.3’s
condition for equilibrium.26

A second way to see the relevance of unimodularity is given by Fact 4.8(2), which
tells us that any vector in the space spanned by a given set of vectors can be created as
an integer combination of the set iff the set is unimodular. The four corners of the square
are vertices of the aggregate demand complex cell, and therefore correspond to UDRs
that each contain the price (30, 20) in their boundary (Prop. 2.21). If we move between
these UDRs, around the price (30, 20), aggregate demand changes by the vector normal
to any facet crossed (Prop. 2.5), that is, by the vector in the direction of one of the
square’s edges. Moreover, at (30, 20), the only possible changes in individual demand,
and hence the only possible changes in aggregate demand, are made up of these vectors.
So the impossibility of demanding (1,1) on aggregate at this price vector, and so (by
Lemma 2.20) anywhere, is equivalent to the impossibility of obtaining (1,1) by starting
at any of the four bundles at the corners of the square ((1,0), (0,1), (2,1), and (1,2)),
and adding integer combinations of the edge vectors (1,1) and (−1,1).

More generally if, by contrast, every demand complex cell’s edge vectors were a
unimodular set, this problem could not arise. So equivalent condition 4.8(2) will be
useful for demonstrating the sufficiency of Thm. 4.3’s condition for equilibrium.

4.2.3 Proof of Necessity Condition for the Unimodularity Theorem

It is now easy to prove the necessity of the condition of our Thm. 4.3 by replicating
the situation of the simple example discussed in Sections 4.2.1-4.2.2 above.

Let agent \( j \) have a concave valuation, and be indifferent between (only) the bundles
\( x^j, x^j + v^j \), at some price \( p \), for \( j = 1, \ldots, s \leq n \). (This requires that \( v^j \) are primitive
integer vectors). So the aggregate demand set of agents \( j = 1, \ldots, s \), at \( p \) is \( D_U(p) = \left\{ \sum_{j=1}^{s} (x^j + \delta_j v^j) : \delta_j \in \{0,1\}; j = 1, \ldots, s \right\} \). If the \( v^j \) are linearly independent, then
this set is precisely the vertices of an \( s \)-dimensional parallelepiped in \( \mathbb{Z}^n \) with edges \( v^j \).
By Fact 4.8(1) there exists a non-vertex integer bundle in this parallelepiped iff the set
\( \{v^1, \ldots, v^s\} \) is not unimodular. So, applying Lemma 2.20:

Lemma 4.9. Under the hypotheses of the paragraph above, the set \( \{v^1, \ldots, v^s\} \) is uni-
modal iff every integer bundle in \( \text{conv}(D_U(p)) \) is demanded for some price.

For Thm. 4.3, we need to show a failure of equilibrium with only two agents. So for
any non-unimodular linearly independent set of primitive integer vectors \( v^1, \ldots, v^s \in D \),
find the \( k \) such that the set \( v^1, \ldots, v^k \) is unimodular, but the set \( v^1, \ldots, v^{k+1} \) is not.27

26When the set of vectors is not unimodular, the number of non-vertex bundles in such a paral-
leelepiped is one less than the determinant (Fact 5.15). So we expect these determinants should give
bounds on the extent to which supply constraints need to be relaxed to achieve equilibrium. (More-
over, the extension of our results to matching theory—see Section 6.5—might then yield results related
to Nguyen and Vohra, 2014 and Nguyen et al., 2015.)

27Clearly \( 1 \leq k \leq s - 1 \); a single primitive integer vector is a unimodular set, by Fact 4.8(2).
Let agents \( j = 1, \ldots, s \) be as above and specify that \( x^j, x^j + v^j \) are the only bundles in the domain of their valuation. Construct an agent \( k^* \)’s valuation by aggregating the valuations of agents \( 1, \ldots, k \). Since these valuations meet only at \( p \), equilibrium could possibly fail for agent \( k^* \) only at \( p \) (see Lemma 4.7). But by Lemma 4.9, this does not happen, since the set \( \{ v^1, \ldots, v^k \} \) is unimodular, so agent \( k^* \)’s valuation is concave (by Lemma 2.13). And the aggregate demand of the two agents \( k^* \) and \( k + 1 \) is obviously the aggregate demand of agents \( 1, \ldots, k + 1 \) for which equilibrium clearly fails, so we have proved:

**Proposition 4.10.** If \( \mathcal{D} \) is not unimodular, then there exists a pair of concave valuations of demand type \( \mathcal{D} \) and a relevant supply bundle for which equilibrium fails.

### 4.2.4 Transverse Intersections

If LIPs only had intersections of the simple form discussed above, a similar argument would demonstrate the sufficiency part of Thm. 4.3. For the general case, we need to consider more complex intersections, but we now show we need only directly prove existence of equilibrium for generic intersections, more precisely, “transverse” LIP intersections:

**Definition 4.11** (see e.g. Maclagan and Sturmfels, 2015, Defn. 3.4.9),

1. The intersection of \( L_{u^1} \) and \( L_{u^2} \) is transverse at \( p \) if \( \dim(C^1 + C^2) = n \), in which \( C^j \) is the minimal cell of \( L_{u^j} \) containing \( p \), for \( j = 1, 2 \).
2. The intersection of \( L_{u^1} \) and \( L_{u^2} \) is transverse if they intersect transversally at every point of their intersection.
3. The intersection of \( L_{u^1}, \ldots, L_{u^k} \) is transverse at \( p \) if \( L_{u^j+1} \) intersects the aggregate LIP of the valuations \( u^1, \ldots, u^j \) transversally at \( p \), for all \( j = 1, \ldots, k - 1 \).

For example, in two dimensions, two lines crossing at a single point are intersecting transversally. So the intersections in Figs. 5c and 5d are both transverse. However, two coincident lines do not intersect transversally, and nor does a line crossing through a 0-cell intersect transversally. So the grey LIP and the black dotted LIP of Fig. 7a, which intersect at \((4, 1)\) and along the line from \((4, 3)\) to \((5, 4)\), do not intersect transversally

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\footnote{This definition is independent of the order in which the LIPs are taken; see e.g. Lemma A.5(2).}
at any price. (For each of the three prices (4, 1), (4, 3) and (5, 4), the minimal cell of the grey LIP containing the price is the 0-cell at the price itself.) In three dimensions, an intersection is transverse at all of the prices where a 1-cell meets a facet in a single point, or two facets meet along a line, or three facets meet in a single point.

We will make use of two important features of transverse intersections.

First, as is intuitive, the intersections of ‘tropical hypersurfaces’, and therefore of LIPs, are generically transverse:

**Proposition 4.12** (Maclagan and Sturmfels, 2015, Prop. 3.6.12). For any \( L_{u_1} \) and \( L_{u_2} \), and generic \( v \in \mathbb{R}^n \), the intersection of \( L_{u_1} \) and \( L_{u_2} + \{ \epsilon v \} \) is transverse for all sufficiently small \( \epsilon > 0 \).

For example, a small translation of the black dotted LIP of Fig. 7a by \( \{ \epsilon(1, 0) \} \) yields the transverse intersection shown in Fig. 7b, which consists of the points \((4, 1 + \epsilon), (4 + \epsilon, 1), (4, 3 - \epsilon)\) and \((5 + \epsilon, 4)\).

The significance of this is that modifying a valuation from \( u(x) \) to \( u(x) + \epsilon v \cdot x \) translates its LIP from \( L_u \) to \( L_u + \{ \epsilon v \} \), since any bundle demanded at \( p \) under valuation \( u(x) \) is demanded at \( p + \epsilon v \) under valuation \( u(x) + \epsilon v \cdot x \).

Furthermore, failure of equilibrium is preserved by a sufficiently small modification of a valuation:

**Proposition 4.13.** If equilibrium does not exist for valuations \( u^1 \) and \( u^2 \), and some relevant supply bundle, then for any \( v \in \mathbb{R}^n \), equilibrium also fails for valuations \( u^1 \) and \( u^2 + \epsilon v \), in which \( u^2(x) = u^2(x) + \epsilon v \cdot x \), for the same supply and all sufficiently small \( \epsilon > 0 \).

To prove Prop. 4.13, consider an allocation of the supply bundle between agents with valuations \( u^1 \) and \( u^2 \). Failure of equilibrium means, by definition, that the two sets of prices at which these agents do demand their respective allocations, must be disjoint. But if the set of prices at which an agent demands a bundle is non-empty, it is either the closure of a UDR or a cell of the LIP, and so is a polyhedron (see Cor. 2.22). And if two polyhedra are disjoint, then there is a positive minimum Euclidean distance between a point in one and a point in the other (even if the polyhedra are not bounded, see, e.g., Gruber, 2007, p59). Let \( \delta \) be the minimum such distance, over the pairs of polyhedra corresponding to the finite set of all possible allocations between the agents.

Now consider \( u^2(x) = u^2(x) + \epsilon v \cdot x \). Since \( D_{u^2}(p + \epsilon v) = D_{u^2}(p) \), the polyhedron of prices in which an agent with valuation \( u^2 \) demands a bundle is shifted by \( \{ \epsilon v \} \), compared with the corresponding prices for \( u^2 \). Suppose \( \| \epsilon v \| < \delta \). Then however we allocate the supply between agents with valuations \( u^1 \) and \( u^2 \), the sets of prices at which these agents demand their allocations are still disjoint. So equilibrium will still fail.

Combining Props. 4.12 and 4.13, we see that we need only prove sufficiency in Thm. 4.3 for transverse intersections:

**Corollary 4.14.** If equilibrium does not exist for a pair of concave valuations of demand type \( \mathcal{D} \) and some relevant supply bundle, then it does not exist for a pair of concave valuations of demand type \( \mathcal{D} \) whose LIP intersection is transverse and some relevant supply bundle.
The second important fact about transverse intersections is that the changes in the bundles considered by agents at any prices in these intersections are in fundamentally different directions, in the sense that:

**Definition 4.15.** The linear span of changes in demand, $K_\sigma$, associated to a demand complex cell $\sigma$, is the set of linear combinations of vectors in $\{y - x : x, y \in \sigma\}$.

**Lemma 4.16.** Suppose $L_{u1}$ and $L_{u2}$ intersect at $p$, and the two agents’ individual demand complex cells at this price are $\sigma^1$ and $\sigma^2$, while the aggregate demand complex cell is $\sigma$. Then the intersection is transverse at $p$ iff every vector in $K_\sigma$ can be written uniquely as a sum of a vector in $K_{\sigma^1}$ and a vector in $K_{\sigma^2}$.

In our example of Sections 4.2.1-4.2.2, at the transverse intersection at price $(30, 20)$, we have $K_\sigma = \mathbb{R}^2$, while $K_{\sigma^1} = \{\lambda(1, -1) : \lambda \in \mathbb{R}\}$ and $K_{\sigma^2} = \{\lambda(1, 1) : \lambda \in \mathbb{R}\}$. And any vector in $\mathbb{R}^2$ can be uniquely written as $\lambda_1(1, -1) + \lambda_2(1, 1)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

The next subsection will use this second fact to complete the proof of the Unimodularity Theorem.

### 4.2.5 Proof of Sufficiency Condition for the Unimodularity Theorem

Cor. 4.14 shows that we need only prove sufficiency of the condition of Thm. 4.3 for concave valuations whose LIP intersection is transverse.

So let $u^1$ and $u^2$ be such valuations of a unimodular demand type. By Lemma 4.7 it suffices to show that the demand set is discrete convex at any price $p$ in their intersection. Write $\sigma^1$ and $\sigma^2$ respectively for the individual demand complex cells at $p$, and let $\sigma$ be the aggregate demand complex cell at $p$. We want to show that any integer supply $y \in \sigma = \text{conv} (D_U(p))$ is demanded, that is, also satisfies $y \in D_U(p)$.

To do this, consider any vertex, $x$, of $\sigma$. By Defn. 4.15, the change in demand, $y - x$, is in $K_\sigma$. Fix a basis for $K_\sigma$, composed of edge vectors of $\sigma$.29 These vectors are, equivalently, the normals to the facets of $L_U$ which contain $p$ (see Prop. 2.21). So this set is unimodular and, by Fact 4.8(2), $y - x$ can therefore be written as an integer combination of them. Furthermore, since $L_U = L_{u1} \cup L_{u2}$, each of these facet normals is normal to a facet of $L_{u1}$ or of $L_{u2}$. So we can separate the basis vectors into two sets, correspondingly. Then our presentation of $y - x$, in terms of this basis, splits as $y - x = z^1 + z^2$, in which $z^j \in K_{\sigma^j}$, for $j = 1, 2$, and the $z^j$ are integer bundles.

On the other hand, since $\sigma = \sigma^1 + \sigma^2$ (see Rockafellar, 1970, Thm. 3.1) there exist $y^j \in \sigma^j$, $j = 1, 2$, such that $y = y^1 + y^2$. And since $x$ is a vertex of $\sigma$, it is demanded in a UDR adjacent to $p$, so $x \in D_U(p)$, i.e. $x = x^1 + x^2$ in which $x^j \in D_{u^j}(p)$ (and so are integer bundles). So $y - x = (y^1 - x^1) + (y^2 - x^2)$, and we also have $y^j - x^j \in K_{\sigma^j}$, $j = 1, 2$.

So, by transversality (Lemma 4.16), $y^j - x^j = z^j$. And, since we already showed $x^j$ and $z^j$ are integer, it follows that $y^j$ are also integer, for $j = 1, 2$.

So we have $y^j \in \sigma^j = \text{conv} (D_{u^j}(p))$ and $y^j \in \mathbb{Z}^n$. But since $u^j$ is concave, its demand sets are all discrete convex (Lemma 2.13) and so $y^j \in D_{u^j}(p)$, for $j = 1, 2$.

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29That such a basis exists follows from, e.g., combining Gruber (2007) Thms 14.2 and 15.8.
Therefore, since $y = y^1 + y^2$, we can conclude $y \in D_U(p)$, as required. So we have proved:\textsuperscript{30}

**Proposition 4.17.** If $D$ is unimodular, then an equilibrium exists for every pair of concave valuations which are of demand type $D$, for all relevant supply bundles.

### 4.3 Related Work

Danilov et al. (2001) have developed results that are very closely related to our Thm. 4.3. In particular, their Thms. 3 and 4 together provide a sufficient condition for equilibrium, which is analogous to our condition on demand types.\textsuperscript{31} However, the interpretation or usefulness of their condition is not made clear. By contrast, our Thm. 4.3 both demonstrates the applicability of the result, and clarifies the connections to existing economic results. We will see in Section 6 that our Thm. 4.3 generalises many results in work subsequent to Danilov et al.’s, including results in Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014).\textsuperscript{32}

Danilov et al. also prove no necessity result. Because they have no concept of “demand types”, and have not developed their definition as a taxonomy of demand, there is no natural result for them to give. But once our concept of demand types is introduced, a necessity result can easily be developed from their work (using Lemma 2.20).

However, our methods seem simpler and more accessible to economists than Danilov et al.’s extremely advanced mathematics. So we prefer the proof we have given above (which we developed before we knew of their work). Tran and Yu (2015) provide another proof, via integer programming, in their recent exposition of our work.

Danilov et al. also state their results under different assumptions from ours. They assume the domain, $A$, of every agent’s valuation is contained in $\mathbb{Z}^n_{\geq 0}$, which precludes, for example, the application to agents who both buy and sell which our more general assumption permits. For example, our model, unlike theirs, applies to (and extends) Hatfield et al. (2013)–see Section 6.1.

Finally, our techniques lead to an additional set of results about when equilibrium exists for specific valuations; the next section turns to these.

\textsuperscript{30}Ex. A.6 in Appendix A.3 provides additional intuition for the sufficiency condition by illustrating the failure of this argument for the simple two-goods substitutes and complements example discussed in Section 4.2.2.

\textsuperscript{31}Their Thm. 3 shows that equilibrium is guaranteed if the valuations are “$\mathcal{D}$-concave” for some “class of discrete convexity” $\mathcal{D}$. (This notation $\mathcal{D}$ is not connected with our use of $D$ to represent demand types.) “$\mathcal{D}$-concave” valuations are concave valuations such that every demand set $D_u(p)$ belongs to a specified set “$\mathcal{D}$” of subsets of $\mathbb{Z}^n$. A collection of such sets is a “class of discrete convexity” if every set it contains, and every sum and every difference of these sets, is discrete convex.

Their Thm. 4 (which is proved by Danilov and Koshevoy, 2004, Thm. 2) is that $\mathcal{D}$ is a class of discrete convexity if the edges of the convex hulls of the sets in $\mathcal{D}$ form a unimodular set of vectors. But this is true if, in our language, all the valuations are of the demand type defined by this unimodular set. So the sufficiency part of our Thm. 4.3 follows from combining their two results.

\textsuperscript{32}Danilov et. al.’s lack of our notion of demand types or of any economic interpretation of their “$\mathcal{D}$-concavity”, and the presentation of their work in relatively unfamiliar terms (namely the relationships between sets of primitive integer vectors which are parallel to edges of specific collections of integral pointed polyhedra and their “classes of discrete convexity”) seem to have resulted in leading economists being unaware of their work or of its implications. (We were also unaware of their work until after we had developed our own results.)
5 The Intersection Count Theorem—when does Equilibrium exist for Specific Valuations?

The Unimodularity Theorem (Thm. 4.3) tells us exactly which demand types always have a competitive equilibrium. But even when equilibrium is not guaranteed to exist for every concave valuation, it of course exists for many specific valuations. Our Intersection Count Theorem (Thm. 5.12) gives results about which these valuations are.

The key to the Unimodularity Theorem was to understand which demand types permitted “over-large” volumes to potentially arise in the demand complex; thus the key to the Intersection Count Theorem is to understand whether such large volumes in fact arise for agents’ actual valuations.

As in Section 4.2’s development of the Unimodularity Theorem, we need only analyse certain isolated points in the LIP intersection. Furthermore, tropical intersection theory bounds the number of such points; remarkably, a simple count of them often suffices to tell us whether there are “over-large” volumes, and hence demonstrate the existence or failure of equilibrium.

Section 5.1 provides a preview and explanation of the theorem. Although much of the intuition is clear from the two-dimensional case, the analysis for higher dimensions requires additional techniques. So Section 5.2 develops the necessary machinery, and provides a technical theorem—the “Subgroup Indices Theorem” (Thm. 5.16)—which is the backbone of all our results. Section 5.3 can then state, and sketch the proof of, the general Intersection Count Theorem (Thm. 5.12). Section 5.4 explains the limitations of the Theorem, but gives a small extension for transverse cases. Section 5.5 summarises the paper’s results on the existence of equilibrium by providing an algorithm that checks whether equilibrium is guaranteed, for any specific valuations.

