(Il)legal Assignments in School Choice*

Lars Ehlers[†] and Thayer Morrill[‡]

September 29, 2017

(Revised version)

Abstract

In public school choice, students with strict preferences are assigned to schools. Schools are endowed with priorities over students. Incorporating different constraints from applications, priorities are often modeled as choice functions over sets of students. It has been argued that the most desirable criterion *for an assignment* is fairness; there should not be a student having justified envy in the following way: he prefers some school to his assigned school and has higher priority than some student who got into that school. Justified envy could cause court cases. We propose the following fairness notion *for a set of assignments*: a set of assignments is legal if and only if any assignment outside the set has justified envy with some assignment in the set and no two assignments inside the set block each other via justified envy. We show that under very basic conditions on priorities, there always exists a unique legal set of assignments, and that this set has a structure common to the set of fair assignments: (i) it is a lattice and (ii) it satisfies the rural hospitals theorem. The student-optimal legal assignment is efficient and provides a solution for the conflict between fairness and efficiency.

JEL C78, D61, D78, I20.

1 Introduction

Centralized admissions procedures are now being used in a wide range of applications ranging from national college admissions, assigning students to public schools, and implementing auxiliary programs such as magnet schools.[1] There has been a great deal of research

*We are grateful to Federico Echenique for his comments and suggestions. The first author acknowledges financial support from the SSHRC (Canada) and the FRQSC (Québec).

†Département de Sciences Économiques and CIREQ, Université de Montréal, Montréal, QC H3C 3J7, Canada. Email: lars.ehlers@umontreal.ca (Corresponding author).

‡North Carolina State University, NC, USA. Email: thayer_morrill@ncsu.edu.

[1] Examples of countries that use a centralized college admissions process are Turkey (Balinski and Sönmez, 1999); China (Chen and Kesten, 2016); and India (Aygün and Turhan, 2016). There is now a long literature devoted to public school assignment beginning with the seminal work of Abdulkadiroğlu and Sönmez (2003). See Pathak (2011) and Abdulkadiroğlu and Sönmez (2013) for surveys of the literature. See Dur, Hammond, and Morrill (2017) for a discussion of centralized magnet school assignment.

focused on the tradeoffs between efficiency, fairness, and strategic properties of candidate mechanisms. These mechanisms have received much attention in the economics literature precisely because for parents and students school choice is an important issue. Assigning objects which are valuable and yet scarce leads to contention, and contention leads to lawsuits. For example, parent groups in Seattle and Louisville filed lawsuits contesting the use of racial status in the tiebreaker of their school district's assignment procedure. These lawsuits eventually led to the Supreme Court ruling (in Parents Involved in Community Schools v. Seattle School District No. 1, 551 U.S. 701, 2007) that race cannot be used explicitly in a school assignment procedure.

This is our basic question: which school assignments are legal? We are concerned with a parent or group of parents who file a lawsuit with the intent of changing the school assignment that is to be made.[2] Consider what seems to be the most straightforward application: college admissions. Typically, all students take a common exam, and a student's score determines her priority when choosing a university. In this environment, legality may appear simple; if a student is denied admissions to a university, each student accepted to that university must have a higher score than her. However, there are two reasons why (at least in the United States) this does not correctly determine which assignments are legal.

Legal standing, or *locus standi*, is the capacity to bring suit in court. As interpreted by the United States Supreme Court:

> Under modern standing law, a private plaintiff seeking to bring suit in federal court must demonstrate that he has suffered "injury in fact," that the injury is "fairly traceable" to the actions of the defendant, and that injury will *"likely be redressed by a favorable decision."* [3]

Therefore, it is not illegal to reject a student from a university (regardless of which students are accepted) unless there exists a legal way of assigning her to the university. This suggests that legality is a set-wise property of assignments. A set of assignments is legal or not as we must be able to determine which assignments are possible in order to know which assignments are legal.

The second reason why a simple comparison of students' scores is not sufficient to determine the legality of an assignment is that typically a school's decision on which students to admit is at least partially based on the composition of the student body. Public schools often reserve seats for minority students or students who live within a "walk-zone".[4] Admission to a magnet school may incorporate a student's income level (Dur, Hammond and Morrill, 2017). The centralized admissions process in India incorporates the caste to which the student belongs to (Aygün and Turhan, 2016). In each case, admission decisions are

[2]In particular, we do not address the separate question of a parent filing a lawsuit with the purpose of receiving monetary damages. Note that a government agency typically has sovereign immunity and would not be liable for damages.

[3]This quote is from Hessick (2007) regarding Supreme Court case Lujan vs. Defenders of Wildlife, 504 U.S. 555, 560-61 (1992).

[4]For examples, see Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu et al. (2005), Kominers and Sönmez (2016), and Dur et al. (2017).

based on a more complicated choice function than a simple rank-order list of students. Is it still possible to determine which assignments are legal in a coherent way?

A generalized choice function is just a more complicated set of rules for determining which students are admitted. We interpret these rules as conveying rights to each student. A student's rights have been violated if the rules dictate that she should have been chosen. However, whether or not this violation is illegal is more subtle; although the student has been harmed, this violation is not illegal unless the harm is redressable.[5] We propose a definition of legality incorporating these two constraints. This is analogous to a "fairness" notion of a set of assignments (where fairness depends on the whole set). More specifically, blocking is only allowed via assignments in the set (which we deem legal). Any assignment outside the set is illegal because it is blocked by some assignment in the set. The important feature is that here blocking is defined in terms of assignments: student i blocks an assignment if i blocks it with some school *and* there exists some assignment in the set where i is assigned to the blocking school. It should be clear that in this assignment the school is not necessarily better off. It has the interpretation that there is some "legal way" of assigning i to the blocking school. More precisely, we call a set of assignments legal iff (i) any assignment not in the set is blocked by some student with an assignment in the set and (ii) no two assignments block each other.

Of course, this is related to stable sets à la von Neumann Morgenstern (vNM). A cursory reading makes one think that the two concepts are identical. They are in the sense of the formulation of (i) and (ii), but most importantly, a school might be worse off under the assignment in the set when compared to the original one. But this is irrelevant as we are here in the context of public school choice where (as it has been argued) students are "active agents" and schools are "objects to be consumed". Any legal set is a vNM-stable set where schools are "objects to be consumed".

Our main results show that there always exists a unique legal set of assignments and that this set shares the following properties with stable assignments: (i) it is a lattice and (ii) the rural hospitals theorem is satisfied. Therefore, there always exists a student-optimal legal assignment and a school-optimal legal assignment. Moreover, we demonstrate that the student-optimal legal assignment is (Pareto) efficient. Therefore, unlike for fairness and efficiency, there is no tension between legality (vNM-stability) and efficiency. Considering first legality and second efficiency or first efficiency and second legality yields the student-optimal legal assignment. This is in contrast to traditional school choice where when stability is more important than efficiency, the DA (deferred-acceptance) assignment was suggested, and considering first efficiency and second fairness, the TTC (top-trading cycles) assignment was suggested. Finally, we relate the student-optimal legal assignment to Kesten's efficiency adjusted deferred-acceptance (DA)-mechanism (Kesten, 2010). The efficiency adjusted DA-mechanism has not been previously defined when schools have general choice functions. We show that when schools have acceptant[6] choice functions that the

[5]It is common, especially among economists, to view all harm as redressable via side payments. However, states and by extension local governments have sovereign immunity from lawsuits for damages unless the state consents to be sued.

[6]A school a's choice function is *acceptant* if there exists a capacity q such that a accepts all students if

mechanism is straightforward to generalize and that the efficiency adjusted DA-mechanism chooses the student-optimal legal assignment. As a byproduct, we offer a foundation for the generalization of Kesten's efficiency adjusted DA-mechanism to school choice environments where priorities are given by substitutable choice functions.