5.1 Preview, and Explanation, of the Theorem

5.1.1 The simple “Hotel Rooms” example

Recall the introduction’s “hotel rooms” example. As discussed in Section 4.2.2, equilibrium fails when the supply is (1, 1) for the valuations $u_s$ and $u_c^r$, illustrated in Figs. 5a-b. This is shown by the fact that the demand complex cell corresponding to the only price in the intersection of these LIPs has area 2 (see Fig. 6c) which exceeds 1.

But equilibrium exists for valuations $u_s^c$ and $u_c^r$ which have the same aggregate demand type. The two demand complex cells corresponding to the two prices at which their LIPs intersect both have area 1 (see Fig 6d). It is not a coincidence that the number of the points in the intersection, weighted by the corresponding areas, is constant: this is an implication of the tropical version of the Bézout-Bernstein Theorem.\footnote{As noted in the introduction, Bézout’s (1779) theorem tells us that in two dimensions the number of intersection points of two (ordinary) geometric curves equals the product of their degrees, if we weight the intersection points by appropriate “multiplicities” (e.g., a tangency between a line and a parabola has multiplicity 2). Bernstein (1975) and Kouchnirenko (1976) extended the theorem, including to higher dimensions. LIPs are particular limits of logarithmic transformations of algebraic hypersurfaces (in complex projective space) (see, e.g., Maclagan and Sturmfels, 2015), and similar intersection theorems hold. Moreover, because a LIP is in real (not complex) space, we can “see” the intersection points.}
Specifically, in 2-dimensions, consider any two valuations with individual domains, \( A^1 \) and \( A^2 \), whose LIPs intersect transversally. The tropical Bézout-Bernstein Theorem tells that the number of price points at which the LIPs intersect, weighting each point by the area of the corresponding demand-complex cell is a constant, \( \Gamma^2(A^1, A^2) \).

Fig. 6 illustrates why: the total area of any 2-dimensional demand complex is the sum of the areas of its 2-cells. These 2-cells correspond, of course, to the 0-cells of its LIP. Furthermore, the 0-cells of the aggregate LIP correspond either to the 0-cells of the individual LIPs, or to points in the intersection between these LIPs (which categories are mutually exclusive when the intersection is transverse). But the 0-cells of the individual LIPs correspond in turn to the 2-cells of their individual demand complexes. That is, the triangular 2-cells in Figs. 6c and 6d are exactly the collection of 2-cells in Figs. 6a and 6b. So the total area of these aggregate demand complex 2-cells is the sum of the areas of the individual demand complexes, that is the sum of the areas of \( \text{conv}(A^1) \) and \( \text{conv}(A^2) \). Furthermore, the total area of the aggregate demand complex is the area of \( \text{conv}(A^1 + A^2) \). And, by definition, \( \Gamma^2(A^1, A^2) \) is the total area of all the aggregate demand complex cells that correspond to intersection prices. So \( \Gamma^2(A^1, A^2) \) equals the area of \( \text{conv}(A^1 + A^2) \) minus the sum of the areas of \( \text{conv}(A^1) \) and \( \text{conv}(A^2) \).

Moreover, in two dimensions, a price at which the intersection of two LIPs is transverse corresponds to a parallelepiped in the aggregate demand complex, so must have area at least 1. It follows that the number of points in the intersection is bounded by \( \Gamma^2(A^1, A^2) \). Furthermore, Sections 4.2.2-4.2.3 showed that equilibrium exists at a supply corresponding to such an intersection of facets of weight 1, if and only if the corresponding area is 1 (combine the discussion immediately below Fact 4.8 with Lemma 4.9). So, in two dimensions, if the valuations \( u^1 \) and \( u^2 \) are concave, all facets are weight 1, and the LIP intersection is transverse: (i) there are at most \( \Gamma^2(A^1, A^2) \) points in the LIP intersection, and (ii) equilibrium exists for every possible supply if and only if there are exactly \( \Gamma^2(A^1, A^2) \) points in this intersection.

In the hotel example, the domain of each individual valuation is \( \{0, 1\}^2 \), so the domain of the aggregate is \( \{0, 1, 2\}^2 \), and \( \Gamma^2(\{0, 1\}^2, \{0, 1\}^2) = 4 - 1 - 1 = 2 \). Moreover, the intersection is transverse and the facets are weight 1. So the previous paragraph tells us that there are at most two points in the intersection, and equilibrium exists for every supply if and only if there are exactly two points in the intersection (which results are consistent with the cases analysed above).

Our full Intersection Count Theorem develops these ideas in several directions:

### 5.1.2 Facet Weights

Reconsider the hotel example, but with weight 2 on every facet: let concave valuations \( u^{2s} \), \( u^{2c} \) and \( u^{2e} \), have the same LIPs as \( u^s \), \( u^c \) and \( u^e \), respectively, but with weight 2 on every facet. (So, for example \( u^{2s} \) is equivalent to the aggregate valuation of two identical copies of Elizabeth.) Let \( u^{2s} \) and \( u^{2c} \) aggregate to \( U^2 \), and \( u^{2s} \) and \( u^{2e} \) aggregate to \( U^{2s} \). The demand complexes of \( u^{2s}, u^{2c}, U^2 \) and \( U^{2s} \) are pictured in Fig. 8. As in Section 2.4, bundles are coloured white if they are uniquely demanded for some price, black if they are never demanded, and grey if they are demanded, but never uniquely.

The demand set \( D_{u^{2s}}(30, 20) \) is \( \{(2, 0), (1, 1), (0, 2)\} \): the non-vertex bundle in the diagonal demand complex 1-cell in Fig. 8a is demanded at this price, because the valuation
is concave–indeed it is locally linear. Similarly, $D_{u^{2c}}(30, 20) = \{(0, 0), (1, 1), (2, 2)\}$. So aggregate demand at the intersection price of $L_{u^{2c}}$ and $L_{u^{2c}}$ is $D_{U^{2c}}(30, 20) = \{(2, 0), (1, 1), (0, 2)\} + \{(0, 0), (1, 1), (2, 2)\}$. Thus the vertices of the cell highlighted in Fig. 8c, $(2, 0), (1, 1), (0, 2), (2, 4), (4, 2)$, the bundles on the mid-points of its edges, $(1, 1), (1, 3), (3, 1), (3, 3)$, and its central bundle, $(2, 2)$, are all demanded at $(30, 20)$, while the cell’s remaining four bundles are never demanded.

Observe that this cell is therefore just a grid of $2 \times 2 = 4$ copies of the central cell of Fig. 6c, as shown by the dashed lines in Fig. 8. This corresponds to the fact that the relevant 1-cells of the individual demand complexes have “length” 2 (Defn. 2.16(4)), that is, each of the facets of the individual LIPs have weight 2. The intuition for which bundles of the central cell of Fig. 8c are demanded is exactly as for the central cell of Fig. 6c in Section 4.2.2—the issue is which bundles can be reached from a vertex by an integer combinations of vectors that are (primitive) edge vectors of the cell.

Likewise, at the two prices at which the LIPs $L_{u^{2c}}$ and $L_{u^{2c}}^*$ meet, the corresponding cells of the aggregate demand complex $U^{2c}$ (shaded in Fig. 8d) are each $2 \times 2 = 4$ copies of their equivalent cell in $U^*$ (see Fig. 6d).

Ex. A.8 gives further details on both these cases.

This result is general: if the intersection is transverse, then multiplying any one facet weight multiplies the area of the demand complex cell by the same factor, without affecting the existence or otherwise of equilibrium for concave valuations. So, applying this to the discussion of the previous subsection, in the two-dimensional transverse intersection case: (i) the weighted count of points in the LIP intersection, where each point is weighted by the product of the weights of the facets passing through it is bounded above by $\Gamma^2(\cdot, \cdot)$; and (ii) equilibrium exists for every relevant supply for two concave

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Footnote 34: A distinction is that bundles that are not at the vertices of the aggregate demand complex cells are not uniquely demanded, when they are demanded—that is, they are “grey”, not “white”. This is because concave valuations are locally linear at facets with weights greater than one.
valuations if and only if this bound holds with equality.

It is easy to check for the “weight-2” hotel example of Fig. 8 that $\Gamma^2(A^{2s}, A^{2c}) = 8$, and that each intersection point has weight 4. So there are at most two points in the LIP intersection, and equilibrium is guaranteed if and only if there are exactly 2, just as before.

5.1.3 Non-Transverse Intersections

As in our development of the Unimodularity Theorem in Section 4, we can handle non-transverse intersections by considering the effects of small perturbations.

Return to the weight-one example of Figs. 5a-b, but modify the complements valuation to $u^\#(x_1, x_2) = \min\{70x_1, 70x_2\}$. Then $L_{u^\#}$ intersects $L_{u^s}$ non-transversely, exactly through its 0-cell at (40,30). This case is intermediate between those illustrated in Figs. 5c-d, since the UDRs of the aggregate LIP in which nothing is demanded just meets the two “bottom left” UDRs in which the agents demand three units in total between them. So the aggregate demand complex is like that of Fig. 6c, but without the edge that separates the bundles (1,1) and (0,0); equivalently, the aggregate demand complex is like that of Fig. 6d, but without any of the three edges that include the bundle (1,1).

If we translate $L_{u^\#}$ by $\epsilon(1, 1)$, for small $\epsilon > 0$, we return to the situation of Fig. 5d. So, by Prop. 4.13, equilibrium exists for all supplies: the bundle (1,1) is now “grey”. So our count will give us the “right” result for this non-transverse case if we weight the intersection point by the sum of the weights that apply after this translation, that is, by 2: then the weighted count equals $\Gamma^2(\{0, 1\}^2, \{0, 1\}^2)$, as with the case of Fig. 5d.

But if we had translated in the other direction, and returned to the Fig. 5c case, the sum of the weights would have been only 1, that is “too low”. We therefore choose the translation that yields the maximum possible sum of weights.

Weighting non-transverse intersection points in this way handles situations like this one. As in the transverse case, $\Gamma^2(\cdot, \cdot)$ is an upper bound on the possible count. That is, a weighted count equal to $\Gamma^2(\cdot, \cdot)$ remains sufficient, though, we will see, no longer necessary for equilibrium to exist for all supplies.

5.1.4 Illustration: the case of strong substitutes

An example the power of our approach is that it provides an elegant illustration of Prop. 4.6 that strong substitutes valuations always have equilibrium:

Consider agents $j = 1, 2$ wanting up to $d_j$ units, respectively, in total of two goods. Figs. 9b and 9c show two different cases for the LIPs of generic “strong substitutes” valuations for $d_1 = 1, d_2 = 3$. Observe that the LIP intersection contains $1 \times 3 = 3$ points in both cases. It is not hard to check that this will remain true after any generic translation of either LIP. Moreover, since all “strong substitutes” facet normals for two goods are in $\pm \{(1,0), (0,1), (-1,1)\}$ (Prop. 3.9), all generic “strong substitutes” LIPs have similar “honeycomb” structures, and it is a general result that any intersection has $d_1 d_2$ points. It is also straightforward that the area of $\text{conv}(A^j)$ is $d_j^2/2$ and that of

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35The LIP of any valuation for up to $d$ units in total is the tropical transformation of an “ordinary” polynomial of degree $d$. So Figs. 9a-c show tropical cubics and lines that intersect $3 \times 1 = 3$ times.
\( \text{conv}(A^1 + A^2) \) is \((d_1 + d_2)^2/2\), so \( \Gamma^2(A^1, A^2) = d_1 d_2 \) (see also Fact A.9(3)). So equilibrium exists for all supplies in the generic case.

\[
\begin{align*}
\text{(a) LIP of a strong substitute valuation for up to 3 units.} & \quad \text{\textbf{(b) The LIP from (a), and a LIP of a valuation for 1 unit}} & \quad \text{\textbf{(c) The LIP from (a), and a LIP of another valuation for 1 unit}} \\
\end{align*}
\]

Figure 9: The LIPs of two generic strong substitutes valuations, one for up to 3 units, and one for a single unit, always intersect exactly \(3 \times 1 = 3\) times.

For non-generic cases, it is obvious from Figs. 9b and 9c, that any translation of a non-transverse LIP intersection to create a transverse intersection yields the count \(d_1 d_2\). Furthermore if, e.g., all the facets of the LIP in Fig. 9a had weight \(w\), we would obtain the “correct” count \(wd_1 d_2\). So the argument can be extended to confirm equilibrium always exists in non-generic cases too.

### 5.1.5 Higher Dimensions

Handling \( n > 2 \) dimensions is harder. First, intersections have dimension at least \( n - 2 \), so do not consist of isolated points when \( n > 2 \). However, we will show it is sufficient to focus our analysis on “intersection 0-cells”. Second, even if the intersection is transverse at such prices, the corresponding cell in the demand complex need not be a parallelepiped. Correspondingly, the individual LIP cells at an intersection point need not be facets, so we need to extend our definition of facet “weights” to cover all cells.

As in two dimensions, our results can be thought of in terms of whether or not there are problematic bundles within appropriate parallelepipeds. But we proceed in a similar way to Section 4.2.5. We there used an alternative equivalent definition of unimodularity (Fact 4.8(2)): whether all integer bundles in a demand complex cell can be reached by combinations of appropriate vectors. (We also used this for the second intuition of Section 4.2.2.) Our key tool here will be the “subgroup indices” that are exactly the tool used by mathematicians to generalise the 2-dimensional tropical Bézout-Bernstein theorem to higher dimensions, but are also, as we show, intimately connected with existence of equilibrium (Thm. 5.16). Thus, for all \( n \), there is an upper bound \( \Gamma^n(A^1, A^2) \) (Defn. 5.11) for an appropriately weighted count of intersection 0-cells (in general, the domains \( A^1, A^2 \subseteq \mathbb{Z}^n \)). And moreover, the weighted count equalling \( \Gamma^n(A^1, A^2) \) is a sufficient, but for \( n \geq 4 \) not a necessary, condition for equilibrium to always exist—hence the general version of the Intersection Count Theorem (Thm. 5.12).
5.2 Definitions for the Theorem

5.2.1 Intersection 0-cells

Recall that the existence of equilibrium depends on whether the aggregate demand set is discrete convex at all prices in the LIP intersection (Lemma 4.7). For more than two goods, and for some non-transverse intersections, these prices are not a set of isolated points. However, it will suffice to focus our analysis on a particular (finite) set of 0-cells in the intersection:

**Definition 5.1.** An intersection 0-cell for LIPs $L_{u1}$ and $L_{u2}$ is a 0-cell of their aggregate LIP, $L_{U}$, contained in the intersection $L_{u1} \cap L_{u2}$.

**Proposition 5.2.** If two individual valuations are concave, and their aggregate demand complex has dimension $n$, then equilibrium exists for every relevant supply bundle iff the aggregate demand set is discrete-convex at every intersection 0-cell.

If the aggregate demand complex is not $n$-dimensional, we can always make a unimodular basis change to reduce the dimension of the goods-space so that the aggregate domain is of full dimension. (Otherwise we would have to analyse higher-dimensional cells.)

5.2.2 Parallel Lattices, and Cell Weights

As we saw in Section 5.1.2, the facets weights provide a measure of the ‘relative size’ of the corresponding demand complex cell. To generalise this to lower-dimensional cells, we need a “lattice-volume” of a polytope analogous to the “length” of Defn. 2.16(4).

**Definition 5.3.**

1. A lattice is a set $\Lambda \subseteq \mathbb{Z}^n$ such that $0 \in \Lambda$ and if $v, v' \in \Lambda$ then $v - v' \in \Lambda$.
2. $\Lambda'$ is a sublattice of $\Lambda$ if $\Lambda' \subseteq \Lambda$ and $\Lambda'$ has the structure of a lattice.
3. An (integer) basis for a lattice $\Lambda$ is a set $\{v_1, \ldots, v_k\}$ such that any $v \in \Lambda$ can be uniquely presented as $v = \sum_j \alpha_j v_j$ for $\alpha_j \in \mathbb{Z}$.
4. The rank of a lattice is the dimension of its basis.

Importantly, every lattice has an integer basis, and the rank is well-defined (see e.g. Cox et al., 2005, p334). The lattices important to us are (recalling–Defn. 4.15–that $K_{\sigma}$ denotes the linear span of changes in demand associated with $\sigma$):

**Definition 5.4.** The parallel lattice to a demand complex cell $\sigma$ is $\Lambda_{\sigma} := K_{\sigma} \cap \mathbb{Z}^n$.

It is easy to see that if $\sigma$ is a $k$-cell then $\Lambda_{\sigma}$ has rank $k$.

For our “hotel room” example (Figs. 5 and 6), if we let $\sigma^*, \sigma^c$ and $\sigma$ be the demand complex cells at $p = (30, 20)$ of $u^*, u^c$ and $U$ respectively, then $\Lambda_{\sigma^*} = \{m(-1, 1) : m \in \mathbb{Z}\}$, and $\Lambda_{\sigma^c} = \{m(1, 1) : m \in \mathbb{Z}\}$, while $\Lambda_{\sigma} = \mathbb{Z}^2$ (see Fig. 10 in Section 5.3.1). The

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36 If the intersection is transverse, then the set of intersection 0-cells is exactly the obvious set of 0-cells of $L_{u1} \cap L_{u2}$. If the intersection is not transverse the set may contain additional points which are not 0-cells in the simplest structure of a polyhedral complex that one might impose on $L_{u1} \cap L_{u2}$.

37 This is the group-theoretic meaning of “lattice” (see, e.g., Cassels, 1971), not the (completely different) order-theoretic meaning of, e.g., Milgrom and Shannon (1994).

38 We will refer to “integer bases” rather than just ‘bases’ when there is ambiguity.
parallel lattices are the same in the weight two version of this example (Section 5.1.2): at \( p = (30, 20) \) and cells \( \sigma^{2e}, \sigma^{2e} \) and \( \sigma^2 \) of \( \Sigma_{u^2}, \Sigma_{u^2} \) and \( \Sigma_{U^2} \) respectively, \( \Lambda_{u^2} = \Lambda_{u^2} \) and \( \Lambda_{u^2} = \Lambda_{u^2} \).

We now generalise the “weight” of Defn. 2.4. Given a rank-\( k \) lattice \( \Lambda \), we can find an invertible \( n \times n \) matrix \( G_\Lambda \) such that \( G_\Lambda \Lambda := \{ G_\Lambda \mathbf{v} : \mathbf{v} \in \Lambda \} = \mathbb{Z}^k \).\(^{39}\) We can use this identification to give volumes relative to the lattice \( \Lambda \) (as usual, the \( n \)-dimensional volume of \( X \subseteq \mathbb{R}^n \) is \( \text{vol}_n(X) := \int \cdots \int_X 1 \, dp_1 \ldots dp_n \)).

**Definition 5.5.** If \( X \subseteq \mathbb{R}^n \) is a polytope with vertices in \( \Lambda \), define the *lattice-volume* of \( X \) in \( \Lambda \) as \( \text{vol}_n(X) := \text{vol}_k(G_\Lambda X) \), where \( G_\Lambda \) is an invertible \( n \times n \) matrix such that \( G_\Lambda \Lambda = \mathbb{Z}^k \).

It is standard that this is independent of the choice of \( G_\Lambda \): if also \( \hat{G}_\Lambda \Lambda = \mathbb{Z}^k \) then the images of \( X \) under the two transformations are related by a change of basis matrix on \( \mathbb{Z}^k \), which must be unimodular, and so the volumes of the images are the same.

For example, given \( \Lambda_{u^*} \) as above, we may set \( G_{\Lambda_{u^*}} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \). Then \( G_{\Lambda_{u^*}}(-1, 1)' = (1, 0)' \) and so \( G_{\Lambda_{u^*}} \Lambda_{u^*} = \mathbb{Z} \). That is, operating with \( G_{\Lambda_{u^*}} \) on the lattice \( \Lambda_{u^*} \) “rotates” it to “match it up” with \( \mathbb{Z} \). So \( G_{\Lambda_{u^*}} \sigma^* \) is just the interval \([0, 1]\), which has length (1-dimensional volume) equal to 1, that is, its facet weight according to Defn. 2.4. Now:

**Definition 5.6.** Let \( C_\sigma \) be an \((n-k)-\)cell of the LIP, dual to the demand complex \( k \)-cell \( \sigma \). The *weight* of \( C_\sigma \) is \( w_u(C_\sigma) := k! \text{vol}_{\Lambda_u}(\sigma) \).

So if \( k = 1 \) the weight of a cell is just its “length” in terms of the primitive integer vector in the direction of its demand-complex cell. Thus this definition generalises Defn. 2.4. For example, \( w_u(C_{\sigma^*}) = w_u(C_{\sigma^*}) = 1 \) and \( w_u(C_{\sigma^*}) = w_u(C_{\sigma^*}) = 2 \).

The factor of \( k! \) ensures that the cell weight is an integer (all cells are measured relative to a lattice simplex). So in our “hotel room” example, the weights of the 0-cells of \( L_{u^*} \) are both 1 (see Figs. 5a and 6a). Since the lattice-volume of the central cell in the aggregate demand complex, \( \sigma \), is 2, this means that \( w_u(C_{\sigma}) = 4 \). Similarly, \( w_u(C_{\sigma^*}) = 16 \).

### 5.2.3 Naïve Multiplicities

**Definition 5.7.** If the intersection is transverse at an intersection 0-cell \( C \), define the *naïve multiplicity* \( \hat{m}(C) := w_{u_1}(C^1) \cdot w_{u_2}(C^2) \), where \( C^j \) is the smallest cell of \( L_{u^j} \) containing \( C \), for \( j = 1, 2 \).

To extend this to non-transverse cases, we first write:

**Definition 5.8.** If \( C \) is an intersection 0-cell for \( L_{u^1} \) and \( L_{u^2} \), and if \( C' \) is an intersection 0-cell for \( L_{u^1} \) and \( \{ \epsilon \mathbf{v} \} + L_{u^2} \) (for any \( \epsilon > 0 \) and \( \mathbf{v} \in \mathbb{R}^n \)), then we say \( C' \) emerges from \( C \) if there exist cells \( C^j \) of \( L_{u^j} \) for \( j = 1, 2 \), such that \( C^1 \cap C^2 = C \) and \( C^1 \cap \{ \epsilon \mathbf{v} \} + C^2 = C' \).

There can be several intersection 0-cells emerging from \( C \) under the same translation; for example, the intersection 0-cells at \((4, 1 + \epsilon)\) and \((4 + \epsilon, 1)\) in Fig. 7b emerge from the intersection 0-cell at \((4, 1)\) in Fig. 7a. There is always at least one:

\(^{39}\)Specifically, if \( H_\Lambda \) is an invertible matrix whose first \( k \) columns give a basis \{ \( \mathbf{v}^1, \ldots, \mathbf{v}^k \) \} for \( \Lambda \), then \( H_\Lambda \mathbf{e}^i = \mathbf{v}^i \) and so \( H_\Lambda^{-1} \mathbf{v}^i = \mathbf{e}^i \) for \( i = 1, \ldots, k \). Thus \( G_\Lambda := H_\Lambda^{-1} \) has the properties described.
Lemma 5.9. For every $v \in \mathbb{R}^n$ and sufficiently small $\epsilon > 0$, there exists an intersection 0-cell for $L_{u_1}$ and $\{\epsilon v\} + L_{u_2}$ emerging from every intersection 0-cell for $L_{u_1}$ and $L_{u_2}$. 

Now if $L_{u_1}$ and $\{\epsilon v\} + L_{u_2}$ intersect transversally, each of their intersection 0-cells has a naive multiplicity. Moreover, for fixed $v$ and small enough $\epsilon > 0$, the set of these multiplicities, and therefore also the sum of these multiplicities, is independent of $\epsilon$. But as there are only finitely many cells in each LIP, there are only finitely many different sums that can be obtained in this way.  

Take, for example, the intersection 0-cell at (4, 1) in Fig. 7a. When $v = (1, 0)$, as shown in Fig. 7b, we obtain intersection 0-cells whose naive multiplicities sum to 2. But for $v = (-1, 0)$, only one intersection 0-cell, with naive multiplicity 1, would have emerged. We will always want the maximum of the sums, so:

Definition 5.10. The naive multiplicity $\hat{m}(C)$ at an intersection 0-cell $C$ for $L_{u_1}$ and $L_{u_2}$ is the maximum number that can be obtained by adding the naive multiplicities of 0-cells emerging from $C$ under a small translation of $L_{u_2}$ which makes the intersection transverse at $C$.

Note that this “naive multiplicity” is not the “true multiplicity” of the tropical Bézout-Bernstein theorem (Defn. 5.18).

5.2.4 General definition of $\Gamma^n(\cdot, \cdot)$

Finally, we extend the definition of $\Gamma^2(\cdot, \cdot)$ to $n > 2$:

Definition 5.11. If $A_1, A_2 \subset \mathbb{Z}^n$ are finite then, for $k = 1, \ldots, n - 1$, define:

\begin{align*}
(1) \quad & \Gamma^n_k(A_1, A_2) := \sum_{r=0}^{k} \sum_{s=0}^{n-k} (-1)^{n-r-s} \binom{k}{r} \binom{n-k}{s} \text{vol}_n \text{conv}(rA_1 + sA_2). \\
(2) \quad & \Gamma^n(A_1, A_2) := \sum_{k=1}^{n-1} \Gamma^n_k(A_1, A_2)
\end{align*}

Some readers will notice that $\Gamma^n_k(A_1, A_2)$ is the “mixed volume” of $k$ copies of the convex hull of $A_1$ with $n-k$ copies of the convex hull of $A_2$. Importantly, it is therefore a non-negative integer. See Fact A.9 for more details and special cases.

5.3 The Intersection Count Theorem

Theorem 5.12 (The Intersection Count Theorem). For $j = 1, 2$, let $u_j$ be concave valuations, on domains $A^j \subset \mathbb{Z}^n$ such that $\dim \text{conv}(A^1 + A^2) = n$. Then the count of intersection 0-cells for $L_{u_1}$ and $L_{u_2}$, counted with naive multiplicities, is bounded above by $\Gamma^n(A_1, A_2)$. If the count equals this bound, equilibrium exists for all relevant supplies. 

Suppose additionally that the intersection is transverse and that $n \leq 3$. The number of intersection 0-cells for $L_{u_1}$ and $L_{u_2}$, counted with naive multiplicities, is equal to $\Gamma^n(A_1, A_2)$ iff equilibrium exists for all relevant supplies.

The remainder of Section 5.3 develops the proof. Full details are in Appendix A.4.
5.3.1 Subgroup Indices

As usual, we will want to investigate whether, starting at a point at which we know equilibrium exists, a change in aggregate supply can be matched by a change in aggregate demand. These changes are within lattices, so we use a standard tool to compare them:

**Definition 5.13.** Let $\Lambda$ be a lattice, and $\Lambda' \subseteq \Lambda$ a sublattice of the same rank as $\Lambda$.

1. A **fundamental parallelepiped** of $\Lambda$ is a set $\Delta_\Lambda := \{\sum_j \lambda_j v^j \in \mathbb{R}^n : 0 \leq \lambda_j < 1\}$, where $\{v^1, \ldots, v^k\}$ are a basis for $\Lambda$.

2. The **subgroup index** $[\Lambda : \Lambda']$ is the lattice-volume in $\Lambda$ of a fundamental parallelepiped of $\Lambda'$, that is, $[\Lambda : \Lambda'] := \text{vol}_\Lambda(\Delta_{\Lambda'})$.

Parallel lattices for individual demand combine to $\Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r}$, so we check:

**Lemma 5.14.** For valuations $u^1, \ldots, u^r$ with aggregate valuation $U$, and respective demand complex cells $\sigma^1, \ldots, \sigma^r$ and $\sigma$ at $p$, the lattice $\Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r}$ is a sublattice of $\Lambda_{\sigma}$ of the same rank as $\Lambda_{\sigma}$.

The parallel lattices for our “hotel room” example (as given in Fig. 5), with $p = (30, 20)$, are given in Fig. 10. By considering $\Lambda_{\sigma^s}$ and $\Lambda_{\sigma^c}$, we see that the sublattice $\Lambda_{\sigma^1} + \Lambda_{\sigma^2} = \{m(1, -1) + m'(1, 1) : m, m' \in \mathbb{Z}\}$. (That is, this sublattice comprises the white bundles in Fig. 10b, while the lattice $\Lambda_{\sigma} = \mathbb{Z}^2$ comprises all the bundles in Fig. 10b.) A fundamental parallelepiped, $\Delta_{\Lambda_{\sigma^1} + \Lambda_{\sigma^c}}$, of this sublattice is shown; its lattice-volume in $\Lambda_{\sigma}$ is 2. So the subgroup index $[\Lambda_{\sigma} : \Lambda_{\sigma^1} + \Lambda_{\sigma^c}] = 2$.

Figure 10: The parallel lattices corresponding to demand complex cells $\sigma^s$, $\sigma^c$ and $\sigma$, of respectively, $u^s$, $u^c$ and their aggregate valuation $U$ (as shown in Fig. 5) at $p = (30, 20)$.

Observe in Fig. 10b that the subgroup index also corresponds to the ratio of the total number of bundles to the number of white bundles. We can in fact understand subgroup indices generally in terms of such ratios (see Fact A.12). Subgroup indices generalise unimodularity, as is seen by comparing Fact 5.15(1), below, with Fact 4.8(1), and Fact 5.15(3) with Defn. 4.2:

**Facts 5.15.** Let $\Lambda$ be a lattice, and $\Lambda' \subseteq \Lambda$ a sublattice of the same rank as $\Lambda$.

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40Different fundamental parallelepipeds are images of one another under unimodular basis changes.

41These are the ordinary group-theoretic subgroup indices (see the proof of Fact 5.15), and so are independent of the choice of parallelepiped.
(1) $[\Lambda : \Lambda'] - 1$ is equal to the number of elements of $\Lambda$ in $\Delta_{\Lambda'}$ which are not vertices.\footnote{Since fundamental parallelepipeds are not closed, we do not count points on the “upper” boundary.}

(2) $[\Lambda : \Lambda'] = 1$ iff $\Lambda = \Lambda'$.

(3) If $\Lambda = \mathbb{Z}^n$, and $\{v^1, \ldots, v^n\}$ are a basis for $\Lambda'$, then $[\Lambda : \Lambda'] = |\det(\{v^1, \ldots, v^n\})|$.\footnote{When $\Lambda \neq \mathbb{Z}^n$, we can calculate $[\Lambda : \Lambda']$ by fixing a matrix identifying $\Lambda$ with $\mathbb{Z}^k$ for some $k$, and then applying the matrix to $\Lambda'$.}

(4) Suppose $\Lambda = K \cap \mathbb{Z}^n$ for some linear subspace $K$ of $\mathbb{R}^n$. Then $[\Lambda : \Lambda'] = 1$ iff any basis for $\Lambda'$ is unimodular.

In our example, there is one element of $\Lambda_{\sigma}$ in $\Delta_{\Lambda_{\sigma} + \Lambda_{\sigma'}}$ which is not at a vertex (see Fig. 10b). This illustrates Fact 5.15(1). As $\Lambda_{\sigma} = \mathbb{Z}^2$ we can use Fact 5.15(3) to calculate $[\Lambda_{\sigma} : \Lambda_{\sigma} + \Lambda_{\sigma'}]$ as the determinant of the matrix with columns $(1, -1), (1, 1)$ (cf. Section 4.2.2). This basis is not unimodular, and so Fact 5.15(4) verifies for us again that $\Lambda_{\sigma} + \Lambda_{\sigma'} \neq \Lambda_{\sigma}$.

### 5.3.2 The Relationship Between Subgroup Indices and Equilibrium

Subgroup indices allow us to determine whether equilibrium exists at a particular price. Following the logic of Section 4.2, it is sufficient that the facet normals at this intersection price are a unimodular set, but the following theorem gives a weaker sufficient condition. Although the Intersection Count Theorem only requires the $r = 2$ case of Thm. 5.16, we give it in the most general form:

**Theorem 5.16 (The Subgroup Indices Theorem).** Let $w^1, \ldots, w^r$ be concave and suppose the intersection of those LIPs which contain $p$ is transverse at $p$. Write $\sigma^j, \sigma$ for the demand complex cell at $p$ of respectively $w^j$ ($j = 1, \ldots, r$) and the aggregate valuation $U$.

1. If $[\Lambda_{\sigma} : \Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r}] = 1$ then $D_U(p)$ is discrete-convex.
2. If $[\Lambda_{\sigma} : \Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r}] > 1$ and if also $\dim \sigma^1 \leq 2$, $\dim \sigma^j \leq 1$ for $j = 2, \ldots, r$, then $D_U(p)$ is not discrete-convex.

To translate part (1) back to the familiar terms of Section 4.2, first let $r = 2$ and suppose both LIPs contain $p$. Recall that $\Lambda_{\sigma} = K_{\sigma} \cap \mathbb{Z}^n$ (Defn. 5.4). So, by Fact 5.15(4), $[\Lambda_{\sigma} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] = 1$ iff any basis for $\Lambda_{\sigma^1} + \Lambda_{\sigma^2}$ is unimodular. Moreover, the combination of a basis for $\Lambda_{\sigma^1}$ with a basis for $\Lambda_{\sigma^2}$ gives a basis for $\Lambda_{\sigma^1} + \Lambda_{\sigma^2}$ (see Fact A.13(3) for more details). The proof in Section 4.2.5 that $D_U(p)$ is discrete-convex depended on a unimodular basis for $K_{\sigma}$, consisting of integer vectors in either $K_{\sigma^1}$ or $K_{\sigma^2}$. Exactly the same arguments prove Thm. 5.16(1).

But the result here is more powerful. It also provides existence of equilibrium when the set of normals to agents’ facets at $p$ is not unimodular: all that is required is an integer basis for the changes in demand that each agent considers, such that the combination of these integer bases forms a unimodular set. When individual agents’ demand sets are of dimension greater than 1, equilibrium does not require that the changes in demand they consider can be broken down as integer combinations of vectors along the edges of these sets.

For example, if Figs. 5c and 6c correspond to an individual agent with a concave valuation, the bundle $(1, 1)$ is demanded at price $(30, 20)$.

In this case the edges to this
demand complex cell do not provide a basis for its parallel lattice. By combining two such agents in 4-dimensional space, Ex. A.14 shows that Thm. 5.16(1) demonstrates competitive equilibrium in situations in which Thm. 4.3 does not.

Now we consider Thm. 5.16(2). If \( \text{dim} \sigma_j \leq 1 \) for \( j = 1, \ldots, r \), then the LIP cells are all facets. If additionally they have weight 1, then the situation at the price \( \mathbf{p} \) is exactly that described in Lemma 4.9. So, just as in Lemma 4.9, unimodularity is necessary (as well as sufficient) for equilibrium; now apply Fact 5.15(4).

But for any weights, \( \sigma \) is a grid of copies of a “small parallelepiped”, as we saw in Section 5.1.2. This “small parallelepiped” is (the closure of) a fundamental parallelepiped of \( \Lambda_{\sigma_1} + \cdots + \Lambda_{\sigma_r} \). So equilibrium fails for some supply if and only if such a fundamental parallelepiped contains a non-vertex integer point. Applying Fact 5.15(1) therefore yields Thm 5.16(2) if \( \text{dim} \sigma_j \leq 1 \) for all \( j \).

But if \( \text{dim} \sigma_j > 1 \) for some \( j \), then \( \sigma \) need not be a parallelepiped. In such cases, even if any fundamental parallelepiped of \( \Lambda_{\sigma_1} + \cdots + \Lambda_{\sigma_r} \) contains a non-vertex integer point, it does not necessarily follow that a corresponding point lies in \( \sigma^1 + \cdots + \sigma^r = \sigma \) itself: it can fall outside the aggregate demand set at this price.

However, when we have one 2-cell, along with 1-cells, then we can capture this point. The reason is that a 2-cell will always contain a triangle, two of whose edges give a basis for its parallel lattice (see Appendix A.4). And this triangle is a copy of half of a fundamental parallelepiped. Meanwhile, while each 1-cell contains an entire fundamental parallelepiped of its parallel lattice. So \( \sigma \), the Minkowski sum of these, contains at least half of a fundamental parallelepiped of \( \Lambda_{\sigma_1} + \cdots + \Lambda_{\sigma_r} \). By symmetry, then, if \( [\Lambda_\sigma : \Lambda_{\sigma_1} + \cdots + \Lambda_{\sigma_r}] > 1 \) in this case, the aggregate demand complex cell \( \sigma \) does contain a bundle not in \( \Lambda_{\sigma_1} + \cdots + \Lambda_{\sigma_r} \), i.e. one that cannot be reached via changes in demand among the two agents.

Examples (A.15 and A.16) show that this is the furthest we can go, that is:  

**Proposition 5.17.** In the notation of Thm. 5.16, there exist examples in which \( [\Lambda_\sigma : \Lambda_{\sigma_1} + \Lambda_{\sigma_2}] > 1 \), and either \( \text{dim} \sigma^1 = \text{dim} \sigma^2 = 2 \) or \( \text{dim} \sigma^1 = 3 \), but in which \( D_U(\mathbf{p}) \) is discrete-convex.