Our paper is most closely related to Morrill (2017) which reinterprets fair school assignments. Typically, an assignment is deemed unfair if a student has justified envy.[7]. Morrill (2017) defines a student to have legitimate envy if she has justified envy at school a and it is possible to assign her to school a. Otherwise, her envy is defined to be petty. There is no direct relationship between which schools are possible and which schools are legal since a legal assignment is allowed to be wasteful whereas in Morrill (2017) non-wasteful assignments are excluded by assumption; however, there is a close relationship between which assignments are possible and which assignments are legal. The analysis in Morrill (2017) relies critically on two assumptions: each school has responsive priorities and the school assignments considered are non-wasteful. However, in many practical applications (such as when incorporating affirmative action) these assumptions are unreasonable. Our paper demonstrates that the legal set of assignments has similar properties even when these restrictions do not hold.

Our paper also relates to several recent contributions that consider alternative fairness notions to eliminating justified envy. Dur, Gitmez, and Yilmaz (2015) introduce the concept of partial fairness. Intuitively, they define an assignment to be partially fair if the only priorities that are violated are "acceptable violations". Kloosterman and Troyan (2016) also introduce a new fairness concept called essentially stable. Intuitively, an assignment is essentially stable if resolving i's justified envy of school a initiates a vacancy chain that ultimately leads to i being rejected from a. Both Dur, Gitmez and Yilmaz (2016) and Kloosterman and Troyan (2016) provide characterizations of EADA using their respective fairness notion. Partial fairness and essential stability are similar in spirit but do not directly relate to legality. Each is a pointwise solution concept while legality is a setwise solution concept. Moreover, the analysis in both Dur, Gitmez and Yilmaz (2015) and Kloosterman and Troyan (2016) relies heavily on the assumption of schools having responsive priorities. It is not clear whether their results continue to hold in the general environment considered in the current paper.[8]

In school choice with responsive priorities, Wu and Roth (2016) study the structure of assignments which are fair and individually rational (i.e. non-wastefulness may be violated). They show that this set has a lattice structure and that the student-optimal assignment of this set coincides with the student-optimal stable assignment.

In contexts where both sides are agents, in one-to-one matching problems Ehlers (2007) studies vNM-stable sets, and Wako (2010) shows the existence and uniqueness of such sets.

fewer than q apply and q students whenever q or more students apply.

[7]Student i has *justified envy* of student j if i prefers j's assignment to her own and i has a higher priority at that school than does j.

[8]Kesten (2004), Alcalde and Romero (2015), and Cantala and Papai (2014) also introduce alternative notions of fairness for the school assignment problem. The concepts they introduce do not directly relate to legality.

Klijn and Masso (2003) study bargaining sets in those problem. Note that all these papers consider one-to-one settings whereas our paper considers the most general many-to-one setting and provides an alternative solution concept to the set of stable assignments.

We proceed as follows. Section 2 introduces school choice and all basic notions for choice functions and assignments. Section 3 defines legal assignments. Section 3.1 generalizes the Pointing Lemma, the Decomposition Lemma and the Rural Hospital Theorem to any two individually rational assignments which do not block each other, and Section 3.2 establishes a Lattice Theorem. We then use these results to show the existence and uniqueness of a legal set in Section 3.3. Section 4 discusses our results. Section 4.1 relates legal assignments to efficiency and non-wastefulness. Section 4.2 provides a general EADA-algorithm for calculating the student-optimal legal assignment. Section 4.3 shows that there is a unique strategy-proof and legal mechanism, namely, the student-proposing DA-mechanism. Section 5 concludes and explains how our results carry over to matching with contracts. The Appendix introduces assignment with contracts and generalizes all our results from school choice to this setting.

2 Model

We consider the following many-to-one matching problem. There is a finite set of students, $A = \{i, j, k, \ldots\}$, to be assigned to a finite set of schools, $O = \{a, b, c, \ldots\}$. Each student i has a strict preference P_i over the schools and being unassigned $O \cup \{i\}$ (where i stands for being unassigned). Then iP_ia indicates that student i prefers being unassigned to being assigned to school a and R_i denotes the weak preference relation associated with P_i.

We allow schools having general choice functions for priorities in order to incorporate various assignment constraints. Let 2^A denote the set of all non-empty subsets of A. Each school a has a choice function $C_a : 2^A \to 2^A$ such that for all $X \in 2^A$, $C_a(X) \subseteq X$. Then $C_a(X)$ denotes the set of students that school a chooses from X. Throughout we assume that C_a satisfies the following standard properties of substitutability and the law of aggregate demand (LAD): (a) substitutability rules out complementarities in the sense that students chosen from larger sets should remain chosen from smaller sets and (b) LAD requires the number of chosen students to be weakly monotonic for bigger sets of students.

Definition 1. *Let $a \in A$ and $C_a : 2^A \to 2^A$ be a choice function.*

(a) The choice function C_a is ***substitutable*** *if for all $X \subseteq Y \subseteq A$ we have $C_a(Y) \cap X \subseteq C_a(X)$.*[9]

(b) The choice function C_a satisfies the ***law of aggregate demand (LAD)*** *if for all $X \subseteq Y \subseteq A$ we have $|C_a(X)| \leq |C_a(Y)|$.*[10]

[9]Note that this is equivalent to $i \in C_a(Y)$ and $j \in Y \backslash \{i\}$ implies $i \in C_a(Y \backslash \{j\})$ (or the same condition formulated in terms of rejected students $Y \backslash C_a(Y)$).

[10]Here $|X|$ denotes the cardinality of a set. LAD was introduced by Hatfield and Milgrom (2005) in a more general model of matching with contracts. Our definition of LAD is equivalent to size monotonicity

Throughout we fix the assignment problem $(A, O, (P_i)_{i\in A}, (C_a)_{a\in O})$.

An assignment is a function $\mu : A \to O \cup A$ from students to schools and students such that for all $i \in A$, $\mu_i \in O \cup \{i\}$. Given assignment μ and $i \in A$, let $\mu_i = a$ indicate student i being assigned to school a (and $\mu_i = i$ indicate student i being unassigned). We use the convention that for each school a the set $\mu_a = \{i \in A | \mu_i = a\}$ denotes the students assigned to school a. Let $\mathcal{A}$ denote the set of all assignments. An assignment μ is **individually rational** if for every student i, $\mu_i R_i i$ and, for every school a, $C_a(\mu_a) = \mu_a$. *Throughout we will consider individually rational assignments only.*[11] Let $\mathcal{IR}$ denote the set of individually rational assignments. An assignment μ is **efficient** (among all individually rational assignments) if there exists no $\nu \in \mathcal{IR}$ such that $\nu_i R_i \mu_i$ for all $i \in A$ and $\nu_j P_j \mu_j$ for some $j \in A$.

Blocking is defined as follows for general choice functions. Given an assignment μ, student i and school a **block** μ if $aP_i\mu_i$ and $i \in C_a(\mu_a \cup \{i\})$. This means that student i prefers school a to his assignment and school a chooses i from its assigned students and i. There are two types of blocking: school a has an empty seat available for i or school a would like to admit i and reject a previously admitted student. These two types are distinguished below in the usual sense. An assignment μ is **non-wasteful** if (it is individually rational and) there do not exist a student i and a school a such that $aP_i\mu_i$ and $C_a(\mu_a \cup \{i\}) = \mu_a \cup \{i\}$. Given an assignment μ, student i has **justified envy** if there is a school a such that $aP_i\mu_i$, $i \in C_a(\mu_a \cup \{i\})$, and $C_a(\mu_a \cup \{i\}) \neq \mu_a \cup \{i\}$. This means that student i prefers a to his assignment and has higher "choice" priority because he is chosen from the set of students assigned to school a and including him (and some other student is rejected). An assignment is **fair** if (it is individually rational and) there is no justified envy. An assignment is **stable** if it is individually rational, non-wasteful and fair.

Stable assignments were introduced by Gale and Shapley (1962) in two-sided matching and adopted to school choice by Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2002). The main difference is that in two-sided matching both sides are "agents" whereas in school choice students are "agents" and schools are "objects to be consumed".

Nevertheless, the set of stable assignments coincide in both interpretations: the set of stable assignments is non-empty, it is a lattice and it satisfies the strong rural hospitals theorem. Furthermore, note that stability is a "point-wise" property specific to one assignment alone.

3 Legal Assignments

We will be interested in "set-wise" blocking which will depend on the whole set of assignments under consideration.

Definition 2. *Let $\mu, \nu \in \mathcal{IR}$ and $i \in A$.*

introduced by Alkan and Gale (2003) and Fleiner (2003). We use the LAD terminology to be consistent with the standard matching literature.

[11]Individual rationality can be alternatively interpreted as "feasibility" of assignments.

(a) *Student i **blocks** assignment μ with assignment ν if for some school $a \in A$: (1) $aP_i\mu_i$, (2) $i \in C_a(\mu_a \cup \{i\})$ and (3) $\nu_i = a$.*

(b) *Assignment ν **blocks** μ if there is a student i who blocks μ with ν.*

Note that in the usual blocking notion, both the blocking student and the school are unambiguously (myopically) better off (with respect to the original assignment) whereas here only the student is unambiguously better off (because the school's priority ranking is not clear between μ and ν). Our main solution concept only allows blocking via assignments which are in the set under consideration: (i) any assignment outside the set is blocked via some assignment inside the set and (ii) any two assignments inside the set do not block each other.

Definition 3. *Let $L \subseteq \mathcal{IR}$. Then L is **legal** if and only if*

(i) for all $\mu \in \mathcal{IR} \backslash L$ there exists $\nu \in L$ such that ν blocks μ, and

(ii) for all $\mu, \nu \in L$, ν does not block μ.

On first sight this is similar to stable sets à la von Neumann-Morgenstern (hereafter vNM-stability). However, under vNM-stability, both sides (often called workers and firms instead of students and schools) are considered to be agents, and all agents must be made better off in order to block. However, in the school assignment problem only students are agents. The important fact in our definition of blocking is that only the student is made better off and the school may be made worse off.[12] One could interpret the legality of a set of assignments as the natural generalization of stable sets to school choice. Of course, this could be done to other contexts in cooperative game theory where sharing problems contain "neutral" agents with priorities.

Throughout we will use the convention that for a given legal set L, any assignment belonging to L is called **legal** and any assignment not belonging to L is called **illegal**.

Remark 1. *In law, standing or locus standi*[13] *is the term for the ability of a party to demonstrate to the court sufficient connection to and harm from the law or action challenged to support that party's participation in the case. In the United States the three standing requirements are*

(1) Injury-in-fact: The plaintiff must have suffered or imminently will suffer injury-an invasion of a legally protected interest that is (a) concrete and particularized, and (b) actual or imminent (that is, neither conjectural nor hypothetical; not abstract). The injury can be either economic, non-economic, or both.

(2) Causation: There must be a causal connection between the injury and the conduct complained of, so that the injury is fairly traceable to the challenged action of the defendant and not the result of the independent action of some third party who is not before the court.

[12]Note that legality and vNM-stability are equivalent when a school can be assigned at most one student.

[13]https://en.wikipedia.org/wiki/Standing_(law).

(3) *Redressability: It must be likely, as opposed to merely speculative, that a favorable court decision will redress the injury.*

In our school choice setting, (1) Injury-in-fact corresponds to envy (student i prefers school a to the school assigned under μ), (2) Causation corresponds to i's envy being justified, and (3) Redressability corresponds to being able to assign student i to school a in a legal way.[14]

Before we continue, we illustrate our (il)legal assignments in the basic example where we have a tradeoff between efficiency and stability.

Example 1. *Let $A = \{1, 2, 3\}$ and $O = \{a, b, c\}$. Let*

P_1	P_2	P_3	$\succ_a$	$\succ_b$	$\succ_c$
a	b	a	2	1	3
b	a	b	3	2	2
c	c	c	1	3	1
1	2	3			

where the choice function C_x ($x \in O$) chooses from any set $X \in 2^A$ the highest $\succ_x$-ranked student. It is easy to verify that

$$\mu = \begin{pmatrix} 1 & 2 & 3 \\ b & a & c \end{pmatrix}$$

is the unique stable assignment. The assignment

$$\nu = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

is efficient and Pareto improves μ (because $\nu_i R_i \mu_i$ for all $i \in A$). At ν, student 3 has justified envy because $aP_3\nu_3$ and $C_a(\{1, 3\}) = \{3\}$. However, we can never assign student 3 to school a in a legal way (i.e., the three standing requirements always hold): first, μ is legal as there is no justified envy; second, if we assign 1 to a under assignment η, 2 has to be assigned to school b as otherwise $\eta_2 \neq a, b$, $aP_2\eta_2$, $C_a(\{2, 3\}) = \{2\}$, and 2 is assigned to a under the legal assignment μ; and third, given that $\eta_3 = a$ and $\eta_2 = b$, we have $bP_1\eta_1$, $C_b(\{1, 2\}) = \{1\}$ and 1 is assigned to b under the legal assignment μ. Indeed, it can be verified that $\{\mu, \nu\}$ is the unique legal set of assignments and it contains a unique student-optimal legal assignment, namely ν.

In school choice, it has been argued when stability is more important than efficiency, then first we consider the set of stable assignments and choose the unique efficient assignment in this set, namely the DA assignment. If efficiency is more important than stability, then first we consider set of efficient assignments and choose the one which minimizes

[14] However, this is one interpretation of our solution concept of a fair set of assignments.

the number of blocking pairs. As it has been recently shown by Abdulkadiroğlu et al. (2017), for one-to-one settings this leads to the TTC (top trading cycles) assignment. As it will turn out, here we will not have this conflict of which order to choose: independently whether we view legality (vNM-stability) more important than efficiency, or efficiency more important than legality, this yields the same assignment, namely the student-optimal legal assignment.

Our main challenge will be to establish the existence and uniqueness of a legal set of assignments.

For this, it will be instrumental to show for any two individually rational assignments μ and ν, which do not block each other, a Pointing Lemma, a Decomposition Lemma and the Rural Hospitals Theorem. Then we go on to show the lattice structure for these assignments. Any reader, who wants to go directly to the main results, may skip Sections 3.1 and 3.2. All proofs except for very short ones are relegated to the Appendix where we generalize all our results to the framework of matching with contracts.

3.1 Pointing, Decomposition and Rural Hospitals Theorem

Two of the classic results in matching theory are the Pointing Lemma and the Decomposition Lemma. The Pointing Lemma (attributed to Conway in Knuth, 1976) is the basis for the proof that the set of stable marriages is a lattice.[15] The Pointing Lemma compares any two stable assignments μ and ν. We ask each man to point to his favorite wife under the two marriages (he is possibly unmarried or married to the same woman), and we ask each woman to point to her favorite husband. The Pointing Lemma says that no man and woman point to each other; no two men point to the same woman; and no two women point to the same man.

Lemma 1 (Classical Pointing Lemma)**.** *Consider a marriage problem where men and women have strict preferences and let μ and μ' be stable matchings. Then:*

(i) no man and woman point at each other unless they are matched under both μ and μ';

(ii) no two women point at the same man; and

(iii) no two men point at the same woman.

The key implication of the Pointing Lemma is that the assignments $\mu \vee \nu$ (defined by each man is assigned to the woman he is pointing at) and $\mu \wedge \nu$ (defined as each woman is assigned to the man she is pointing at) are well defined. This is the basis of the Lattice Theorem as all that remains is to show that $\mu \vee \nu$ and $\mu \wedge \nu$ are also stable.

The Pointing Lemma is closely related to the Decomposition Lemma which is due to Gale and Sotomayor (1985).

[15] Following the exposition in Roth and Sotomayor (1992), we refer to it as the Pointing Lemma.