### 5.3.3 The Tropical Bézout-Bernstein Theorem

We now define the “true multiplicity”, \( \text{mult}(C_\sigma) \), of a cell, \( C_\sigma \):\(^{44}\)

**Definition 5.18** (Bertrand and Bihan, 2013, Defn. 5.2). For valuations \( u_j, j = 1, 2 \), and an intersection 0-cell for their LIPs at \( \mathbf{p} \), write \( \sigma_j := D_{u_j}(\mathbf{p}) \) and \( \sigma := D_U(\mathbf{p}) \) for their aggregate, and write \( C_{\sigma_1} \) and \( C_{\sigma_2} \) for the corresponding cells of their LIPs.

1. If the intersection of \( \mathcal{L}_{u_1} \) and \( \mathcal{L}_{u_2} \) is transverse at \( C_\sigma \), then the true multiplicity of \( C_\sigma \) is \( \text{mult}(C_\sigma) := [\Lambda_\sigma : \Lambda_{\sigma_1} + \Lambda_{\sigma_2}] \cdot w_{u_1}(C_{\sigma_1}) \cdot w_{u_2}(C_{\sigma_2}). \)

2. If the intersection is not transverse at \( C_\sigma \), then the true multiplicity, \( \text{mult}(C_\sigma) \), of \( C_\sigma \), is the sum of the multiplicities at all the intersection 0-cells emerging from \( C_\sigma \) after any small translation of \( \mathcal{L}_{u_2} \) which makes the intersection transverse.

So the distinction between “true” and “naive” multiplicities is embodied in the subgroup indices that capture our results on equilibrium (Thm. 5.16). The use of subgroup

\(^{44}\)We follow the conventions of Bertrand and Bihan (2013) although these ideas and theorems are older (see, e.g., McMullen, 1993, Huber and Sturmfels, 1995 and Fulton and Sturmfels, 1996).
indices means that true multiplicities, unlike naïve multiplicities, are unchanged by small translations in any directions. Note that both multiplicities are positive integers, since this is true of cell weights (and by Lemma 5.9).

A version of the “tropical Bézout-Bernstein theorem” is now:

\textbf{Theorem 5.19} (Bertrand and Bihan, 2013, Thm. 6.1).

1. The number of intersection 0-cells for $\mathcal{L}_{u_1}$ and $\mathcal{L}_{u_2}$, counting with their true multiplicities, is equal to $\Gamma^n(A_1, A_2)$.

2. If the intersection is transverse whenever a $k$-cell of $\mathcal{L}_{u_1}$ meets an $(n-k)$-cell of $\mathcal{L}_{u_2}$, then the number of such intersection 0-cells, counting with their true multiplicity, is equal to $\Gamma^n_k(A_1, A_2)$.

5.3.4 Sketch Proof of the Intersection Count Theorem

It is now easy to prove the sufficiency result for the transverse case. From Defns. 5.7 and 5.18, and Fact 5.15(1) (which implies that the subgroup index is always at least 1) we know that $\hat{m}(C) \leq \text{mult}(C)$ for any intersection 0-cell $C$. Equality holds iff the subgroup index at this price is 1. So, by Thm. 5.19(1), the number of intersection 0-cells, counted with naïve multiplicities, is bounded above by $\Gamma^n(A_1, A_2)$, and this bound holds with equality iff the subgroup index at every intersection 0-cell is 1. If this is true, then the aggregate demand is discrete-convex at every intersection 0-cell (Thm. 5.16(1)) and so equilibrium exists for every relevant supply (Prop. 5.2).

For necessity, we also assume $n \leq 3$. Every intersection 0-cell is the intersection of a 2-cell and a 1-cell of the two individual LIPs. So apply Thm. 5.16(2): if the number of intersection 0-cells, counted with naïve multiplicities, is strictly below $\Gamma^n(A_1, A_2)$ then the subgroup index is greater than 1 for some intersection 0-cell, and this implies a failure of equilibrium for some relevant supply.

When the intersection is not transverse, the logic is similar, but we need a little more care. We need to translate in the “right direction” to find the naïve multiplicity for each intersection 0-cell, but these directions need not be the same across all intersection 0-cells. However, we can combine Prop. 4.13 with Thms. 5.16(1) and 5.19(1) by working locally at each 0-cell. See Appendix A.4 for details.

5.4 Limitations, and Extension, of the Theorem

5.4.1 Necessity

The condition of the Intersection Count Theorem is necessary for equilibrium only when the intersection is transverse and $n \leq 3$.

If the intersection is not transverse, it is possible for equilibrium to exist for any relevant supply, but for there to be no small translation of the LIPs which retains existence (see Ex. A.19). However, if an intersection is not transverse, our results relating unimodularity and subgroup indices to equilibrium fail to hold (recall, for example, the proof given in Section 4.2.5). So we cannot hope to identify these “fragile” equilibria with these tools, and so cannot give a necessary result in the non-transverse case.

The failure of our condition to give necessity for $n \geq 4$ is immediate from Prop. 5.17. (Recall we needed Thm. 5.16(2) to prove necessity for $n \leq 3$.)
Note, however, that our upper bound $\Gamma^n(\cdot, \cdot)$ is tight (strong substitutes valuations illustrate this for both non-transverse cases and arbitrary $n$; see Prop. 4.6 and Section 5.1.4) so this bound cannot be improved on.

5.4.2 Sufficiency

If the bound in Thm. 5.12 is not met with equality, the combination of Thm. 5.19(2) with Thm. 5.16 provides an alternative sufficient condition for equilibrium existing at particular prices. (Note that an intersection is transverse at a 0-cell iff the minimal cells of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ that contain that 0-cell have dimensions $k$ and $(n - k)$ for some $k$.)

**Proposition 5.20.** Let $u^j : A^j \to \mathbb{R}$ be concave valuations for $j = 1, 2$, suppose $\dim \text{conv}(A^1 + A^2) = n$, and fix $k \in \{1, \ldots, n-1\}$. Suppose the intersection is transverse at every intersection 0-cell $C$ such that the minimal cell of $\mathcal{L}_{u^1}$ containing $C$ has dimension $k$. Then the number of these intersection 0-cells, counted with naïve multiplicities, is bounded above by $\Gamma_k^n(A^1, A^2)$. If the number equals this bound, then equilibrium exists for all supplies in the convex hull of demand at each of these intersection 0-cells.

5.5 An Algorithm to Check for Existence of Equilibrium

The following algorithm summarises the results of this paper on existence of equilibrium.

**Algorithm 5.21.** Suppose we are given concave valuations $u^j : A^j \to \mathbb{R}$, where $A^j \subset \mathbb{Z}^n$ is finite, for $j = 1, 2$.

1. Are $u^1, u^2$ of the same unimodular demand type?
   (i) If yes, then equilibrium exists for all relevant supplies. End here.
   (ii) If no, continue.

2. If $\dim \text{conv}(A^1 + A^2) = n' < n$, make a unimodular change of basis so that $A^1, A^2 \subset \mathbb{Z}^{n'}$, and use $n'$ instead of $n$ in the following steps.

3. Calculate $\Gamma_k^n(A^1, A^2)$ for $k = 0, \ldots, n$ and $\Gamma^n(A^1, A^2)$. Find the intersection 0-cells.

4. Does the number of intersection 0-cells equal $\Gamma^n(A^1, A^2)$?
   (i) If yes, then equilibrium exists for all relevant supplies. End here.
   (ii) If no, continue.

5. Find the naïve multiplicity of each intersection 0-cell $C$; note if the intersection is transverse at $C$, and the dimension of the minimal cell of $\mathcal{L}_{u^1}$ containing $C$.

6. Does the weighted count of all intersection 0-cells equal $\Gamma^n(A^1, A^2)$?
   (i) If yes, then equilibrium exists for all relevant supplies. End here.
   (ii) If no, if $n \leq 3$, and if the intersection is transverse, then equilibrium fails for some relevant supply; end here. Otherwise continue.

7. Identify all intersection 0-cells $C$ such that the dimension $k$ of the minimal cell of $\mathcal{L}_{u^1}$ containing $C$ does not satisfy both:
   (i) the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is transverse at every intersection 0-cell, $C'$, for which the minimal cell of $\mathcal{L}_{u^1}$ containing $C'$ has dimension $k$, and
   (ii) the count of all intersection 0-cells $C'$ as in (7)(i), counted with naïve multiplicities, is equal to $\Gamma_k^n(A^1, A^2)$.

41
(8) Is $D_U(p)$ discrete-convex for all $p$ at the intersection 0-cells identified in Step (7)?

(i) If yes, equilibrium exists for every relevant supply.
(ii) If no, equilibrium fails for some relevant supply.

(To determine equilibrium existence for more than two concave valuations, we can use the algorithm for the first two, then use the algorithm for the third valuation and the aggregate of the first two valuations, etc.)

Step (1) summarises the Unimodularity Theorem (Thm. 4.3). Step (4) uses the special case of the Intersection Count Theorem (Thm. 5.12) that, if all naive multiplicities equal 1, then the weighted count of intersection 0-cells is just the number of these cells. (This can save computations.) Step (6) is the Intersection Count Theorem itself. Finally, Steps (7)-(8) use Prop. 5.20 (with Prop. 5.2). It is not surprising that we cannot rule out needing Step (8); the question of competitive equilibrium with indivisible goods is closely related to long-studied problems (see e.g. Haase et al., 2007). However, we know (from combining Thm. 5.12 with Prop. 5.2) that we have to check at most $\Gamma^n(A^1, A^2)$ prices.

Appropriate tools exist for each step of this Algorithm. These tools often behave well in “usual” conditions, but may have bad worst-cases. (See Remark A.21 for details.) So it seems hard to give clean results of computational complexity, or to compare the efﬁciency of using earlier stages of our Algorithm with simply resorting to many “brute force” calculations, as in Step (8). And, because we may often need to resort to Step (8), it is also hard to compare with Bikhchandani and Mamer’s (1997) algorithm which checks for equilibrium by the alternative “brute force” approach of comparing the solution of an integer programming problem with the corresponding linear programming problem for every relevant supply.

Of course, an important virtue of the Unimodularity and Intersection Count Theorems is that, when the earlier Steps do sufﬁce to determine existence, they increase our understanding and intuition as to why equilibrium exists or fails.

6 Applications

6.1 Interpreting Classic Models in a Unified Framework

Our model encompasses some classic studies as special cases, so clarifies connections between them. It also facilitates our understanding of them. In particular, it makes many of their equilibrium-existence results straightforward.

It is straightforward that Bikhchandani and Mamer (1997)’s model is the special case of ours in which each agent’s domain is restricted to $\{0, 1\}^n$.

Kelso and Crawford’s (1982) seminal analysis of $n_1$ firms, each of which is interested in hiring some of $n_2$ workers, can be understood as a model with $n_1n_2$ distinct “goods”, each of which is the “transfer of labour” by a specified worker (a “seller”) to a speciﬁed firm (a “buyer”); the “price” of a good is the salary to be paid. So the full set of bundles is $\{-1, 0, 1\}^n$, in which $n = n_1n_2$. However, each worker’s valuation is deﬁned only over a subset of this domain of the form $\{-1, 0\}^{n_1}$ (that is, only over the $n_1$ goods that correspond to its own labour), and only over the subset of these vectors that have at most one non-zero entry (it can work for at most one ﬁrm). Obviously, their only
possible demand complex edges are the strong substitute vectors (non-zero vectors with at most one +1 entry, at most one −1 entry, and no other non-zero entries, see Prop. 3.9). Similarly, each firm’s valuation is defined only over a subset of the form \(\{0, 1\}^{n_2}\) (that is, it has preferences only over the \(n_2\) goods that correspond to workers it can employ). Kelso and Crawford assume firms have ordinary substitutes preferences over workers but, since there is only one unit of each good, the only substitute changes of demand are the strong substitutes vectors, so all valuations are of this type.\(^{45}\)

It may be less obvious that Hatfield et al.’s (2013) model of networks of trading agents, each of whom can both buy and sell, both fits into our framework, and is also closely related to Kelso and Crawford’s model. To show this, we again treat each transfer of a product from a specified seller to a specified buyer as a distinct good, so each agent again has preferences over a subset of \(-1, 0, 1\)^n, where \(n\) is the number of distinct goods.

Since Hatfield et al. restrict each agent to be either a seller or a buyer (or neither) on any one good, an agent \(j\) which is the specified seller in \(n_j^1\) potential trades and is the specified buyer in \(n_j^2\) potential trades simply has preferences over a subset of the domain which, after an appropriate re-ordering of the goods for that agent, is of the form \((-1, 0)^{n_j^1} \times \{0, 1\}^{n_j^2}\). (As in Hatfield et al., we can restrict an agent’s domain of preferences further so that, e.g., it cannot sell good 1 unless it also buys one of goods 2 or 3. They do this by using “−∞” valuations, while we simply exclude bundles from the domain, but the effect is the same—see Section 2.1.) Furthermore, although Hatfield et al. describe goods to be sold as complements of goods to be bought, this is because they measure both buying and selling as non-negative quantities. So, since in our framework selling is just “negative buying”, the “complementarities” disappear and it is clear that the condition they impose is exactly ordinary substitutes.\(^{46}\) Just as for Kelso and Crawford’s model, the only demand complex edges of such a domain that are vectors of the ordinary substitutes demand type are also vectors of the strong substitutes demand type.

Trivially, any valuation over any subset of \((-1, 0)^{n_1}\) or \(\{0, 1\}^{n_2}\) or \((-1, 0)^{n_j^1} \times \{0, 1\}^{n_j^2}\) is concave so, in both Kelso and Crawford’s and Hatfield et al.’s models, the existence of equilibrium follows immediately from Section 4.1’s discussion.

Another model that is easy to analyse using our methods is the “cycle of complements” model in which each of \(n\) kinds of agent is only interested in a single, specific, pair of goods, and these pairs form a cycle. The Unimodularity Theorem immediately tells us equilibrium is guaranteed to exist if and only if \(n\) is even. Furthermore, we can use our method of proof of Prop. 4.10 to find examples of equilibrium failure if \(n\) is odd; Ex. A.22 in Appendix A.5 gives details. More generally, our proof of Prop. 4.10 shows how to easily construct an explicit example of failure of equilibrium for any model that permits valuations from a non-unimodular demand type.

Reformulating models in our framework also shows clearly how we can generalise them. It is immediate, for example, that as long as we retain concavity and the strong

\(^{45}\)Kelso and Crawford make a superficially more restrictive assumption than ordinary substitutes, but it is equivalent in their context: see Danilov et al. (2003), Baldwin and Klemperer (2014), and Baldwin, Klemperer and Milgrom (in preparation).

\(^{46}\)Their “choice language” definition differs superficially from Defn. (1), but Hatfield et al., 2015, Thm B.1 confirms the equivalence.
substitutes demand type, we can remove Hatfield et al.’s restriction that an agent cannot be both a buyer and a seller on any one good (by simply extending their domain to be any subset of \{-1, 0, 1\}^n) and can also permit their agents to trade multiple units of the same products (by enlarging the domain to any subset of \(\mathbb{Z}^n\)).

6.2 Basis Changes of LIPs and Demand Types

Obviously, many properties are preserved if we re-package goods so that any integer bundle can still be obtained by buying and selling an integer selection of the new packages. This corresponds, of course, to a unimodular change of basis;\(^{47}\) it simply distorts the LIP by a linear transformation which leaves its important structure unaffected.

Specifically, for a unimodular \(n \times n\) matrix \(G\), it is standard to define the “pullback” of a valuation \(u : A \to \mathbb{R}\) to be \(G^*u : G^{-1}A \to \mathbb{R}\) via \(G^*u(x) := u(Gx)\). Then:

**Proposition 6.1** (cf. e.g. Gorman, 1976, p. 219-220).

1. \(x \in D_u(p) \iff G^{-1}x \in D_{G^*u}(G^T p)\).
2. \(L_{G^*u} = G^T L_u := \{G^T p : p \in L_u\}\).
3. \(u(\cdot)\) is of demand type \(D\) iff \(G^*u(\cdot)\) is of demand type \(G^{-1}D := \{G^{-1}v : v \in D\}\).

Ex. A.23 gives an illustration.

Baldwin and Klemperer (2014, especially Section 5; and in preparation-b) use the fact that important properties of demand are unaffected by unimodular basis changes in their analysis of the comparative statics of individual valuations. But the most immediate pay-off from this and our method of categorising valuations into “demand types” is that it is straightforward (see Appendix A.5) that:

**Proposition 6.2.** “Having equilibrium for every finite set of concave valuations and any relevant supply bundle” is a property of a demand type that is preserved under unimodular basis changes.

So we can find demand types which always have equilibrium. For example, it is immediate that equilibrium is guaranteed for the following demand types:

“Interval Package Valuations” (see Danilov et al., 2008, 2013). Premultiplying the strong substitutes vectors, \(e_i\) and \((e_i - e_j)\), by the upper triangular matrix of 1s (of the appropriate dimension) yields the vectors \(\sum_{k=1}^{i-1} e_k\) and \(\sum_{k=j+1}^{i} e_k\) for \(i > j\), respectively (and their negations). This is the demand type for goods which have a natural fixed order, and for which any contiguous collection of goods may be considered as complements by any agent. For example, valuations for bands of radio spectrum, or for “lots” of sea bed to be developed for offshore wind (see Ausubel and Cramton, 2011) may be of this form.

\(^{47}\)A unimodular matrix is an integer matrix with integer inverse, or, equivalently, an integer matrix with determinant \(\pm 1\).
“Generalised Gross Substitutes and Complements Valuations” Premultiplying the strong substitutes vectors by a matrix formed of \( \{ e_i : i \leq k \} \cup \{-e_i : i > k \} \), for some \( k \), yields the demand type for which goods can be separated into two groups, with goods within the same group being strong substitutes, and each good also may exhibit 1:1 complementarities with any good in the other group.\(^{48}\)

6.3 New Demand Types which Guarantee Equilibrium

Our Unimodularity Theorem helps identify new demand types of economic interest which are not unimodular basis changes of ordinary substitutes, or any subset thereof, for which equilibrium is guaranteed. For example, consider the demand type whose vectors are the columns of:

\[
D := \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 
\end{pmatrix}.
\]

This might, for example, model a firm’s demand for “bundles of” four kinds of workers–three sorts of specialist (the first three goods) and a supervisor (the fourth good). The first three columns of \( D \) show that any one of the three kinds of specialist has value on his own; the middle three columns of \( D \) show that a supervisor increases the value of any specialist (that is, there are pairwise complementarities between any one of the first three “goods” together with the fourth); the last three columns of \( D \) show that there are also complementarities between any pair of different specialists if (but only if) a supervisor is also present; but a supervisor on her own is worthless.\(^{49}\)

This demand type is pure complements, and is not a basis change of ordinary substitutes, or any subset thereof, as we show in the supplementary material to this paper (O’Connor, 2015). (Baldwin and Klemperer (2014) give a mathematical proof that it is not a basis change of strong substitutes.) It is routine to check that it is unimodular.