Lemma 2 (Classical Decomposition Lemma)**.** *Consider a marriage problem where men and women have strict preferences and let μ and μ' be stable matchings. Let $M(\mu')$ be the set of men who prefer μ' to μ and let $W(\mu)$ be the set of women who prefer μ to μ'. Then μ' and μ map $M(\mu')$ onto $W(\mu)$.*

The Pointing Lemma generalizes to many-to-one problems in a straightforward way when schools have responsive priorities with quotas: instead of a choice function, each school a has a strict priority over sets of students, say $\succ_a$, and a quota q_a (of available seats at a). Then $\succ_a$ is responsive iff for any students i, j and any set $H \subseteq A \backslash \{i, j\}$ such that $|H| \leq q_a - 1$, we have (i) $H \cup \{i\} \succ_a H \cup \{j\}$ iff $i \succ_a j$, and (ii) $H \cup \{i\} \succ_a H$ iff $i \succ_a \emptyset$; and (iii) $\emptyset \succ_a H$ for any $H \subseteq A$ with $|H| > q_a$. Now we know that the set of stable assignments of the many-to-one market corresponds to the set of stable assignments of the one-to-one market where any school a is split into q_a copies. A similar construction can be done for two assignments which do not block each other,[16] and hence the pointing lemma carries over in a straightforward manner from one-to-one to many-to-one.

We will show that when schools have general choice functions that only the first two conditions of the Pointing Lemma generalize. However, the Decomposition Lemma continues to hold. To the best of our knowledge, we are the first to generalize the Pointing and Decomposition Lemmas when schools have choice functions instead of responsive priorities (or responsive preferences).

Since pointing indicates that the student is willing to form a blocking pair, the most natural way to adapt pointing to non-responsive preferences is, given two assignments μ and ν and given a student $i \in \mu_a \setminus \nu_a$, a points to i if $i \in C_a(\nu_a \cup \{i\})$.

For later purposes, instead, we define pointing using a seemingly stronger condition. We will later show (in Corollary 2) that this condition is equivalent to the weaker version of pointing.

Definition 4. *Given two assignments μ and ν, student i points to μ_i (ν_i) if $\mu_i R_i \nu_i$ ($\nu_i R_i \mu_i$), and school a points to student i if $i \in C_a(\mu_a \cup \nu_a)$.*

It is clear that under this definition of pointing that when a school points to a student, then she is willing to form a blocking pair with that student. However, it is less clear that each student will be pointed at. We first establish a weak version of the Pointing Lemma.

Lemma 3 (Weak Pointing Lemma)**.** *Let μ and ν be two individually rational assignments which do not block each other. Then:*

(i) no student and school point at each other unless they are assigned under both μ and ν, and

(ii) no two schools point to the same student.

[16]Simply consider $\mu \backslash \nu$ (where any school a receives students $\mu_a \backslash \nu_a$)and $\nu \backslash \mu$ with appropriately reduced capacities (where for any school a we reduce q_a by $|\mu_a \cap \nu_a|$ and the set of students is shrunk to $A \backslash (\cup_{a \in O}(\mu_a \cap \nu_a)))$.

Notice that we are missing the third conclusion of the Classical Pointing Lemma. The generalization to the school assignment problem would be as follows: no two students point at the same school unless they are classmates (i.e. they are both assigned to that school under either μ or ν). The following example is taken from Ehlers and Klaus (2014) and demonstrates that this result does not hold when a school does not have responsive priorities.

Example 2. *Let $O = \{a, b\}$ and $A = \{s_1, s_2, j_1, j_2\}$. University a and university b are both hoping to hire two economists. They are considering two senior candidates, s_1 and s_2, and two junior candidates, j_1 and j_2. Candidates s_x and j_x are in the same field. University a would prefer to hire seniors to juniors, but if it must hire a mixture of the two, it would prefer to hire candidates in the same field. Specifically:*

$$\{s_1, s_2\} \succ_a \{s_1, j_1\} \succ_a \{s_2, j_2\} \succ_a \{j_1, j_2\} \succ_a \{s_1, j_2\} \succ_a \{j_1, s_2\}.$$

If a is only able to hire one economist, then its preferences are: $s_1 \succ_a s_2 \succ_a j_1 \succ_a j_2$. Note that the choice function C_a induced by $\succ_a$ satisfies substitutability and LAD, but $\succ_a$ is not responsive because $\{s_2, j_2\} \succ_a \{s_2, j_1\}$ and $j_1 \succ_a j_2$.

University b has the opposite preferences:

$$\{j_1, j_2\} \succ_b \{s_1, j_1\} \succ_b \{s_2, j_2\} \succ_b \{s_1, s_2\} \succ_b \{s_1, j_2\} \succ_b \{j_1, s_2\},$$

and $j_1 \succ_b j_2 \succ_b s_1 \succ_b s_2$. Again the choice function C_b induced by $\succ_b$ satisfies substitutability and LAD, but $\succ_b$ is not responsive.

Both junior candidates prefer a to b whereas both senior candidates prefer b to a. Consider the assignments

$$\mu = \begin{pmatrix} a & b \\ \{s_1, j_1\} & \{s_2, j_2\} \end{pmatrix} \text{ and } \nu = \begin{pmatrix} a & b \\ \{s_2, j_2\} & \{s_1, j_1\} \end{pmatrix},$$

where under assignment μ, a receives $\{s_1, j_1\}$ and b receives $\{s_2, j_2\}$ (and similar for ν). It is straightforward to verify that μ and ν are both stable (and therefore, do not block each other). Note that both junior candidates point to a. Similarly, both senior candidates point to b, whereas university a points to the two senior candidates and university b points to the two junior candidates.

Our objective is to show that the pointing procedure still leads to two well-defined assignments: assigning each student to the school she points to, and assigning each student to the school pointing to her. Eventually, we will show that if the original assignments are legal, then the induced reassignments are legal. But it is interesting to note that this construction applies to any two individually rational assignments which do not block each other.

Definition 5. *Given assignments μ and ν, define $\mu \wedge \nu$ by $\mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$ for all $a \in O$.*

Our main focus is on any two individually rational assignments μ and ν which do not block each other. Then $\mu \wedge \nu$ is the reassignment resulting from assigning a student to the school that is pointing to her. The following lemma demonstrates that this yields a well-defined assignment.

Lemma 4. *Let μ and ν be two individually rational assignments which do not block each other. Then:*

(i) $\mu \wedge \nu$ is an individually rational assignment;

(ii) if i is assigned a school under μ, then i is assigned a school under $\mu \wedge \nu$; and

(iii) every school receives the same number of students under μ and $\mu \wedge \nu$.

An immediate corollary of Lemma 4 is our version of the Rural Hospitals Theorem (where hospitals correspond to schools in our context).[17] The Rural Hospitals Theorem is an important result for the residency matching program (Roth and Sotomayor, 1992). It says that under any stable assignment, each hospital receives the same number of doctors. It turns out that this result holds far more generally than when it is just applied to stable assignments. In any two individually rational assignments which do not block each other, each school is assigned the same number of students.

Corollary 1 (Rural Hospitals Theorem)**.** *Let μ and ν be two individually rational assignments which do not block each other. Then*

(i) for any school a, $|\mu_a| = |\nu_a|$; and

(ii) for any student i, $\mu_i = i$ if and only if $\nu_i = i$.

Lemma 4 allows us to strengthen the Pointing Lemma.

Corollary 2 (Strong Pointing Lemma)**.** *Let μ and ν be two individually rational assignments which do not block each other.*

(i) If a student is assigned a school under either μ or ν, then she points to one school and is pointed to by one school.

(ii) For any school a, a points to $|\mu_a| = |\nu_a|$ students and $|\mu_a| = |\nu_a|$ students point to a.

(iii) Let $i \in A$ be such that $\mu_i = b$ and $\nu_i = a$. Then $i \in C_a(\mu_a \cup \{i\})$ if and only if $i \in C_a(\mu_a \cup \nu_a)$.

[17] One could also refer to this as the "Rural Schools Theorem" in our context with the appropriate interpretation.

Proof. We show (i) and (ii) in the Appendix.