6.4 Relationships between Equilibrium and Complements; and between Equilibrium and Substitutes

Understanding the relationship between equilibrium and unimodularity shows that much conventional wisdom is mistaken in connecting the existence of equilibrium to

\(^{48}\)The demand type’s vectors are \( \{ e_i, e_j, e^i - e^j, e^j - e^i + e^j, e^j - e^j' : i, i' \in \{1, \ldots, k\}, j, j' \in \{k + 1, \ldots, n\} \} \). This extends the valuations introduced by Sun and Yang (2006, 2009) to permit multiple units of goods, and sellers as well as buyers. Shioura and Yang (2015) have independently made the same extension, and shown that equilibrium always exists for it.

\(^{49}\)The reason is that \( e^i \notin D \); perhaps each firm’s owner is a supervisor, and an additional supervisor without any workers would merely “spoil the broth”. (There are, of course, many basis changes of \( D \)–and of any unimodular demand type–that include all the coordinate vectors.)

This unimodular set is a basis change of the first ‘cographic’ unimodular set; see Seymour (1960) but we do not think it has previously been presented as an example of purely complementary preferences in this way.

Another interesting unimodular demand type is given in Ex. A.24.
substitutabilities: taking advantage of existing mathematical work on unimodular sets, demonstrates a link between equilibrium and complements that is, if anything, stronger.\textsuperscript{50}

First, we can easily extend the previous section’s example to obtain:

**Proposition 6.3.** With $n > 3$, there exist demand types which are not a unimodular basis change of ordinary substitutes, or a subset thereof, and for which an equilibrium exists for any finite set of concave valuations of the demand type and any relevant supply bundle. For $n \leq 3$ there exist no such demand types.

To see the result for $n > 3$, consider the demand type defined by the matrix $D$ of Section 6.3, with its vectors extended by $n - 4$ zeros, and with the coordinate vectors $e_i$, $i = 5, \ldots, n$ appended. Any basis change taking such a demand type to a substitutes demand type would restrict to a basis change taking $D$ to substitutes, contradicting the result given in Section 6.3. The result for $n \leq 3$ follows from Seymour’s (1960) characterisation.

We can also make use of mathematical results from Grishukhin et al. (2010) that imply that every unimodular set of vectors is a unimodular basis change of a set that contains only vectors in $\pm\{0, 1\}^n$. Since demand types containing these vectors contain only complements valuations, Prop. 6.2 tells us:

**Proposition 6.4.** Every demand type for which an equilibrium is guaranteed (that is, exists for any finite set of concave valuations of the demand type and any relevant supply bundle) is a unimodular basis change of a demand type which contains only complements valuations and for which equilibrium is guaranteed.

Observe that Prop. 6.3 shows the corresponding statement cannot be made about substitutes.\textsuperscript{51} It is true (and easy to show) that the strong substitutes vectors are maximal as a unimodular set of vectors. So, given any one valuation not for strong substitutes, we can find valuations which are strong substitutes such that equilibrium fails (see Section 4.2.3). Some have interpreted this result as a necessity of substitutes, but this overlooks the fact that not all strong substitutes valuations need be within the demand type in question.\textsuperscript{52}

\textsuperscript{50}Seymour (1960) developed a characterisation of unimodular sets of vectors; Danilov and Grishukhin (1999) extended this to give a full characterisation of all maximal such sets.

\textsuperscript{51}For two goods (but not more—see Section 3.2), substitutes are a basis change of complements via the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (see Section 6.2) so (only) for two goods, equilibrium fails “as often” for substitutes as for complements.

\textsuperscript{52}Gul and Stacchetti demonstrate this maximality result using $n$-dimensional variants of our “simple substitutes” valuation (Fig. 5a). Such valuations are not, of course, in (e.g.) any purely-complements demand type, but Gul and Stacchetti’s description of them as “unit demand” might suggest they should be in any set of interest (cf., the paper’s remark (p.96) “in a sense, the GS [gross substitutes] condition is necessary to ensure existence of a Walrasian equilibrium”). See also Kelso and Crawford (1982). And Azevedo et al.’s (2013, p.286) remark “adding a continuum of consumers . . . eliminates the existence problems created by complementarities” can also be misinterpreted. (However, Bikhchandani and Mamer, 1997, Prop. 3, previously noted the existence of non-substitute preferences which guarantee equilibrium if just two agents each value at most one unit of each good.)
6.5 Multi-Party Matching

Section 6.1 showed Kelso and Crawford’s (1982) model of bipartite matching was a special case of our model.

We can also apply our model to other forms of matching and coalition-formation. The example in Section 6.3 can be interpreted as a multi-player matching problem in which the columns of $D$ are the coalitions of workers that create value: Baldwin and Klemperer (2014, Thm. 6.7) show that, assuming perfectly transferable utility, a stable matching in which no subset of workers can gain from re-matching (that is, an allocation in the core of the game among the workers) corresponds exactly to an equilibrium allocation of workers in our model (in which every worker receives its competitive wage, and no further gains from trade are possible). So, since the demand type is unimodular, it describes a class of multi-player matching problems for which a stable match always exists.

More generally, Baldwin and Klemperer (2014 Sections 2.5, 6.2, 6.3.1) show that any model of coalition formation with perfectly transferable utility can be modelled using our tools; it corresponds to a demand type containing some subset of vectors in $\pm\{0,1\}^n$. And a stable match always exists if this demand type is unimodular (Baldwin and Klemperer, 2014, Cor. 6.11), as it is in the case of the “workers” example above. If the demand type is not unimodular, the methods of Section 5 tell us for what coalitions’ valuations there are stable matchings. Moreover, applying Prop. 6.3 to matching problems shows that stable matchings can be guaranteed for a broader class of preferences than many people assume.\footnote{Consider, e.g., Hatfield and Milgrom’s (2005) statement that “preferences that do not satisfy the substitutes condition cannot be guaranteed always to select a stable allocation”, though the Proposition (p.921) that their introductory remark loosely summarises is, of course, correct in its context.}

Baldwin and Klemperer (in preparation-a) develops the applications to matching and coalition-formation in more detail.\footnote{Since our framework allows us to consider multiple players of each kind, it easily yields results along the lines of Chiappori, Galichon, and Salanié (2014, see also Balinski, 1970).}

6.6 Understanding Individual Demand

Our techniques are powerful tools for understanding individual demand. In particular, Thm. 2.10 tells us that any balanced rational polyhedral complex is the LIP of some quasilinear valuation and conversely. This allows us to explore properties of valuations by drawing and analysing appropriate geometric diagrams without needing to undertake the typically much more challenging task of constructing valuations that generate these diagrams.

Baldwin and Klemperer (2014, especially Section 5; and in preparation-b) further explores the comparative statics of individual demand, in order to better understand demand changes at non-UDR prices.

Unimodularity turns out to have important implications for the structure of individual demand, as well as (as we saw in our discussion of the existence of equilibrium) for aggregate demand. This work also leads to a generalisation of Gul and Stacchetti’s (1999) “Single Improvement Property”.

\footnote{Consider, e.g., Hatfield and Milgrom’s (2005) statement that “preferences that do not satisfy the substitutes condition cannot be guaranteed always to select a stable allocation”, though the Proposition (p.921) that their introductory remark loosely summarises is, of course, correct in its context.}

\footnote{Since our framework allows us to consider multiple players of each kind, it easily yields results along the lines of Chiappori, Galichon, and Salanié (2014, see also Balinski, 1970).}
Related work (joint with Paul Milgrom) uses our framework to help understand implications of different notions of substitutability for indivisible goods that have been suggested in the literature.

6.7 Auction Design

Practical auctions need to restrict the kinds of bids that can be made, thus restricting the preferences that bidders can express. Restricting to a demand type is often natural, since the economic context often suggests appropriate trade-offs between goods. For example, the Bank of England expected bidders to have £1:£1 trade-offs between any pair of the several different “kinds” of money it loaned in the financial crisis. Such trade-offs represent a form of strong substitutes preferences. So the Bank chose auction rules that made it easy for bidders to communicate such preferences, and was also unconcerned about ruling out the expression of other preferences.

Knowing that the bids in an auction must all express preferences of a demand type also clarifies the meaning, and the implications, of the restrictions that have been imposed on the bidders. In particular, the motivation of the Product-Mix Auction is to find competitive equilibrium, given bidders’ and the bid-taker’s reported preferences. Since the Bank of England’s implementation of the Product-Mix auction allows rationing (which makes “goods” divisible) ensuring the existence of equilibrium is not too hard. But in many contexts rationing is less sensible. For example, a too-small piece of radio spectrum may not be useful. Similarly, a government may be interested in offers to build gas-fired plants, nuclear-power stations, wind farms, etc., and these may be indivisible. So results about equilibrium with indivisibilities tell us when Product-Mix Auctions can easily be used.

Our techniques also facilitate the analysis of Product-Mix Auctions. Individuals’ bids in these auctions are aggregated in exactly the same simple way that LIPs are combined to find aggregate demand. This also makes the auctions more “user-friendly”, which is critical for getting them implemented in practice. Moreover, geometric analysis can develop methods for finding equilibrium in new versions of the Product-Mix Auction; this may help resolve problems currently facing regulators such as the U.S. Federal Communications Commission, the U.K.’s Ofcom and the U.K. Department for Energy and Climate Change.

55 Baldwin, Klemperer, and Milgrom (in preparation). This paper also develops the relationship between the existence of equilibrium for substitutes and properties of the Vickrey auction and the core.

56 The different “goods” were long-term loans (repos) against different qualities of collateral.

57 Any strong substitutes preference could be expressed if the Bank’s “Product-Mix” Auctions were augmented by permitting “negative” bids (see Klemperer, 2010, and Baldwin and Klemperer, in preparation-c). Product-mix auctions are “one-shot” auctions for allocating heterogeneous goods. Their equilibrium allocations and prices are similar to those of clock or Simultaneous Multiple-Round Auctions in private value contexts, but they permit the bid-taker to express richer preferences; they are more robust against collusive and/or predatory behaviour; and they are, of course, much faster. (Other work in auction development using geometric techniques includes Candogan et al., 2015, and Lee, 2015.)

58 Restricting to a demand type also permits relatively complex “bids” while still checking that they satisfy the restrictions, since there are easy software solutions to calculate the normal vectors of the LIP for any valuation and so reveal the demand “type” (see Remark A.21).

59 So the updated (2014) implementation of the Bank’s auction also permitted some complements preferences while maintaining the existence of equilibrium.
7 Conclusion

An agent’s demand is completely described by its choices at all possible price vectors. So it can also be described by the divisions between the regions of price space in which the agent demands different bundles, and hence by the vectors that define these divisions. This suggests a natural way of classifying valuations into “demand types”.

Using this classification, together with the duality between the geometric representations of valuations in price space and in quantity space, yields significant new insights into when competitive equilibrium exists.

A demand type’s vectors also encode the possible comparative statics of demand, and we expect many other results can be understood more readily, and developed more efficiently, using our geometric perspective.

Companion papers\textsuperscript{60} use our framework and tools to obtain new results about the existence of stable matchings in multiple-agent matching models; about individual demand; and further develop the Product-Mix Auction implemented by the Bank of England in response to the 2007 Northern Rock bank run and the subsequent financial crisis.

References


\textsuperscript{60}See Baldwin and Klemperer (in preparation-a and b) and Baldwin, Goldberg and Klemperer (in preparation). Preliminary work is in Baldwin and Klemperer (2014).


A Proofs and Additional Examples

Note: results are given below in the order in which it is most convenient to prove them.

A.1 Proofs for Section 2

Proof of Prop. 2.7. Write Π for the set of both cells of the LIP and closures of its UDRs. There are finitely many objects in Π, by definition (as $A \subseteq \mathbb{Z}^n$ is finite). They are defined by a collection of equalities $p \cdot (x - x') = u(x) - u(x')$ and inequalities $p \cdot (x - x') \geq u(x) - u(x')$, so it follows (since $A \subseteq \mathbb{Z}^n$) that they are rational polyhedra. Conversely, a set defined by equalities and inequalities of this form is in Π. A face of a polyhedron in Π is defined by the fact that at least one of these inequalities holds with equality: it is therefore a cell of the LIP. To consider the intersection of two polyhedra $C \neq C'$ in Π, suppose $X, X'$ are such that $C = \{ p \in \mathbb{R}^n : X \subseteq D_u(p) \}$ and $C' = \{ p \in \mathbb{R}^n : X' \subseteq D_u(p) \}$. Then $X \neq X'$ and $C \cap C' = \{ p \in \mathbb{R}^n : X \cap X' \subseteq D_u(p) \}$. Either this is empty, or it is itself a cell of the LIP. Moreover it is easily seen to be a face of both $C$ and $C'$. These facts prove that Π is a polyhedral complex.

If more than one bundle is demanded then a linear equation must be satisfied (as above); as there are only finitely many bundles it is thus generic for only one bundle to be demanded, and hence the closures of UDRs are all $n$-dimensional. Moreover, the union of these closures gives the whole of $\mathbb{R}^n$. So the LIP is equal to the boundary of the closures of the UDRs, which thus is contained in its $(n - 1)$-dimensional components, and any cell of Π of dimension less than $n$ is a cell of the LIP. Thus the set of cells of the LIP forms an $(n - 1)$-dimensional rational polyhedral complex. □

Proof of Lemma 2.8. It is easier to prove these statements in the opposite order. The set $\{ p \in \mathbb{R}^n : D_u(p^o) \subseteq D_u(p) \}$ is non-empty (it contains $p^o$ itself) and so defines a cell, $C'$. If $X \subseteq A$ is such that $C = \{ p \in \mathbb{R}^n : X \subseteq D_u(p) \}$ then $X \subseteq D_u(p^o)$ and so any $p \in C'$ satisfies $X \subseteq D_u(p)$, so $C' \subseteq C$. From Prop. 2.7, we know that $C \cap C'$ is a face of $C$ and of $C'$. However $C \cap C'$ contains a point, $p^o$, that is in the interior of $C$. This is only consistent if $C \cap C' = C$. As $C' \subseteq C$ we conclude $C' = C$.

So $D_u(p^o) \subseteq D_u(p)$ iff $p \in C$. But since this holds for any $p^o \in C^o$, we may reverse the roles of $p$ and $p^o$ to see that $D_u(p^o) = D_u(p)$ if $p \in C^o$. □

Proof of Lemma 2.11. We saw in the proof of Prop. 2.7 above that the set of all cells of the LIP together with the closures of UDRs forms a polyhedral complex, and in particular therefore the intersection of two or more distinct closures of UDRs is a cell of the LIP.

Conversely, given any cell $C$ of the LIP and $p^o \in C^o$, let $X$ be the set of all bundles uniquely demanded at some price in a small neighbourhood of $p^o$. Then $C' = \{ p \in \mathbb{R}^n : X \subseteq D_u(p) \}$ is a cell of the LIP. By continuity of the indirect utility in $p$, that is, of $p \mapsto u(x) - p \cdot x$, every bundle in $X$ is demanded at $p^o$, so $p^o \in C'$ and hence (since $p^o$ is in the interior of $C$) $C \subseteq C'$. So $C$ is a face of $C'$: there exists $v \in \mathbb{R}^n$ such that $p' \cdot v \leq p \cdot v$ for all $p' \in C'$, $p \in C$. Suppose for a contradiction that $C \not\subseteq C'$, so there exists $p' \in C'$ such that $p' \cdot v < p^o \cdot v$. But because the UDRs are dense in $\mathbb{R}^n$, for any $\epsilon > 0$ there exists a UDR price $p \in \mathbb{R}^n$ such that $0 < v \cdot (p - p^o) < \epsilon$. Thus, if
\( \{x\} = D_u(p) \), we must have \( x \in X \). However, again by continuity of indirect utility, \( x \) is only demanded in the closure of the UDR containing \( p \), and by construction, \( p' \) is not in this set. This contradiction completes the proof. \(\square\)

**Proof of Lemma 2.18.** By definition \( \text{conv}(u) \) is weakly concave, so the set bounded above by its graph is convex. By the supporting hyperplane theorem, for every \( x \in \text{conv}(A) \) there exists a supporting hyperplane to this graph at \( x \). Both \( \text{conv}(D_u(p)) \) and \( D_{\text{conv}(u)}(p) \) are equal to the projection of the intersection between the hyperplane and the graph, to its first \( n \) coordinates (by minimality of \( \text{conv}(u) \)), and thus are equal. \(\square\)

**Proof of Lemma 2.13.** Suppose that \( A \) is discrete-convex. It is a standard application of the supporting hyperplane theorem that a function \( u : A \to \mathbb{R} \) is concave iff, for all \( x \in A \), there exists \( p \in \mathbb{R}^n \) such that \( x \in D_u(p) \). As we have defined concavity to require discrete-convexity of domain, this provides the first equivalence. Moreover, since the intersection between a supporting hyperplane and a convex set is always itself convex, discrete-convexity of \( D_u(p) \) also follows.