(iii): By substitutability of C_a, if $i \in C_a(\mu_a \cup \nu_a)$, then $i \in C_a(\mu_a \cup \{i\})$. In showing the other direction, suppose that $i \in C_a(\mu_a \cup \{i\})$ but $i \notin C_a(\mu_a \cup \nu_a)$. Because μ and ν do not block each other, we must have $b = \mu_i P_i \nu_i = a$ and i does not point to a. Thus, i points to $\mu_i = b$. Because $i \notin C_a(\mu_a \cup \nu_a)$, school a does not point to i. But then by (i), school b must point to i meaning $i \in C_b(\mu_b \cup \nu_b)$. Now by substitutability of C_b, we have $i \in C_b(\nu_b \cup \{i\})$. But then i blocks ν with μ, a contradiction. □

We have already established that if we reassign each student to the school pointing to her, then this results in a well-defined assignment. It is immediate from Corollary 2 that reassigning students to the school they are pointing to is an individually rational assignment. We refer to this assignment as $\mu \vee \nu$.

Definition 6. *Let μ and ν be two individually rational assignments which do not block each other. Define the assignment $\mu \vee \nu$ as follows: for all $i \in A$,*

$$\mu \vee \nu_i = \max_{P_i} \{\mu_i, \nu_i\}.$$

Lemma 5. *Let μ and ν be two individually rational assignments which do not block each other. Then $\mu \vee \nu$ is an individually rational assignment.*

We conclude by showing that the Classical Decomposition Lemma generalizes to our environment. In the classical formulation, the Decomposition Lemma asks the men and women "Do you prefer μ or ν?". We do not know the preferences of the schools but instead know their choice functions. The analogous question (for the students) in choice language is "Do you choose your assignment under μ or ν?". Note that by construction, student i's answer is $\mu \vee \nu_i$. We cannot ask a school "Do you choose μ or ν?" since we do not know the schools preferences. However, we can ask them the following question: "Which students do you choose among all the students you were assigned?" Note that by construction, school a's answer is $\mu \wedge \nu_a$. Our generalization of the Classical Decomposition Lemma is to show that there is a one-to-one mapping between the two answers.

Lemma 6 (Generalized Decomposition Lemma)**.** *Let μ and ν be two individually rational assignments which do not block each other, and let i be a student such that $\mu_i \neq \nu_i$. Student i chooses school a if and only if school a rejects i. Formally, $\mu \vee \nu_i = a$ if and only if $i \notin \mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$.*

3.2 Lattice Theorem

Since school choice problems have a non-empty set of stable assignments (the core), the following heuristic way of finding a set of legal assignments (as already suggested by von Neumann-Morgenstern) is plausible.

Recall that $\mathcal{IR}$ denotes the set of all individually rational assignments. We call a function $f : 2^{\mathcal{IR}} \to 2^{\mathcal{IR}}$ an **operator**. We define an operator f to be increasing if $X \subseteq Y \subseteq \mathcal{IR}$ implies $f(X) \subseteq f(Y)$, and analogously, f is decreasing if $X \subseteq Y$ implies $f(X) \supseteq f(Y)$.

The following operator will be central for finding legal assignments. Given any set of assignments $X \subseteq \mathcal{IR}$, $\pi(X)$ is the set of individually rational assignments which are not blocked by any assignment in X:

$$\pi(X) = \{\mu \in \mathcal{IR} \mid \nexists\, \nu \in X \text{ such that } \nu \text{ blocks } \mu\}\,. \tag{1}$$

The following three properties are straightforward to verify but will be useful.

Lemma 7. *The operator π defined in (1) satisfies:*

(i) π is decreasing.

(ii) π^2 is increasing.

(iii) If J is the set of stable assignments, then $J \subseteq \pi(M)$ for any set $M \subseteq \mathcal{IR}$.

Proof. If a student is able to block with more assignments, then fewer assignments will remain unblocked. Therefore, π is a decreasing operator. Consider two sets of assignments X and Y such that $X \subseteq Y \subseteq \mathcal{IR}$. Since π is decreasing, $\pi(Y) \subseteq \pi(X)$. Again, since π is decreasing, $\pi(\pi(X)) \subseteq \pi(\pi(Y))$. Therefore, π^2 is increasing. Finally, stable assignments are not blocked by any assignment. Therefore, they are not blocked by any assignment in $\mathcal{IR}$. □

As it turns out, any legal set of assignments is a fixed point of the operator π (and vice versa).

Lemma 8. *Let $L \subseteq \mathcal{IR}$. Then L is a legal set if and only if $\pi(L) = L$.*

Proof. Suppose L is legal. If $\mu \in L$, then μ is not blocked by any $\nu \in L$. Therefore, $L \subseteq \pi(L)$. Similarly, if $\mu \in \pi(L)$, then by construction there does not exist $\nu \in L$ such that ν blocks μ. Therefore, $\pi(L) \subseteq L$. For the other direction, suppose that $\pi(L) = L$. Then $\mu \notin L$ if and only if $\mu \notin \pi(L)$ (since $L = \pi(L)$) if and only if there exists a $\nu \in L$ such that ν blocks μ (by the definition of π). Therefore, L is legal. □

It is not obvious that a legal set of assignments must exist (we will show this later). Suppose that a legal set of assignments does exist. We define $S^0 = \emptyset$, and we set $B^0 = \pi(S^0)$. Note that $B^0 = \mathcal{IR}$, the set of all individually rational assignments. Continuing, we let $S^1 = \pi(B^0)$. Note that S^1 is the set of stable assignments. In general, we define:

$$\begin{aligned} S^0 &= \emptyset \\ B^k &= \pi(S^k) \\ S^{k+1} &= \pi(B^k) = \pi^2(S^k) \end{aligned}$$

Let L be a legal set of assignments. It is trivially true that $S^0 \subseteq L \subseteq B^0$. If μ is a stable assignment, then μ is not blocked by any assignment. Therefore, the set of stable assignments, S^1, must be contained in L. Moreover, a legal set of assignments must be internally consistent. Since S^1 is contained in any legal set, no assignment blocked by an assignment in S^1 can be part of any legal set. Therefore, $L \subseteq B^1$. Similarly, if L is a legal set of assignments, and μ is not blocked by any assignment in B^1, then μ is not blocked by any assignment in L. Therefore, by external stability, μ must be legal. Therefore, it must be that $S^2 \subseteq L$, and so on.

In general, for any k, if L is a legal set of assignments then:

$$S^0 \subseteq S^1 \subseteq \ldots \subseteq S^k \subseteq L \subseteq B^k \subseteq \ldots \subseteq B^1 \subseteq B^0$$

We seek a fixed point of the operator π; however, it is not obvious that such a fixed point exists. However, since π^2 is an increasing function, a fixed point of π^2 must exist. In particular, since there are only a finite number of possible assignments, there must be a n such that $S^n = S^{n+1}$.[18]

Furthermore, for this fixed point we have $S^n \subseteq \pi(S^n)$: Trivially, $S^0 \subseteq \pi(S^0) = B^0$. Now suppose by induction that we have $S^{k-1} \subseteq \pi(S^{k-1})$. Because π^2 is increasing, we have $\pi^2(S^{k-1}) \subseteq \pi^3(S^{k-1})$. Thus,

$$S^k = \pi^2(S^{k-1}) \subseteq \pi^3(S^{k-1}) = \pi(\pi^2(S^{k-1})) = \pi(S^k),$$

which yields the desired conclusion $S^k \subseteq \pi(S^k)$.

Thus, if S^n is a fixed point of π^2, then the two key properties of S^n are:[19]

$$(1)\, S^n \subseteq \pi(S^n) \text{ and } (2)\, S^n = \pi^2(S^n).$$

Our main challenge will be to show that in fact $S^n = B^n$. This will establish the existence and uniqueness of a legal set of assignments. However, we first establish properties of S^n that will be used in our proof. We will show that any set with properties (1) and (2) is a lattice and satisfies the Rural Hospitals theorem.

So far we have only compared individually rational assignments which do not block each other. Next we strengthen our results by considering the additional structure inherent in S^n. We will show that S^n is a lattice under the following partial order which was inspired

[18] This follows from Tarski's fixed point theorem because $2^{\mathcal{IR}}$ is a partially ordered set with respect to set inclusion and π^2 is increasing. Moreover, Tarski's theorem says that the set of fixed points of π^2 is a lattice with respect to unions and intersections of sets. However, his result does not tell us anything about the structure of the assignments belonging to a fixed point of π^2.

[19] This is very closely related to the concept of a subsolution defined in Roth (1976). What is now called a vNM-stable set was originally referred to by von Neumann and Morgenstern as a **solution**. Roth (1976) introduced a generalization of vNM-stability called a subsolution: A subsolution is any set S such that (1) $S \subseteq \pi(S)$ and (2) $S = \pi^2(S)$ (and we used above Roth's argument to show the existence of a subsolution). The reason we do not call our set S^n a subsolution is that the definition of blocking is different in our framework than under the traditional vNM-stability. We thank Federico Echenique for pointing out this connection.

by Blair (1988) and Martinez et al. (2001).[20] Strikingly, our results are analogous to the properties of the stable set of assignments (Roth and Sotomayor, 1990) and the set of individually rational assignments that eliminate justified envy (Wu and Roth, 2016).

$$\mu \geq \nu \text{ if for every school } a \in O, C_a(\mu_a \cup \nu_a) = \nu_a \tag{2}$$

Lemma 9. *Let μ and ν be two individually rational assignments which do not block each other. Then*

$$\mu \vee \nu \geq \mu \geq \mu \wedge \nu.$$

Lemma 10. *Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $\pi^2(S) = S$. For any $\mu, \nu \in S$, $\mu \vee \nu \in S$ and $\mu \wedge \nu \in S$. In particular, S with partial order $\geq$ is a lattice.*

Let μ^I be the student-optimal assignment in S and let μ^O be the school optimal assignment in S. The key step for the proof of Theorem 1 is to show that any individually rational assignment which is not blocked by S, must lie in between μ^I and μ^O with respect to students' preferences.

Lemma 11. *For every $\lambda \in \pi(S)$ and every student i, $\mu^I_i \; R_i \; \lambda_i \; R_i \; \mu^O_i$.*

3.3 Existence and Uniqueness

We are now ready to prove the main theorem. As a reminder, we set $S^0 = \emptyset$, $S^1 = \pi^2(\emptyset)$, $S^k = \pi^2(S^{k-1})$ and $B^k = \pi(S^k)$. We defined S as the first fixed point of our construction, i.e. $S = \pi^2(S)$. Let $B = \pi(S)$. By Lemma 10, S is a lattice, and we may let μ^I denote the student-optimal assignment in S and μ^O denote the school-optimal assignment in S.

In the Appendix we establish that any such fixed point must be a legal set of assignments.

Theorem 1. *There exists a legal set of assignments.*

We can now prove that there exists a unique legal set of assignments.

Theorem 2. *There exists a unique legal set of assignments.*

Proof. By Lemma 8, L is a legal set of assignments if and only if $\pi(L) = L$.

Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $S = \pi^2(S)$. By the proof of Theorem 1, we have $S = \pi(S)$. Thus, S is legal.

To show uniqueness, let L be any legal set of assignments. By (iii) of Lemma 7, $S^1 \subseteq \pi(L) = L$. By (i) of Lemma 7, π is decreasing. Therefore, $\pi(L) = L \subseteq \pi(S^1) = B^1$. Repeating this argument, we find that

$$S^0 \subseteq S^1 \subseteq \dots S^n \subseteq L \subseteq B^n \subseteq \dots B^1 \subseteq B^2.$$

Since there exists n such that $S^n = B^n$, we conclude that $L = S^n$. □

[20]Note that it is an immediate corollary of Tarski's Fixed Point Theorem that S^n is a lattice. However, we will be able to prove the stronger properties of S^n by using first principles.

4 Discussion

4.1 Efficiency and Non-Wastefulness

First, we discuss various properties of the student-optimal legal assignment. Because any individually rational assignment outside L is illegal, it must be that μ^I is not Pareto dominated by any individually rational assignment.

Proposition 1. *The student-optimal legal assignment μ^I is efficient.*

Proof. Suppose that there exists $\nu \in \mathcal{IR}$ such that for all $i \in I$, $\nu_i R_i \mu_i^I$ and for some $j \in I$, $\nu_j P_j \mu_j^I$. By Lemma 11 and $L = S$, $\nu \notin L$. Since ν is illegal, there exists $\mu \in L$ which blocks ν. Thus, for some $i \in A$ we have $\mu_i P_i \nu_i R_i \mu_i^I$. But again by Lemma 11, $\mu_i^I R_i \mu_i$, which is a contradiction to transitivity of P_i. □

It is well-known that the student-optimal stable assignment is weakly efficient among all individually rational assignments. Hence, Proposition 1 describes the important advantage of the student-optimal legal assignment over the student-optimal stable assignment: the student-optimal legal assignment is "ideal" as it is efficient among all individually rational assignments and legal (or fair à la vNM when students are the only active agents).

As the example below shows, efficiency of the student-optimal legal assignment is not guaranteed when Pareto domination is allowed via non-individually rational assignments (and as it is known, the student-optimal stable assignment is not necessarily weakly efficient). Furthermore, the example establishes that non-individually rationally assignments are not necessarily blocked by legal assignments, and the Pointing Lemma may be violated.

Example 3. *Let $A = \{1, 2\}$ and $O = \{a, b\}$. Let*

P_1	P_2	$\succ_a$	$\succ_b$
a	b	2	1
1	2	a	b
b	a	1	2

where the above stands for $aP_1 1 P_1 b$ and $\succ_b$ stands for $C_b(\{1\}) = C_b(\{1,2\}) = \{1\}$ and $C_b(\{2\}) = \emptyset$, and similarly for $\succ_a$. Let μ^0 be such that $\mu^0_1 = 1$ and $\mu^0_2 = 2$. Then $\mathcal{IR} = \{\mu^0\}$ and $L = \{\mu^0\}$, and μ^0 is the unique stable assignment. Considering μ such that $\mu_1 = a$ and $\mu_2 = b$ we see that μ^0 is not (weakly) efficient. In addition, μ and μ^0 do not block each other but the pointing lemma is violated for these two assignments: 1 and 2 would point to a school but no school would point to a student.

One would expect legal assignments to be non-wasteful. The following example shows that wasteful assignments may be legal. Of course, by Proposition 1, the student-optimal legal assignment is non-wasteful (as otherwise it would not be efficient among individually rational assignments).

Example 4. *Let* $A = \{1, 2\}$ *and* $O = \{a, b, c\}$*. Let*

P_1	P_2	$\succ_a$	$\succ_b$	$\succ_c$
b	a	1	2	1
c	c	2	1	2
a	b	a	b	c
1	2			

.

Let $\mu_1 = b$ *and* $\mu_2 = a$*. It is easy straightforward to verify that* μ *is the unique stable assignment. There is no legal assignment where 1 is assigned school c: to see this, let* ν *be any assignment such that* $\nu_1 = c$*; if* $\nu_2 \neq b$*, then* $\nu_b = \emptyset$*,* $C_b(\nu_b \cup \{1\}) = \{1\}$ *and* bP_1c *meaning that 1 blocks* ν *with* μ*; thus,* $\nu_2 = b$ *and we have* $\nu_a = \emptyset$*,* $C_a(\nu_a \cup \{2\}) = \{2\}$ *and* aP_2b *meaning that 2 blocks* ν *with* μ*. A similar argument shows that there is no legal assignment where 2 is assigned school c. Consider the assignment* μ' *defined by* $\mu'_1 = a$ *and* $\mu'_2 = b$*. There is no legal assignment where 1 is assigned to c and 1 cannot block* μ' *with any assignment where 1 is assigned school b because* $2 \succ_b 1$*. Therefore, 1 cannot block* μ'*, and similarly 2 cannot block* μ'*. Therefore,* μ' *is legal and* $L = \{\mu, \mu'\}$ *is the unique legal set of assignments.*

But the legal assignment μ' *is wasteful because* $cP_1\mu'_1 = a$ *and* $C_c(\mu'_c \cup \{1\}) = C_c(\{1\}) = \{1\}$*.*

Non-wastefulness allows for blocking of students and "empty" slots (in the sense that adding a student to a school would result in the choice of this student and all previously assigned students). However, as we show below, legal assignments satisfy a weaker property of non-wastefulness (where blocking is only allowed with unassigned students and "empty" slots): μ is **weakly non-wasteful** if there exist no student i and school a such that $\mu_i = i$, aP_ii and $C_a(\mu_a \cup \{i\}) = \mu_a \cup \{i\}$.