Conversely, suppose \( D_u(p) \) is discrete-convex for all \( p \). Let \( u' : \text{conv}(A) \cap \mathbb{Z}^n \to \mathbb{R} \) be the restriction of \( \text{conv}(u) \) to \( \text{conv}(A) \cap \mathbb{Z}^n \). Consider any \( x \in \text{conv}(A) \cap \mathbb{Z}^n \). By the previous equivalence, there exists \( p \) such that \( x \in D_u(p) \). As \( \text{conv}(u) = \text{conv}(u') \), it follows by Lemma 2.18 that \( \text{conv}(D_u(p)) = \text{conv}(D_{u'}(p)) \). But then, by assumption, \( x \in D_u(p) \). So the second property holds (and in particular \( A \) is discrete-convex). \(\square\)

**Proof of Lemma 2.14.** Restrict \( u \) to \( \text{conv}(D_u(p)) \cap \mathbb{Z}^n \) to see that Part (1) follows from Lemma 2.13. For (2), recall that \( \text{conv}(u) = \text{conv}(u') \) and so, by Lemma 2.18, \( D_u(p) = \{x\} \) iff \( D_{u'}(p) = \{x\} \), for any bundle \( x \). So \( |D_u(p)| = 1 \) iff \( |D_{u'}(p)| = 1 \). \(\square\)

**Proof of Prop. 2.17.** The “roof” is \( \{(x, \text{conv}(u)(x)) \in \mathbb{R}^{n+1} : x \in \text{conv}(A)\} \) (it is the graph of \( \text{conv}(u) \)). It is clearly the upper boundary, with respect to the final coordinate, of the convex hull of the points \( \{(x, u(x)) \in \mathbb{Z}^n \times \mathbb{R} : x \in A\} \). The set of faces of the “roof” thus has the structure of a polyhedral complex (see e.g. Grünbaum and Shephard, 1969). Moreover, there is a clear bijection between \( \text{conv}(A) \) and the “roof”, which is linear on each of these faces. So the projections of these faces to their first \( n \) coordinates also form a polyhedral complex. Now the result follows from Lemma 2.19 (which is shown in the text to be a consequence of Lemma 2.18, which is proved above). \(\square\)

**Proof of Prop. 2.21.** By Lemma 2.8, the demand set is constant in a cell interior; it is clearly also constant in a UDR. So the correspondence in (1) is well-defined. Moreover, the affine span of \( C_\sigma \) is given by the set of prices \( p' \) such that \( u(x) - p \cdot x = u(x') - p \cdot x' \) for all \( x, x' \in D_u(p) \), i.e. all prices such that \( p' \cdot (x - x') = u(x) - u(x') \) for all such \( x, x' \). If \( \sigma = \text{conv}(D_u(p)) \) is \( k \)-dimensional, there are \( k \) linearly independent vectors of the form \( x - x' \), and so \( k \) linearly independent constraints \( p' \). So \( \dim C_\sigma = n - k \).

For (2), recall from Lemma 2.8 that \( C_\sigma = \{p \in \mathbb{R}^n : D_u(p) \subseteq D_u(p)\} \), where \( p^o \in C_\sigma \). But for such \( p^o \), we show \( D_u(p^o) \subseteq D_u(p) \) iff \( \sigma = \text{conv}(D_u(p^o)) \subseteq \text{conv}(D_u(p)) \). Necessity is obvious, and sufficiency follows from Lemma 2.20: if we assume \( \text{conv}(D_u(p^o)) \subseteq \text{conv}(D_u(p)) \), then any \( x \in D_u(p^o) \subseteq \text{conv}(D_u(p)) \) must satisfy \( x \in D_u(p) \). So \( p \in C_\sigma \) iff \( \sigma \subseteq \text{conv}(D_u(p)) \). Now (3) follows from the combination of (1) and (2).
For (4), we saw already that \( p' \cdot (x - x') = u(x) - u(x') \) for all \( x, x' \in D_u(p) \) and all \( p' \in C_{\sigma} \) where \( \sigma = \text{conv}(D_u(p)) \). Thus \( (p'' - p') \cdot (x' - x) = 0 \) for all \( p', p'' \in C_{\sigma} \), also for any \( x, x' \in \text{conv}(D_u(p)) = \sigma \). Now (5) is immediate from Defns. 2.4 and 2.16(4). □

The following result is used in Section 5 but it is natural to state and prove it here:

**Corollary A.1.** If the demand complex is \( n \)-dimensional then every \( k \)-cell \( C_{\sigma} \) of \( L_u \) has some 0-cell \( C_{\tau} \) in its boundary, with \( \sigma \subseteq \tau \). Moreover if \( x \in \sigma \) but \( x \notin D_u(p_{\sigma}^\circ) \) for \( p_{\sigma}^\circ \in C_{\sigma}^o \), then also \( x \notin D_u(p_{\tau}^\circ) \) for \( p_{\tau}^\circ \in C_{\tau}^o \).

**Proof.** If \( \sigma \) is an \((n-k)\)-cell of an \( n \)-dimensional demand complex, then \( \sigma \) is contained in an \( n \)-cell \( \tau \). So \( C_{\tau} \) is a 0-cell of the LIP and \( C_{\tau} \subseteq C_{\sigma} \) (Prop. 2.21(3)). By Prop. 2.21(1) we know \( \sigma = \text{conv}(D_u(p_{\sigma}^\circ)) \), so by Lemma 2.20, \( x \notin D_u(p_{\tau}^\circ) \). □

**A.2 Examples for Sections 2 and 3**

**Additional Discussion of Figs. 2 and 3.** Observe that if the “black bundles’”s value was greater, so the corresponding bar in Figs. 2b and 2c just touched the roof, then it would still not be a vertex of Fig. 3a but it would be demanded at the price corresponding to the wavy-shaded 0-cell (that is, (1, 2)) that is a vertex of the LIP. And if it had an (even) higher valuation (so “poked through” the current roof), then the corresponding demand complex point would become a vertex, and the corresponding LIP 0-cell would “open up” to form a new UDR corresponding to the range of prices at which the bundle (1, 1) would then be demanded.

To find the exact LIP of Fig. 2a’s valuation using the demand complex of Fig. 3a, we compare the values of bundles in adjacent UDRs: note the valuations of bundles (1, 0) and (0, 1) show that the dotted 0-cell of the LIP is at \( p = (4, 8) \), since 4 and 8 are the prices below which the agent will first buy any of goods 1 and 2, respectively, when the other good’s price is very high. And the wavy-shaded 0-cell must be at (1, 2) since \( 10 - 8 = 2 \) is the incremental value of a second unit of good 2, when the agent has no unit of good 1, and \( 11 - 10 = 1 \) would be the incremental value from then adding a unit of good 1, etc.

**Example A.2.** For a fixed \( A \), it is easy to draw every possible demand complex and so obtain every possible combinatorial type of LIP, thus enumerating all possible “essentially-different” structures of demand. We do this for \( A = \{0, 1\}^2 \) in Fig. 11.

![Figure 11: All the possible demand complexes, and examples of their corresponding combinatorial types of LIP, when \( A = \{0, 1\}^2 \).](image-url)
It is not hard to see that Fig. 11a applies when \( u(0, 0) + u(1, 1) < u(1, 0) + u(0, 1) \), so represents substitutes; Fig. 11b applies when \( u(0, 0) + u(1, 1) = u(1, 0) + u(0, 1) \), so is additively separable demand; and Fig. 11c applies when \( u(0, 0) + u(1, 1) > u(1, 0) + u(0, 1) \), so is complements. (See Section 3.2 for these distinctions). Importantly, it is clear that these are the only possibilities.

Observe that Fig. 11b can be seen as a limit of Fig. 11a (or, equivalently, Fig. 11c). In the LIP, the two 0-cells become arbitrarily close and then coincide in the limit; in quantity space, the faces of the “roof” tilt until they are coplanar, meaning that the demand complex edge distinguishing them disappears.

Likewise, any demand complex in which the subdivision is not maximal (that is, additional valid \((n-1)\)-faces could be added) can be recovered by deleting \((n-1)\)-faces from some demand complex whose subdivision is maximal; the corresponding LIP is a limit (or “degeneration”). Even for larger domains than \( A = \{0, 1\}^2 \), we can go on to enumerate all those combinatorial types of demand for which the demand complex subdivision is maximal, knowing we can recover the remainder as their limits, as in the following example.

**Example A.3.** For \( A = \{0, 1\} \times \{0, 1, 2\} \), we list the maximal subdivisions which correspond to LIPs in Fig. 12.

![Figure 12: All the possible demand complexes with maximal subdivision, and examples of their corresponding combinatorial types of LIP, when \( A = \{0, 1\} \times \{0, 1, 2\} \).](image)

**Example A.4.** To illustrate why the conditions for indivisible goods to be substitutes are so restrictive, consider a consumer who regularly makes three kinds of trips: journey A can be made only by bus or train; journey B can be made only by car or train; journey C can be made only by car or bus. Thought of as divisible goods, the three modes of transport are all mutual substitutes. But if the price of *either* bus tickets or train tickets is slightly raised, a consumer might buy a car and reduce her use of both forms of public transport, which are therefore locally complements—that is, the car takes the role of good 2 in the situation pictured in Fig. 4. \(^{61}\)

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\(^{61}\)Even if all goods are mutual substitutes, there can never be trade-offs between more than two of them across the whole of a facet. One mutual substitute might trade-off against two others at prices where more than one facet meet, if at least one of those facets has weight greater than 1. For example,
A.3 Proofs and Additional Examples for Section 4

Proof of Fact 4.8. That (2) is equivalent to unimodularity follows from Cassels (1971) Lemma I.1 and Cor. I.3. For (1)⇒(2) consider a parallelepiped $P$ whose vertices are $y + \sum_{j=1}^{s} a_j v^j$ for $a_j \in \{0, 1\}$. If $z$ is a non-vertex integer point in $P$, then $z - y$ exhibits the failure of (2). Conversely, if failure of (2) is exhibited by an integer $\sum_{j=1}^{s} b_j v^j$ where $b_j$ are not all integers, then $y + \sum_{j=1}^{s} a_j v^j$ exhibits failure of (1), where $a_j$ is the non-integer part of $b_j$ in each case. □

Lemma A.5. For valuations $u^1, \ldots, u^r$, write $\sigma^j = \text{conv}(D_{u^j}(p))$ for $j = 1, \ldots, r$ and $\sigma = \text{conv}(D_U(p))$, where $D_U(p)$ is their aggregate demand.

(1) $\sigma = \sigma^1 + \cdots + \sigma^r$ and $K_\sigma = K_{\sigma^1} + \cdots + K_{\sigma^r}$.

(2) If the LIPs at $p$, they are transverse at $p$ iff $K_\sigma = K_{\sigma^1} + \cdots + K_{\sigma^r}$.

Proof. (1): by definition $D_U(p) = D_{u^1}(p) + \cdots + D_{u^r}(p)$, from which $\sigma = \sigma^1 + \cdots + \sigma^r$ easily follows from elementary properties of polytopes (see e.g. Cox et al 2005, Section 7.4, Exercise 3). Thus, by Defn. 4.15, $K_\sigma = K_{\sigma^1} + \cdots + K_{\sigma^r}$.

(2): First suppose $r = 2$ and write $k^j = \dim \sigma^j = \dim K_{\sigma^j}$, and $k = \dim \sigma = \dim K_\sigma$. By Lemma A.5 it follows that $k = k^1 + k^2 - \dim(K_{\sigma^1} \cap K_{\sigma^2})$. But also, $\dim(C_{\sigma^1} + C_{\sigma^2}) = \dim C_{\sigma^1} + \dim C_{\sigma^2} - \dim C_\sigma = (n-k^1) + (n-k^2) - (n-k)$. By Defn. 4.11, the intersection is transverse at $p$ iff this is equal to $n$, that is, iff $k^1 + k^2 = k$. So we conclude that the intersection is transverse at $p$ iff $K_{\sigma^1} \cap K_{\sigma^2} = \{0\}$, which, together with (1), is the definition of $K_\sigma = K_{\sigma^1} \oplus K_{\sigma^2}$. The $r \geq 3$ now follows: as in Defn. 4.11, we check for transversality incrementally, so applying the $r = 2$ case yields the result. □

Proof of Lemma 4.16 This is a standard consequence of Lemma A.5(2). Every vector $x$ in $K_\sigma$ can be written as a sum of a vector $x^1 \in K_{\sigma^1}$ and a vector $x^2 \in K_{\sigma^2}$. If we also write $x = x^1 + x^2$, where $x^j \in K_{\sigma^j}$, $j = 1, 2$, then $x^1 - x^j = x^2 - x^j \in K_{\sigma^1} \cap K_{\sigma^2} = \{0\}$ and so $x^1 - x^j = x^2 - x^j = 0$.

Example A.6. (The “hotel rooms” example, and the argument of Section 4.2.5.) It helps intuition to see which parts of the argument of Section 4.2.5 are, and are not, valid for our simple two-goods substitutes, $u^1(x_1, x_2) = \max\{40x_1, 30x_2\}$ (Fig. 5a), and complements, $u^2(x_1, x_2) = \min\{50x_1, 50x_2\}$ (Fig. 5b), valuations, for which the failure of equilibrium was discussed in Section 4.2.2.

Consider the (transverse) intersection price $p = (30, 20)$, and the bundle $y = (1, 1) \in \sigma = \text{conv}(D_U(p))$.

Here $\sigma$ is the central square 2-cell in Fig. 6c and, for this example, we can easily see that the corresponding cells, $\sigma^s = \text{conv}(D_{u^1}(p))$ and $\sigma^c = \text{conv}(D_{u^2}(p))$, of Figs. 6a and 6b, respectively, are the two diagonal 1-cells. (The general procedure for identifying $\sigma^s$ and $\sigma^c$ is to examine the LIP intersection at $(30, 20)$ in Fig. 5c, hence to identify the relevant cells of the LIPs in Figs. 5a, 5b and so the relevant cells in Figs. 6a, 6b.)

We have $K_\sigma = \mathbb{R}^2$, and the edges of $\sigma$ are $\{(1, -1), (1, 1)\}$ (see Fig. 6c), and in this case the whole set of edges is needed to form a basis of $K_\sigma$. And (see Figs. 5a, 5b) $K_\sigma$ an agent might switch 2 units of some good A for 1 each of two other goods, B and C, which it treats as indistinguishable, in the intersection of all three weight-2 facets where the agent switches between two of the three goods.

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and \( K_{\sigma^c} \) are the sets of scalar multiples of \((1,-1)\) and \((1,1)\), respectively (and these vectors give a basis for each).

Now say, for example, \( x = (1,0) \in \sigma \). We can write \( y - x = (0,1) \) as \( \lambda_1(1,-1) + \lambda_2(1,1) \) by choosing \( \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2} \). So \( y - x = z^* + z^c \) in which \( z^* = \left( \frac{1}{2}, \frac{1}{2} \right) \) and \( z^c = \left( \frac{1}{1}, \frac{1}{2} \right) \). (These are not integer bundles because unimodularity fails in this example.)

On the other hand, we use \( \sigma = \sigma^* + \sigma^c \) to write \( y = y^* + y^c \) with \( y^* \in \sigma^* \) and \( y^c \in \sigma^c \). We can see from Figs. 6a, 6b that the unique way to do this is \( y^* = y^c = \left( \frac{1}{2}, \frac{1}{2} \right) \). Also, since \( x \in D_U(p) \), we can write \( x = x^* + x^c \) in which \( x^* \) and \( x^c \) are integer bundles. In fact, \( x^* = (1,0) \in D_{\sigma^*}(p) \) and \( x^c = (0,0) \in D_{\sigma^c}(p) \), as can be seen by using Fig. 5c to see that the UDR of \( x \) is the region above the price \((30,20)\), hence the relevant UDRs in Figs. 5a and 5b are \((1,0)\) and \((0,0)\), respectively. So \( y^* - x^* = \left( -\frac{1}{2}, \frac{1}{2} \right) \) and \( y^c - x^c = \left( \frac{1}{2}, \frac{1}{2} \right) \).

The intersection is transverse at \( p \) and so, as predicted, \( y^* - x^* = z^* \), and \( y^c - x^c = z^c \). However, in this example, the set of edges of \( \sigma \), that is, \( \{(1,-1),(1,1)\} \) is not unimodular, with the result that \( z^* \) and \( z^c \), and hence also \( y^* \) and \( y^c \), are not integer bundles, so the method does not demonstrate equilibrium. (Indeed, in this case it demonstrates the failure of equilibrium.)

**Fact A.7** (see, e.g. Cassels, 1971, Lemma I.2). A set of \( s \leq n \) linearly independent vectors in \( \mathbb{Z}^n \) are unimodular iff, among the determinants of all the \( s \times s \) matrices consisting of \( s \) rows of the \( n \times s \) matrix whose columns are these \( s \) vectors, the greatest common factor is 1.

### A.4 Proofs and Additional Examples for Section 5

**Example A.8** (The “hotel rooms” example with weight 2 facets–further discussion of Section 5.1.2.). We can understand “mid-point bundles” such as \((1,1)\) and \((1,3)\) in Fig. 8c as being reached by starting from a vertex bundle, and then changing supply to move one “step” along an edge. This is exactly as a change in demand part-way along a diagonal edge is allowed in Figs. 8a and 8b. Likewise the “central bundle” \((2,2)\) can be reached by a combination of these part-way diagonal steps, as shown by the dashed grey lines. Meanwhile the remaining four bundles (the black grid points in Fig. 8c) cannot be reached by any such moves, so are not demanded.

As in the discussion of Fig. 6c in Section 4.2.2, this illustrates the relevance of Fact 4.8(2). Unimodularity is equivalent to the ability to create any vector in the space spanned by the vectors of the demand type as an integer combination of vectors of the demand type. \( U^2 \) is not of a unimodular demand type; there is therefore the possibility that some bundles cannot be reached using combinations of the edge vectors of the relevant demand complex cell; and indeed these four bundles are unreachable.

Fig. 8d can be understood similarly to Fig. 8c. Now all bundles can be reached via changes of demand that take the agents part-way along their individual demand complex cells (so are edge vectors of the relevant aggregate demand complex cell; these are again denoted by dashed lines). So equilibrium always exists in this case.

**Proof of Prop. 5.2.** Necessity of this condition follows from Lemma 2.13.

Conversely, suppose that equilibrium fails for some relevant supply. By Lemma 4.7, there exists a price \( p \) in \( L_u \cap L_u \) such that \( D_U(p) \) is not discrete-convex. So there
exists \( x \in \mathbb{Z}^n \) such that \( x \in \sigma := \text{conv}(D_U(p)) \), but \( x \notin D_U(p) \). By Lemma 2.8 the demand set is constant in the interior of the corresponding LIP cell \( C_\tau \), and thus both individual demand sets must be constant in the interior of \( C_\sigma \). So \( C_\sigma \subseteq \mathcal{L}_u^1 \cap \mathcal{L}_u^2 \).

Since the aggregate demand complex is \( n \)-dimensional, we may apply Cor. A.1: there exists a 0-cell \( C_\tau \) of \( \mathcal{L}_U \) such that \( \sigma \subseteq \tau \) and which also satisfies \( x \notin D_U(p_\tau) \) for \( p_\tau \in C_\tau \). Moreover, \( C_\tau \subseteq C_\sigma \) (by Prop. 2.21(3)) and so \( C_\tau \subseteq \mathcal{L}_u^1 \cap \mathcal{L}_u^2 \).

**Fact A.9.** The quantity we have defined as \( \Gamma_k^n(A^1, A^2) \) is the \( n \)-dimensional mixed volume of \( k \) copies of \( \text{conv}(A^1) \) with \((n-k)\) copies of \( \text{conv}(A^2) \). In particular:

1. \( \Gamma_k^n(A^1, A^2) \in \mathbb{Z}_{\geq 0} \) for finite \( A^1, A^2 \subseteq \mathbb{Z}^n \).
2. \( \Gamma_k^n(A, A) = n! \text{vol}_n \text{conv}(A) \) for \( k = 0, \ldots, n \).
3. If \( A^j = \{ x \in \mathbb{Z}^n : \sum x_i \leq d_j \} \) for \( j = 1, 2 \) then \( \Gamma_k^n(A^1, A^2) = d_1^k d_2^{n-k} \).
4. \( \Gamma^n(A^1, A^2) = 0 \) if \( \text{dim conv}(A^1 + A^2) < n \).