Proposition 2. *If* μ *is legal* ($\mu \in L$)*, then* μ *is weakly non-wasteful.*

Proof. Let $\mu \in L$. Suppose there exists a student i and a school a such that $\mu_i = i$, aP_ii and $C_a(\mu_a \cup \{i\}) = \mu_a \cup \{i\}$. Let μ' be such that $\mu'_i = a$ and $\mu'_j = \mu_j$ for all $j \in A\backslash\{i\}$. Then by the previous facts and $\mu \in \mathcal{IR}$, it follows that $\mu' \in \mathcal{IR}$. Since $|\mu'_a| = |\mu_a| + 1$ and the Rural Hospitals Theorem holds for all assignments in L, we have $\mu' \notin L$. Thus, there exist $j \in A$ and $\nu \in L$ such that j blocks μ' with ν. Thus, $\nu_jP_j\mu'_j$ and (letting $\nu_j = b$) $j \in C_b(\mu'_b \cup \{j\})$. Since $\mu_b \subseteq \mu'_b$, substitutability of C_b implies $j \in C_b(\mu_b \cup \{j\})$. If $j \neq i$, then $\mu'_j = \mu_j$ and j blocks μ with ν, a contradiction to $\mu, \nu \in L$. If $j = i$, then bP_iaP_ii and by $\mu_i = i$, i blocks μ with ν, again a contradiction to $\mu, \nu \in L$. □

4.2 General EADA

Below we provide an algorithm for calculating the student-optimal legal assignment. The deferred-acceptance (DA) assignment is the student-optimal stable assignment and it is found by the (student-proposing) deferred-acceptance (DA) algorithm.[21] To the best of

[21]The proof of Lemma 11 contains a formal description of the DA-algorithm.

our knowledge, Kesten's efficiency adjusted DA (EADA) has only been defined for responsive choice functions. Kesten's original EADA mechanism and the simplified EADA mechanism (hereafter sEADA) introduced by Tang and Yu (2014) produce the same assignment when schools have responsive choice functions. The sEADA is based on the concept of an underdemanded school.

For a given assignment μ, a school a is **underdemanded** if for every student i, $\mu_i R_i a$. For responsive priorities, sEADA is defined as follows.

The (simplified) Efficiency Adjusted Deferred Acceptance Mechanism (sEADA) when choice functions are acceptant:
Round 0: Run DA for the problem P. For each underdemanded school[22] a and each student i assigned to a, permanently assign i to a and then remove both i and a.
Round k: Run DA on the remaining population. For each underdemanded school a and each student i assigned to a, permanently assign i to a and then remove both i and a.
Stop when no school is underdemanded.

Tang and Yu (2014) note two facts which are critical for their mechanism. First, under the DA assignment, there always exists an underdemanded school. For example, the last school that any student applies to is an underdemanded school. Second, a student assigned by DA to an underdemanded school cannot be part of a Pareto improvement. However, as Example 5 demonstrates, when choice functions are not responsive, there does not necessarily exist an underdemanded school. In this case, sEADA no longer produces an efficient assignment.

Example 5. *Let $O = \{a, b, c, d\}$ and $A = \{1, 2, 3, 4, 5\}$, and suppose $q_a = 2$ while all other schools have a capacity of 1. Suppose the preferences of the students and the priorities of the schools (other than a) are defined as below (where we specify the two highest ranked elements only):*

P_1	P_2	P_3	P_4	P_5	$\succ_b$	$\succ_c$	$\succ_d$
b	a	a	c	d	3	2	4
a	c	b	d	a	1	4	5

School a has more complicated preferences. Intuitively, a chooses at most one student from students 1, 2, and 3 (where the students are ranked $\succ_a$: $1, 2, 3$) and at most one student from 4 and 5 (where $\succ'_a$: $4, 5$). More formally, given a set of students X,

$$C_a(X) = (\max_{\succ_a} X \cap \{1, 2, 3\}) \cup (\max_{\succ'_a} X \cap \{4, 5\})$$

Note that C_a is substitutable and satisfies LAD. The DA-assignment is given by:

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & c & b & d & a \end{pmatrix}$$

[22]Note that a student may also be unassigned. For expositional convenience, we interpret being unassigned as being assigned to the null school which has unlimited capacity. Since the DA assignment is individually rational, every student weakly prefers her assignment to being unassigned. Therefore, the null school is underdemanded.

However, there is no underdemanded school as 2 would prefer a, 1 would prefer b, 4 would prefer c, and 5 would prefer d. Further, the DA assignment is Pareto dominated by the following individually rational assignment:

$$\nu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ b & c & a & d & a \end{pmatrix}$$

Note that ν is not Pareto dominated by any other individually rational assignment (because assigning 2 and 3 to a is not individually rational). It is straightforward to verify that ν is legal.[23]

As Example 5 demonstrates, eliminating underdemanded schools with their assigned students does not work and sEADA does not find an efficient assignment. The following notion will turn out to be important for our algorithm.

Definition 7. *Let $\mu \in \mathcal{IR}$ and $i \in A$. Then student i is **irrelevant** for μ if for $\mu_i = a$ we have*

$$C_a(\{j \in A | aR_j\mu_j\} \backslash \{i\}) \subseteq \mu_a.$$

In words, student i is irrelevant for μ if student i is assigned to a under μ and school a chooses from the set of students, who weakly prefer a to their assignment, excluding i, a subset of the students assigned to a under μ. Then it is irrelevant whether student i is present, because from the set of students, who weakly prefer a, school a does not choose any new ones. Notice that in contrast to underdemanded schools, this is a condition in terms of students.

Example 5 (continued). *In Example 5, student 5 is irrelevant for μ because we have $\mu_i = a$ and*

$$C_a(\{j \in A | aR_j\mu_j\} \backslash \{5\}) = C_a(\{1,2,3,5\} \backslash \{5\}) = C_a(\{1,2,3\}) = \{1\} \subseteq \{1,5\} = \mu_a.$$

It is easy to see that 5 is the only student who is irrelevant for μ. As it turns out, the following iterative procedure will work: identify all irrelevant students for μ replace their preferences with ones where their assigned school is the unique acceptable school. Let P^1_5 denote the preference relation such that a is the only acceptable school, and let $P^1 = (P^1_5, P_{-5})$ (i.e. we replace 5's preference with P^1_5 and leave all other preferences unchanged). Now running the DA algorithm for P^1 gives us still μ. Now student 4 is irrelevant for μ (under P^1) because we have $\mu_4 = d$ and (noting $aP^1_5 d$)

$$C_d(\{j \in A | dR^1_j\mu_j\} \backslash \{4\}) = C_d(\{4\} \backslash \{4\}) = \emptyset \subseteq \mu_d.$$

[23] To see that it is not blocked by any legal assignment, note that the only student with justified envy is 2. However, if 2 is assigned to a, then 1 must be assigned to b or else 1 will block with the DA assignment. But if 1 is assigned to b, then 3 must be assigned to a or else she will block with the DA assignment. However, it is not individually rational to assign both 2 and 3 to a.

Let P_4^2 denote the preference relation such that d is the only acceptable school, and let $P^2 = (P_4^2, P_{-4}^1)$. Then we obtain the preference profile

P_1	P_2	P_3	P_4^2	P_5^1
b	a	a	d	a
a	c	b		

Running now DA for P^2 gives us the assignment

$$\eta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & c & b & d & a \end{pmatrix}.$$

Now student 2 is irrelevant for η (under P^2) because we have $\eta_2 = c$ and (noting dP_4^2c)

$$C_c(\{j \in A | dR_j^2 \eta_j\} \backslash \{2\}) = C_c(\{2\} \backslash \{2\}) = \emptyset \subseteq \mu_c.$$

Let P_2^3 denote the preference for 2 such that c is the only acceptable school, and let $P^3 = (P_2^3, P_{-2}^2)$. Then we obtain the preference profile

P_1	P_2^3	P_3	P_4^2	P_5^1
b	c	a	d	a
a	a	b		

Running now DA gives us the assignment ν, which is the desired Pareto improvement over μ (and all students are irrelevant for ν under P^3).