**Proof.** Cox et al. (2005, Thm. 7.4.12.d) shows that \( \Gamma_k^n(A^1, A^2) \) is the mixed volume described. (1) is now Cox et al. (2005, Exercise 7.9.a). (2) is Cox et al. (2005, Exercise 7.7.b). (3) is an elementary calculation. (4) is clear by Defn. 5.11.

**Lemma A.10.** Fix \( p \in \mathbb{R}^n \) and write \( u^* \) for the restriction of \( u \) to \( D_\omega(p) \). Then there exists \( \delta > 0 \) such that \( D_\omega(p^*) = D_\omega^*(p^*) \) for all \( p^* \in B_\delta(p) = \{ p^* \in \mathbb{R}^n : \| p^* - p \| < \delta \} \).

In particular the LIPs coincide for these prices.

**Proof.** There exists some \( \epsilon > 0 \) such that \( u(x) - p \cdot x > u(y) - p \cdot y + \epsilon \) for every \( x \in D_\omega(p), \ y \notin D_\omega(p) \). So if we choose \( \delta < \frac{\epsilon}{\|x\|} \) for all \( x \in D_\omega(p) \) then it is easy to show that \( u(y) - p^* \cdot y < u(x) - p^* \cdot x \) for all \( p^* \in B_\delta(p), \ x \in D_\omega(p), \ y \notin D_\omega(p) \). Thus, at prices \( p^* \in B_\delta(p) \), only bundles in \( D_\omega(p^*)^* \) might be demanded; \( D_\omega(p^*) = D_\omega^*(p^*) \).

**Lemma A.11.** Suppose \( \mathcal{L}_{u^1} \) and \( \mathcal{L}_{u^2} \) have an intersection 0-cell \( C \) at \( p \). Suppose \( v \in \mathbb{R}^n \) is such that \( \mathcal{L}_{u^1} \) and \( \{ \epsilon v \} + \mathcal{L}_{u^2} \) intersect transversally for sufficiently small \( \epsilon > 0 \), and write \( C \) for the set of their intersection 0-cells emerging from \( C \). Write \( U, U_\epsilon \) for the respective aggregate valuations, and write \( x^1, x^2 \) for the bundles such that \( \{ x^1 = D_\omega(p + \epsilon v), \ x^2 = D_\omega(p - \epsilon v) \} \). Then the demand complex cell \( \sigma C \in \Sigma_U \) corresponding to \( C \) is equal to \( \{ x^2 \} + \sigma_1 \cup \{ x^1 \} + \sigma_2 \cup \{ \sigma_{C'} : C' \subseteq C \} \), where \( \sigma_1 = \text{conv}(D_w(p)) \) and \( \sigma_{C'} \) is the demand complex cell in \( \Sigma_{U_\epsilon} \) corresponding to \( C' \subseteq C \).

**Proof.** Use the notation of Lemma A.10; using that lemma, choose \( \delta > 0 \) sufficiently small so that, for all \( p^* \in \mathbb{R}^n \), both \( D_\omega^*(p^*) = D_\omega^*(p')^* \) and \( D_\omega^*(p^*) = D_\omega^*(p')^* \).

For any \( \epsilon > 0 \), we write \( u^2_\epsilon \) for the valuation given by \( u^2_\epsilon(x) = u^2_\epsilon(x) + \epsilon v \cdot x \). Since \( B_\delta(p) \) is topologically open, and since \( \mathcal{L}_{u^2_\epsilon} = \{ \epsilon v \} + \mathcal{L}_{u^2} \), we can choose \( \epsilon \) sufficiently small that \( D_\omega^*(p') = D_\omega^*(p') \) for all \( p^* \in B_\delta(p) \). Moreover, by Defn. 5.8, emerging 0-cells get arbitrarily close to \( C \) for arbitrarily small \( \epsilon \), we can choose \( \epsilon \) sufficiently small that all intersection 0-cells emerging from \( C \) are contained in \( B_\delta(p) \). Now, it is sufficient to prove the result for \( \mathcal{L}_{u^1}, \mathcal{L}_{u^2}, \) and \( \{ \epsilon v \} + \mathcal{L}_{u^{2\epsilon}} = \mathcal{L}_{u^{2\epsilon}} \).

The cell \( \sigma C \) is, by definition, the convex hull of the domain of the aggregate valuation \( U^* \) of \( u^{1\epsilon} \) and \( u^{2\epsilon} \). Hence it is also the convex hull of the domain of the aggregate \( U^* \) of \( u^{1\epsilon} \) and \( u^{2\epsilon} \). So it is equal to the union of the top-dimensional cells in the demand complex \( \Sigma_{U^\epsilon} \). So \( \sigma C \) is the union of the demand complex cells corresponding to 0-cells of
Proof of Lemma 5.9. Write $\sigma^1$, $\sigma^2$ and $\sigma$, respectively, for the demand complex cells at $p$ of valuations $u^1$, $u^2$ and their aggregate valuation. For any $v$ and $\epsilon$, if there exist no intersection 0-cells emerging from $C$ then, by Lemma A.11, $\text{vol}_n(\sigma^1) + \text{vol}_n(\sigma^2) \geq \text{vol}_n(\sigma)$. But this is impossible since $\sigma = \sigma^1 + \sigma^2$, since both $\sigma^1$ and $\sigma^2$ contain more than one element, and since $\text{vol}_n(\sigma) \neq 0$.

Fact A.12. $[\Lambda : \Lambda']$ is the number of disjoint “cosets” $\{v\} + \Lambda'$ where $v \in \Lambda$.

Proof of Facts 5.15 and A.12. Cassels (1971) assumes that all lattices $\Lambda \subseteq \mathbb{Z}^n$ have rank $n$. To adapt his results to our conventions, first fix an invertible matrix $G_A$ such that $G_A\Lambda = \mathbb{Z}^k$. Then $G_A\Lambda'$ is a rank-$k$ sublattice of $\mathbb{Z}^k$. From our definitions it is clear that $[G_A\Lambda : G_A\Lambda'] = [\Lambda : \Lambda']$. Now Cassels (1971) Lemma 1.1 gives Fact A.12. And Fact 5.15(1) follows: if $v, w \in \Delta_{\Lambda'} \cap \Lambda$ but $\{v\} + \Lambda' = \{w\} + \Lambda'$ then $v - w \in \Lambda'$, so $v - w = \sum \alpha_i v^i$ where $v^i$ are our basis for $\Lambda'$, and $\alpha_i \in \mathbb{Z}$; then $w \in \Delta_{\Lambda'}$ and $v = w + (v - w) \in \Delta_{\Lambda'}$ are consistent only if $v = w$; the converse is argued similarly. Fact 5.15(2) follows immediately from Fact A.12. Fact 5.15(3) is given by Cassels (1971, p69). Fact 5.15(4) follows from (2) and Fact 4.8(1).

Facts A.13. Write $K_\Lambda$ for the linear span in $\mathbb{R}^n$ of a lattice $\Lambda \subseteq \mathbb{Z}^n$, so $K_\Lambda$ the set of all linear combinations of finite subsets of $\Lambda$.

1. $K_\Lambda$ is the minimal vector subspace of $\mathbb{R}^n$ containing $\Lambda$.
2. If $\Lambda' \subseteq \Lambda$ is a sublattice, then $K_{\Lambda'} = K_\Lambda$ iff $\text{rank}(\Lambda') = \text{rank}(\Lambda) = \dim K_\Lambda$.
3. If $\Lambda', \Lambda \subseteq \mathbb{Z}^n$ are lattices then $\Lambda' + \Lambda$ is a lattice. If additionally $K_{\Lambda'} \cap K_\Lambda = \{0\}$ then a basis for $\Lambda'$ and a basis for $\Lambda$ together give a basis for $\Lambda' + \Lambda$.

Proof. (1) is clear. An integer basis for $\Lambda$ provides a set spanning $K_\Lambda$, and these are linearly independent by Cassels (1971, Thm. III.VI; our definition of lattices is in line with the characterisation given there), so (2) follows. (3) is also easy to see.

Proof of Lemma 5.14. From Lemma A.5(1), $K_{\sigma^1} + \cdots + K_{\sigma^r} = K_\sigma$. It follows that $K_{\sigma^1}, \ldots, K_{\sigma^r} \subseteq K_\sigma$, and so $\Lambda_{\sigma^1}, \ldots, \Lambda_{\sigma^r} \subseteq \Lambda_\sigma$. Thus, as $\Lambda_\sigma$ is additively closed, $\Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r} \subseteq \Lambda_\sigma$. Moreover $\Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r}$ is a sublattice by Fact A.13(3).

The linear span of $\Lambda_\sigma$ is $K_\sigma$. The linear span of $\Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r}$ contains $K_{\sigma^1} + \cdots, K_{\sigma^r}$ and so it contains their sum; on the other hand $\Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r} \subseteq K_{\sigma^1} + \cdots + K_{\sigma^r}$ and the latter is linear; so $K_{\sigma^1} + \cdots + K_{\sigma^r}$ is the linear span of $\Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r}$. Since, again, $K_{\sigma^1} + \cdots + K_{\sigma^r} = K_\sigma$, the result follows by Fact A.13(2).

Proof of Thm. 5.16. Part (1) is very similar to the proof of “sufficiency” for Thm. 4.3, given in Section 4.2.5. Let $x \in \sigma \cap \mathbb{Z}^n$. By Lemma A.5, $\sigma = \sigma^1 + \cdots + \sigma^r$ and so $x = x^1 + \cdots + x^r$ where $x^j \in \sigma^j$ for $j = 1, \ldots, r$. We wish to show that each $x^j \in \mathbb{Z}^n$; then $x^j \in D_{w^j}(p)$ by concavity of $w^j$ (Lemma 2.13) and so $x \in D_{w}(p)$.

Fix $y \in D_{w}(p)$, so $y = y^1 + \cdots + y^r$ where $y^j \in D_{w^j}(p) \subseteq \mathbb{Z}^n$ for $j = 1, \ldots, r$. Then $y - x \in \Lambda_\sigma$. But $\Lambda_\sigma = \Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r}$ (by assumption and Fact 5.15(2)) and so $y - x = z^1 + \cdots + z^r$, where $z^j \in \Lambda_{\sigma^j} \subseteq \mathbb{Z}^n$. But now $\Lambda_{\sigma^1} + \cdots + \Lambda_{\sigma^r} = (y^1 - x^1) + \cdots + (y^r - x^r)$. By transversality of the intersection and Lemma A.5(2), $y^j - x^j = z^j$ for $j = 1, \ldots, r$. Thus $x^j = y^j - z^j \in \mathbb{Z}^n$ for $j = 1, \ldots, r$, as required.
We prove Part (2) in the case when $\dim \sigma' = 2$. The case in which $\dim \sigma' = 1$ may be seen from the following argument by ignoring the role of $\sigma'$. So suppose $\dim \sigma' = 2$.

Assume that $0 \in \sigma'$ for $j = 1, \ldots, r$ (otherwise the following arguments are simply augmented by a fixed shift). If $\dim \sigma' = 0$ for any $j = 2, \ldots, r$ then $\Lambda_{\sigma'} = \{0\}$ and inclusion of $\sigma'$ has no effect on $\sigma$. So we assume $\dim \sigma' = 1$ for all such $j$.

For $j = 2, \ldots, r$, fix a minimal integer non-zero vector $v^j \in \sigma'$. In each case this vector then forms a basis for the corresponding lattice $\Lambda_{\sigma'}$.

For $j = 1$ we will need to find a basis for $\Lambda_{\sigma'}$ consisting of vectors $v^0, v^1$ which are actually contained inside $\sigma'$ (note that this is not immediate). Start by taking $w^0, w^1 \in \sigma'$ which are linearly independent integer vectors. If these are a basis for $\Lambda_{\sigma'}$, we are done. If not, they span a sublattice $\Lambda'$ of $\Lambda_{\sigma'}$, such that $[\Lambda_{\sigma'} : \Lambda'] > 1$, and so there must exist $w \in \Lambda_{\sigma'}$ which is a non-vertex point of $\Delta_{\Lambda'}$. Then $w = \alpha^0 w^0 + \alpha^1 w^1$ with $\alpha^0, \alpha^1 \in [0, 1)$. If $\alpha^0 + \alpha^1 < 1$ then we fix $w^2 := w$; if $\alpha^0 + \alpha^1 > 1$ then let $w^2 = w^0 + w^1 - w$. In either case now $w^2 = \beta^0 w^0 + \beta^1 w^1$ with $\beta^0 + \beta^1 \leq 1$. As $\sigma'$ is a convex set we conclude that $w^2 \in \sigma'$.

Recalling by definition that $w \notin \Lambda'$, we know $w^2$ is distinct from $w^0, w^1, 0$. So $w^2$ is a non-vertex point of the convex hull $\Delta^0$ of $w^0, w^1$. Hence the convex hull $\Delta^1$ of $0, w^1, w^2$ has strictly smaller area than $\Delta^0$. Moreover, the parallelepipeds spanned by $w^0, w^1$ and by $w^1, w^2$ have areas equal to twice the areas of $\Delta^0, \Delta^1$, respectively. So, if $\Lambda_2$ is the sublattice of $\Lambda_{\sigma'}$ spanned by $w^1, w^2$, then $[\Lambda_{\sigma'} : \Lambda_2] > [\Lambda_{\sigma'} : \Lambda']$.

As all subgroup indices are positive-integer-valued, after a finite number of repetitions of this process, the subgroup index is 1, and hence (by Fact 5.15(2)) we have obtained vectors $v^0, v^1 \in \sigma'$ which are a basis of $\Lambda_{\sigma'}$, as required.

Now, by Fact 5.15(2), there exists a bundle $x \in \Lambda_{\sigma'}$, $x \notin \Lambda_{\sigma'} + \cdots + \Lambda_{\sigma'}$. And by Fact A.13(3), our identified vectors $v^0, v^1, \ldots, v^r$ are an integer basis for this sublattice of $\Lambda_{\sigma'}$. By Lemma 5.14 and Fact A.13(2), they are thus a (vector space) basis for $K_{\sigma'}$, so we can write $x$ as a real-valued linear combination of these vectors. Moreover, since subtracting integer multiples of the $v^j$ from $x$ yields a new element of $\Lambda_{\sigma'}$, we can assume that $x$ is in a fundamental parallelepiped of the sublattice with respect to this basis. So we can write $x = \sum_{j=0}^r \alpha^j v^j$ with $\alpha^j \in [0, 1]$. Furthermore, we can assume that $\alpha^0 + \alpha^1 \leq 1$: if $\alpha^0 + \alpha^1 > 1$ then replace $x$ with $\sum_{j=0}^r \alpha^j v^j - x \in \Lambda_{\sigma'}$.

Now $\alpha^0 v^0 + \alpha^1 v^1 \in \sigma'$ and, for all $j$, also $\alpha^j v^j \in \sigma^j$. So, $x \in \sigma^1 + \cdots + \sigma^r = \sigma = \text{conv}(D_U(p))$. Moreover, $x \in \Lambda_{\sigma'} \subseteq \mathbb{Z}^{n}$. But, by assumption, $x \notin \Lambda_{\sigma'} + \cdots + \Lambda_{\sigma'}$, and so $x \notin D_{\alpha^1}(p) + \cdots + D_{\alpha^r}(p) = D_U(p)$.

**Example A.14.** Suppose $n = 4$ and that $D_{\alpha^1}(p) = \{e^1, e^2, e^1 + e^2, 2e^1 + e^2, e^1 + 2e^2\}$ while $D_{\alpha^2}(p) = \{e^3, e^4, e^3 + e^4, 2e^3 + e^4, e^3 + 2e^4\}$ (as these sets are convex, for any $p$ we can construct explicit $u^1, u^2$ with these properties using Prop. 2.21 and Thm. 2.10). Write $\sigma^j = \text{conv}(D_{\omega^j}(p))$ for $j = 1, 2$ and $\sigma$ for the aggregate. Then $\Lambda_{\alpha^1} = \{v \in \mathbb{Z}^4 : n_1 = n_2 = 0\}$ and $\Lambda_{\alpha^2} = \{v \in \mathbb{Z}^4 : n_3 = n_4 = 0\}$, while $\Lambda_{\sigma} = \mathbb{Z}^4$, so $[\Lambda_{\sigma} : \Lambda_{\alpha^1} + \Lambda_{\sigma}] = 1$, and $D_U(p)$ is discrete-convex (as may also be checked directly).

However, the facets to Agent 1’s demand at $p$ have normal vectors $(1, 1, 1, 0)$ and $(1, -1, 0, 0)$, while the facets normals for Agent 2 are $(0, 0, 1, 1)$ and $(0, 0, 1, -1)$. This set is not unimodular, and so equilibrium is not assured by application of Thm. 4.3.

**Proof of Prop. 5.17.** This is provided by Exs. A.15 and A.16. □
Example A.15. Let \( n = 4 \). Agent 1 has valuation \( u^1(0, 0, 0, 0) = 0, u^1(1, 1, 0, 0) = 6 \), \( u^1(0, 0, 1, 1) = 6 \). So for prices in the LIP 2-cell \( \{ \mathbf{p} \in \mathbb{R}^4 : p_1 + p_2 = 6, p_3 + p_4 = 6 \} \), Agent 1 is indifferent between these three bundles; its corresponding demand complex cell is \( \sigma^1 = \text{conv}((0, 0, 0, 0), (1, 1, 0, 0), (0, 0, 1, 1)) \).

Agent 2 has valuation \( u^2(0, 0, 0, 0) = 0, u^2(0, 1, 1, 0) = 9, u^2(4, 0, 0, 1) = 6 \). So for prices in the LIP 2-cell \( \{ \mathbf{p} \in \mathbb{R}^4 : p_2 + p_3 = 9, 4p_1 + p_4 = 6 \} \), Agent 2 is indifferent between these three bundles; its corresponding demand complex cell is \( \sigma^2 = \text{conv}((0, 0, 0, 0), (0, 1, 1, 0), (4, 0, 0, 1)) \).

These two individual LIP 2-cells intersect at \( \mathbf{p} = (1, 5, 4, 2) \). At this price, the individual demand complex cells are \( \sigma_1 \) and \( \sigma_2 \) as above. The aggregate demand complex cell \( \sigma \) is 4-dimensional and so \( \Lambda_\sigma = \mathbb{Z}^4 \). Meanwhile \( \Lambda_{\sigma_1} \) and \( \Lambda_{\sigma_2} \) are rank-2 lattices, and we check that the non-zero vectors we already know in each lattice do give a basis in each case, by checking that the sets \( \{(1, 1, 0, 0), (0, 0, 1, 1)\} \) and \( \{(0, 1, 1, 0), (4, 0, 0, 1)\} \) are unimodular (use Fact A.7). Thus the union of these sets is a basis for \( \Lambda_{\sigma_1} + \Lambda_{\sigma_2} \) (use Fact A.13(3)). But this union is not unimodular, and calculating its determinant tells us (by Fact 5.15(3)) that \( [\Lambda_\sigma : \Lambda_{\sigma_1} + \Lambda_{\sigma_2}] = 3 \).

So, by Fact 5.15(1), we now know that there are exactly 2 non-vertex integer points to a fundamental parallelepiped of \( \Lambda_{\sigma_1} + \Lambda_{\sigma_2} \). In terms of the basis vectors, these are:

\[
\frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \tag{1}
\]

and

\[
\frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \tag{2}
\]

These expressions show clearly that \((2, 1, 1, 1)\) can be decomposed to give a part in \( \sigma_2 \) and a part not in \( \sigma_1 \), whereas \((3, 1, 1, 1)\) can be decomposed to give a part in \( \sigma_1 \) and a part not in \( \sigma_2 \). Moreover, by linear independence of this set of four vectors, these are the only possible decompositions into sums of bundles in the affine spans of \( \sigma_1, \sigma_2 \). So neither is in \( \sigma_1 + \sigma_2 = \sigma = \text{conv} D_U(1, 5, 4, 2) \). But this means that the only integer vectors in \( \text{conv} D_U(1, 5, 4, 2) \) are in fact in \( D_U(1, 5, 4, 2) \) itself: it is discrete-convex.

Example A.16. This is a modification of the previous example. This time \( D_{u^1}(\mathbf{p}) = \{(0, 0, 0, 0), (1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1)\} \) while \( D_{u^2}(\mathbf{p}) = \{(0, 0, 0, 0), (4, 0, 0, 1)\} \). So this time \( \dim \sigma^1 = \dim \text{conv}(D_{u^1}(\mathbf{p})) = 3 \) and \( \dim \sigma^2 = \dim \text{conv}(D_{u^2}(\mathbf{p})) = 1 \). Again the aggregate demand complex cell \( \sigma \) is 4-dimensional and so \( \Lambda_\sigma = \mathbb{Z}^4 \). The non-zero vectors we know in \( \Lambda_{\sigma_1} \) and \( \Lambda_{\sigma_2} \) again give bases, so again, \( [\Lambda_\sigma : \Lambda_{\sigma_1} + \Lambda_{\sigma_2}] = 3 \). Moreover, the interior points to a fundamental parallelepiped of \( \Lambda_{\sigma_1} + \Lambda_{\sigma_2} \) are exactly those given in Ex. A.15. The expressions there in terms of these same basis vectors show that in neither case do these vectors decompose to give a part in \( \sigma_1 \). Again, then, the only integer vectors in \( \sigma = \text{conv}(D_U(\mathbf{p})) \) are in fact in \( D_U(\mathbf{p}) \) itself: it is discrete-convex.

Example A.17. By changing which combinations of the vectors in Ex. A.15 pertain to the two agents, we obtain an example in which equilibrium fails.
Suppose Agent 1 has valuation \( u^1(0,0,0,0) = 0, u^1(1,1,0,0) = 6, u^1(0,1,1,0) = 9 \) and Agent 2 has valuation \( u^2(0,0,0,0) = 0, u^2(0,0,1,1) = 6, u^2(4,0,0,1) = 6 \). Now \( \sigma^1 = \text{conv}((0,0,0,0), (1,1,0,0), (0,1,1,0)) \) and \( \sigma^2 = \text{conv}((0,0,0,0), (0,0,1,1), (4,0,0,1)) \). The analysis proceeds as before, but in this case the decompositions (1) and (2) show that both \((2,1,1,1)\) and \((3,1,1,1)\) are in \( \sigma^1 + \sigma^2 \). Thus they are in \( \text{conv}(D_U(1,5,4,2)) \) but not in \( D_U(1,5,4,2) \): this set is in this case not discrete-convex.

**Lemma A.18.** \( \hat{m}(C) \leq \text{mult}(C) \) for an intersection 0-cell \( C \), with equality holding iff there exists a small translation such that the subgroup indices are 1 at all emerging intersection 0-cells.

**Proof.** If the intersection is transverse at \( C \) then \( \text{mult}(C) \) is the product of \( \hat{m}(C) \) with a subgroup index (Defns. 5.7 and 5.18). Moreover, \( \hat{m}(C) > 0 \) by definition, and any subgroup index is at least 1 (by Fact 5.15(1)), so the result follows. The non-transverse case is similar; here \( \hat{m}(C) > 0 \) because emerging 0-cells always exist (Lemma 5.9). \( \Box \)

**Proof of Thm. 5.12.** It is clear from Theorem 5.19(1) and Lemma A.18 that \( \Gamma^n(A^1, A^2) \) is an upper bound for intersection 0-cells counted with naïve multiplicities, and that the count equals this bound iff \( \hat{m}(C) = \text{mult}(C) \) for every intersection 0-cell \( C \).

Suppose that equilibrium does not exist for all relevant supplies. By Prop. 5.2 there exists an intersection 0-cell \( C \) at price \( p \), and a bundle \( y \in \mathbb{Z}^n \), such that \( y \not\in D_U(p) := D_{u^1}(p) + D_{u^2}(p), \) but \( y \in \sigma := \text{conv}(D_U(p)) \). Let \( v \in \mathbb{R}^n \) and small \( \epsilon > 0 \) be such that \( \mathcal{L}_{u^1} \) and \( \{\epsilon v\} + \mathcal{L}_{u^2} \) intersect transversally and the sum of the naïve multiplicities of intersection 0-cells emerging from \( C \) is equal to \( \hat{m}(C) \). By Lemma A.11, there exists an intersection 0-cell \( C' \) at \( p' \) for \( \mathcal{L}_{u^1} \) and \( \{\epsilon v\} + \mathcal{L}_{u^2} \) emerging from \( C \), such that \( y \not\in \text{conv}(D_{\mathcal{U}_\epsilon}(p')) \), where \( \mathcal{U}_\epsilon \) is the aggregate valuation for \( u^1 \) and \( u^2 \) where \( w^2(x) = u^2(x) + \epsilon \cdot x \); the fact \( y \notin D_U(p) \) rules out the other cases. By Prop. 4.13 we know \( y \not\in D_{\mathcal{U}_\epsilon}(p') \). By Theorem 5.16(1) we conclude that the subgroup index for \( u^1 \) and \( u^2 \) and \( \mathcal{U}_\epsilon \) at \( p' \) is greater than 1. By Lemma A.18, \( \hat{m}(C) < \text{mult}(C) \). So as stated above, the upper bound \( \Gamma^n(A^1, A^2) \) does not bind. This gives sufficiency of the upper bound. Necessity when \( n \leq 3 \) and the intersection is transverse is presented in the text. \( \Box \)

**Example A.19.** Consider two identical agents whose demand sets at price \( (2,2) \) are the bundles \((0,0),(1,2),(2,1)\) and \((1,1)\). (For example, this is consistent with a valuation \( u(x,0) = x; u(0,y) = y; u(1,1) = 4; u(1,2) = u(2,1) = 6; u(2,2) = 7) \). Then the bundle \((2,2)\) is in the aggregate demand set at this price: we assign bundle \((1,1)\) to both agents. But observe that bundle \((1,1)\) is an interior point of each agent’s demand complex cell with the vertices \((0,0),(1,2)\), and \((2,1)\), so if we make a small perturbation to either agent’s valuation so their LIPs intersect transversally, then for any prices close to \((2,2)\) the perturbed agent’s demand must be some subset of these vertices, and it is easy to see that \((2,2)\) cannot be an aggregate demand.

**Example A.20.** In both Exs. A.15 and A.17, we can calculate \( \Gamma^4(A^1, A^2) \) to be 3 (indeed \( \Gamma^4_2(A^1, A^2) = 3 \) and \( \Gamma^4_2(A^1, A^2) = 0 \) for all other values of \( k \)). In both cases, the weights of the individual’s 2-cells that meet at \((1,5,4,2)\) are both 1. This illustrates again that the condition of Thm. 5.12 is sufficient, but not necessary, for existence of equilibrium when \( n \geq 4 \).

**Proof of Prop. 5.20.** Follows from Lemma A.18, Thms. 5.16(1) and 5.19(2), and Prop. 5.2 in the same way as is described in the text for Thm. 5.12. \( \Box \)
**Remark A.21.** The difficulty of finding the minimal demand type for a valuation depends on how that valuation has been expressed. In related work (Baldwin, Goldberg and Klemperer, in preparation) we provide a method for expressing only strong substitute valuations. But if the valuation is expressed as a function \( w^j : A^j \to \mathbb{R} \) as usual, then its LIP is the associated “tropical hypersurface”; algorithms to calculate this are implemented by “gfan” (see Jensen, 2011), “polymake” (see e.g. Gawrilow and Joswig, 2000). Once the tropical hypersurface is obtained, the primitive normal vectors may be found in a practical way using the “a-tint” extension to polymake (see Hampe, 2014a, Prop. 3.3 and Remark 3.4; this is also possible using gfan via an unsupported function).

Checking unimodularity of a demand type containing \( r \) vectors, requires \((r^n)\) calculations of determinants. However, Seymour (1960) gives a characterisation of all such sets up to unimodular change of basis. There are also many other ways to check for unimodularity (see Schrijver, 2000, Chapter 19).

Intersection 0-cells may be found using a-tint (Hampe, 2014a). As discussed in more detail by Hampe (2014b), these methods provide answers in reasonable time for \( n \leq 10 \) as long as \(|A|\) is relatively small.\(^{62}\)

The terms \( \Gamma^n_k(A^1, A^2) \) are “mixed volumes”, and the calculation of these in general has high complexity (see Cox et al., 2005, Section 7.6) although recent work develops ever-faster algorithms (see e.g. Chen et al., 2014). However, they are extremely easy to calculate in certain cases of relevance to economics. Firstly, if we assume every agent has the same domain of valuation, and moreover that this domain has a simple geometric shape whose volume is easy to calculate, then calculation of \( \Gamma^n_k \) is trivial. And if every agent’s valuation considers all bundles containing at most \( d \) goods, for some fixed \( d \) (which need not be the same across agents) then again there is no complexity. See Fact A.9 for more details.

Calculating the weights of cells requires finding the volumes of lattice polytopes (Defn. 5.6). This, again, has high complexity in general (Dyer and Frieze, 1988) but more efficient methods are likely to help in our context (e.g. Dyer et al., 1991). Finally, checking for discrete convexity in Step (8)(ii) is aided by the “Quickhull” algorithm (Barber et al., 1996).

### A.5 Proofs and Additional Examples for Section 6

**Example A.22.** *(The Cycle of Complements Model)* Consider “complements” consumers, each of whom is only interested in a single, specific, pair of goods, and that these pairs form a cycle. Thus there are \( n \) kinds of consumers and \( n \) goods, and we can number both goods and consumers \( 1, \ldots, n \), such that every consumer of kind \( i < n \) demands goods \( i \) and \( i + 1 \), which it sees as perfect complements, while consumers of

\(^{62}\)Indeed, this will be faster than the case described by Hampe (2014a), since every intersection 0-cell for two LIPs is in the “stable intersection” (see e.g. Maclagan and Sturmfels, 2015, Defn. 3.6.5).
kind $n$ demand goods $n$ and 1. It is easy to check that:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
\cdot & 0 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\begin{cases}
0 & \text{if } n \text{ is even} \\
2 & \text{if } n \text{ is odd}.
\end{cases}
$$

So if $n$ is odd, our Unimodularity Theorem tells us equilibrium does not always exist. Furthermore, the proof of Prop. 4.10 showed how to easily construct an explicit example of equilibrium failure for any non-unimodal demand type. Here we simply select a single agent of each kind, each of which values its desired pair at $v$, so that they are all indifferent between purchase and no purchase (and hence their facets all intersect) if every good’s price is $v/2$.

If $n$ is even, the columns of this matrix are not linearly independent. However, if we exclude the $i$th column, for any $i$, the remaining $n-1$ columns are then linearly independent, and can trivially be extended to $n$ linearly independent vectors with determinant 1 by adding the column $e_i$. So using Thm. 4.3, equilibrium always exists if $n$ is even, since the valuations are, trivially, concave.

**Proof of Prop. 6.1.** (1): by definition, $x \in D_u(p)$ if $p^T(x-x') \leq u(x) - u(x')$ for all $x' \in A$, with equality iff $x' \in D_u(p)$ also. For any invertible matrix $G$, we may re-write $p^T(x-x') = p^TGG^{-1}(x-x') = (G^T p)^T(G^{-1} x - G^{-1} x')$. If $G$ is additionally unimodular, then $G^{-1} x$ and $G^{-1} x' \in \mathbb{Z}^n$. If we write $y = G^{-1} x$ and $y' = G^{-1} x'$ then $(G^T p)^T (y-y') \leq G^*u(y) - G^*u(y')$ holds iff $p^T (x-x') \leq u(x) - u(x')$.

(2): since $L_u = \{ p \in \mathbb{R}^n : |D_u(p)| > 1 \}$, this follows from (1).

(3): if $F$ is a facet of $L_u$, then by (2), $G^T F = \{G^T p : p \in F \}$ is a facet of $L_{G^* u}$. If $v$ is normal to $F$, then $p^T v$ is constant for $p \in F$, so $p^T G G^{-1} v = (G^T p)^T G^{-1} v$ is constant for $G^T p \in G^T F$. So $G^{-1} v$ is a demand type vector for $G^* u$. As $G$ has an integer inverse, the converse is also true.

**Proof of Prop. 6.2.** Suppose there is always an equilibrium for every finite set of agents with concave valuations of type $G^{-1} D$, and any relevant supply.

---

63So equilibrium fails if aggregate supply is exactly 1 unit of each good (the “middle of the parallelepiped”) since the minimum and maximum aggregate demands are zero, and 2 units of each good, respectively, at this price. (It is easy to check failure of equilibrium for $x_i = 1$, for all $i$, by contradiction. At least one good, w.l.o.g. good 1, would not be part of a pair being allocated together. So good 1 has value 0 to whoever receives it, hence $p_1 \leq 0$. Therefore $p_2 \geq v$, since otherwise consumer 1 would demand both goods 1 and 2. Therefore $p_3 \leq 0$, since otherwise good 2 would not be demanded, and consumer 2 therefore buys goods 2 and 3. Therefore $p_4 \geq v$, etc., so $p_j \leq 0$ if $j$ is odd. But consumer $n$ then wishes to buy goods $n$ and 1, which is a contradiction.)

64For example, the aggregate demand of 1 unit of each good is supported by price $v/2$ for every good, when there is exactly one consumer of each kind, each of which values its preferred pair at $v$.

65Sun and Yang (2011) and Teytelboym (2014) have independently used alternative methods to show these results for a version of this model; the even $n$ case is a special case of the “generalised gross substitutes and complements” demand type that we discuss in Section 6.2; one can also use the relationship with matching (see Section 6.5), together with Pycia (2008), to obtain the $n = 3$ case.
Let $u^1, \ldots, u^k$ be concave valuations of type $\mathcal{D}$ and let $x$ be a relevant supply bundle, i.e. in the convex hull of the domain of their aggregate valuation. Then, by Prop. 6.1(3) the valuations $G^*u^1, \ldots, G^*u^k$ are of demand type $G^{-1}\mathcal{D}$. It is obvious they are also all concave. By definition of pullback, $y := G^{-1}x$ is in the convex hull of the domain of their aggregate valuation, i.e. is a relevant supply. By assumption, equilibrium exists in the latter case: there exists a price $p$ at which the agent with valuation $G^*u^j$ demands $y^j$ and $\sum_{j=1}^k y^j = y$. Define $x^j := G y^j \in D_u (G^{-T} p)$ (see Prop. 6.1(1)). Then $x = \sum_{j=1}^k x^j \in D (G^{-T} p)$. As $G$ is invertible, the converse is shown similarly. □

Example A.23. To illustrate Prop. 6.1, consider the example of Fig. 5. Create a new good, 3, from two units of good 1 plus one unit of good 2, and consider the economy in which the goods traded are 1 and 3. Note that we can recreate one unit of good 2 by buying one unit of good 3 and selling two units of good 1, and we can convert any bundle expressed in terms of goods 1 and 2 (as a column vector) to a bundle of goods 1 and 3 by pre-multiplying by
\[
\begin{pmatrix}
1 & -2 \\
0 & 1
\end{pmatrix}
\] –this matrix plays the role of “$G^{-1}$” in Prop. 6.1.

Observe that the “substitutes” agent of the original economy (who bought either (1, 0) or (0, 1) at price (30, 20)) corresponds to an agent in the new economy who would “buy” either (1, 0) or (−2, 1). We can interpret this as an agent with an endowment of −2 units of good 1 (a contract to sell), and who buys either three units of good 1 or one unit of good 3. Thus this agent treats goods 1 and 3 as 3:1 substitutes. Similarly, the “complements” agent of the original economy (who bought neither or both of goods 1 and 2) corresponds to an agent in the new economy with an endowment of −1 unit of good 1, who buys one unit of either good 1 or good 3 (so is indifferent between bundles (0, 0) and (−1, 1)) that is, an agent who treats goods 1 and 3 as 1:1 substitutes. So this is a pure substitutes economy in which equilibrium fails. Since the demand type of the original economy contained the columns of
\[
\begin{pmatrix}
-1 & 1 \\
1 & 1
\end{pmatrix}
\], which have determinant 2, the demand type in the new economy contains the columns of
\[
\begin{pmatrix}
-3 & -1 \\
1 & 1
\end{pmatrix}
\], that is, by
\[
\begin{pmatrix}
1 & -2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 1 \\
1 & 1
\end{pmatrix}
\], which also have determinant −2.

Example A.24. The demand type that contains all vectors in $\mathbb{Z}^6$ with three +1 entries and three -1 entries–thus incorporating all valuations for six distinct goods in which all changes in an agent’s demand involve swapping three of the goods for the other three goods as prices move between UDRs–is unimodular, and is neither a basis change of strong substitutes nor of the demand type of section 6.3.

\[66\]An elementary example of basis-change logic is Section 6.1’s observation that Hatfield et al.’s (2013) model of complements corresponds to a strong substitutes restriction. For other examples see Sun and Yang (2006, 2009).