The following result show the important facts of irrelevant students.

Given assignment μ and student i, we say that student i is **Pareto improvable** if there exists $\nu \in \mathcal{IR}$ such that $\nu_i P_i \mu_i$ and for all $j \in A$, $\nu_j R_j \mu_j$. This simply means that there exists a Pareto improvement over μ where i strictly prefers his assigned school to the one from μ.

Lemma 12. *Let μ be the DA assignment.*

(i) There always exists a student who is irrelevant for μ.

(ii) If student i is irrelevant for μ, then i is not Pareto improvable.

We will show that the algorithm below works for any choice functions satisfying substitutability and LAD.

The general Efficiency Adjusted Deferred Acceptance Mechanism (gEADA):
Round 0: Run DA for the problem P. Let μ^0 denote the DA assignment, $I^0 = \emptyset$ and $P^0 = P$.
Round k: This round consists of two steps.

1. Let I^k denote the set of all students who are irrelevant for μ^{k-1}. If $i \in I^k$, then let P_i^k be the preference for i where μ_i^{k-1} is the only acceptable school. If $i \notin I^k$, then let $P_i^k = P_i^{k-1}$, and let P^k denote the resulting profile.

2. Let μ^k denote the DA assignment obtained from P^k.

Stop when $I^k = A$.

The following shows that the gEADA algorithm is well defined.

Lemma 13. *(i) For all $k \geq 1$, $I^{k-1} \subseteq I^k$ and μ^k Pareto dominates μ^{k-1}.*

(ii) If $A \backslash I^{k-1} \neq \emptyset$, then $I^k \backslash I^{k-1} \neq \emptyset$.

The following captures the two key features of the gEADA algorithm: the output of gEADA is efficient and it coincides with the student-optimal legal assignment. Thus, the gEADA algorithm offers a polynomial algorithm to determine the student-optimal legal assignment. In the Appendix, we generalize this result to the setting of assignment with contracts, i.e. even in this general setting we are able to determine the student-optimal legal assignment.

Theorem 3. *(i) The gEADA assignment is efficient.*

(ii) The output of gEADA algorithm coincides with the student-optimal legal assignment.

Below we show that gEADA and sEADA coincide when schools have responsive priorities with quotas: a student is assigned to an underdemanded school if and only if the student is irrelevant.

Lemma 14. *Let schools have responsive priorities with quotas, μ be the DA assignment and $i \in A$. Then student i is irrelevant for μ if and only if μ_i is underdemanded.*

Proof. Because school a has responsive priorities with quota q_a and strict priority $\succ_a$ over students, C_a chooses from any set the q_a highest $\succ_a$-ranked students (all if there are fewer than q_a students in the set).

If student i is irrelevant for μ and $\mu_i = a$, then $C_a(\{j \in A | aR_j\mu_j\} \backslash \{i\}) \subseteq \mu_a$. Because $i \in \mu_a$, we have $C_a(\{j \in A | aR_j\mu_j\} \backslash \{i\}) \subseteq \mu_a \backslash \{i\}$ and $|\mu_a \backslash \{i\}| \leq q_a - 1$. Because C_a chooses the first q_a elements according to $\succ_a$ (if possible), we must have $\{j \in A | aR_j\mu_j\} \backslash \{i\} = \mu_a \backslash \{i\}$. Thus, for all $j \in A$, $\mu_j R_j a$ and school a is underdemanded.

In showing the other direction, let a be underdemanded and $i \in \mu_a$. Then for all $j \in A$, $\mu_j R_j a$, and $C_a(\{j \in A | aR_j\mu_j\} \backslash \{i\}) = C_a(\mu_a \backslash \{i\}) \subseteq \mu_a$ where the last inclusion follows from the fact that C_a is responsive with quota q_a. Thus, i is irrelevant for μ. □

Remark 2. *First, by Lemma 14 and Theorem 3, for responsive priorities, the student-optimal legal assignment and EADA coincide. Thus, the student-optimal legal assignment and the student-optimal "possible" assignment by Morrill (2017) coincide with the assignment made by EADA. Second, we generalize the sEADA by Tang and Yu (2014) from* *responsive priorities to substitutable and acceptant choice functions. Third, the student-optimal legal assignment offers a foundation for the extension of Kesten's EADA from responsive priorities to choice functions satisfying substitutability and LAD.*[24]

4.3 Strategy-Proofness

Below we consider centralized mechanism design where students have to report their preferences to the clearinghouse. We keep everything fixed except for students' preferences. Let $\mathcal{P}^i$ denote the set of all i's strict preferences over $O \cup \{i\}$, and $\mathcal{P}^A = \times_{i \in A} \mathcal{P}^i$.

A mechanism is a function $\varphi : \mathcal{P}^A \to \mathcal{A}$ choosing for profile P assignment $\varphi(P)$. Then φ is strategy-proof if for all $i \in A$, all $P \in \mathcal{P}^A$ and all $P'_i \in \mathcal{P}^i$ we have $\varphi_i(P) R_i \varphi_i(P'_i, P_{-i})$. This means that reporting the truth is a weakly dominant strategy. A mechanism is legal if for all profiles P, $\varphi(P)$ is a legal assignment.

Let DA denote the student-proposing deferred-acceptance mechanism.

Theorem 4. *DA is the unique strategy-proof and legal mechanism.*

Proof. Because DA is stable, we have that DA is legal. Strategy-proofness of DA has been established by Roth (1982) and Dubins and Freedman (1982).

In showing the converse, let φ be strategy-proof and legal. We show that for all $P \in \mathcal{P}^A$ and all $i \in A$, $\varphi_i(P) R_i DA_i(P)$. Suppose not. Then there exists $i \in A$ such that $DA_i(P) P_i \varphi_i(P)$. Thus, by individual rationality, $DA_i(P) \neq i$, say $DA_i(P) = a$. Let $P'_i \in \mathcal{P}^i$ be such that for all $b \in O$, (i) if $b R_i a$, then $b R'_i a$ and (ii) if $a P_i b$, then $a P'_i i P'_i b$. By construction, stability of $DA(P)$ under P implies stability of $DA(P)$ under (P'_i, P_{-i}). Thus, $DA(P)$ is legal under (P'_i, P_{-i}). Then by the rural hospitals theorem of legal assignments, we have $\varphi_i(P'_i, P_{-i}) \neq i$. Thus, by construction of P'_i,

$$\varphi_i(P'_i, P_{-i}) R_i a P_i \varphi_i(P),$$

which implies that φ is not strategy-proof, a contradiction.

Hence, we have shown for all $P \in \mathcal{P}^A$ and all $i \in A$, $\varphi_i(P) R_i DA_i(P)$. If $\varphi \neq DA$, then φ must Pareto dominate DA. This is a contradiction to Abdulkadiroğlu, Pathak and Roth (2009) who show that no strategy-proof mechanism can Pareto dominate DA. □

Thus, by Theorem 4, any strategy-proof mechanism different than DA must be illegal. In particular, the top-trading cycles algorithm is illegal (and it is easy to see that all variants of the Boston mechanism are illegal).[25]

5 Conclusion

When a school board chooses an assignment mechanism, it typically balances efficiency and fairness. However, a critical pragmatic consideration for any board is which assignments are legal. We show that there is a unique set of legal assignments, and that there is

[24] Note that it is even not clear what the right formulation of Kesten's EADA is for these environments.
[25] One may also use Alva and Manjunath (2016) to show Theorem 4.

always a unique efficient assignment that is legal. Prior to our work, it was thought that there was no "ideal" solution to the school assignment problem as it is impossible for a mechanism to be both efficient and eliminate justified envy. When elimination of justified envy is more important than efficiency, the DA mechanism was recommended, and when efficiency is more important than elimination of justified envy, the TTC mechanism was recommended. However, we show for the set of individually rational assignments, when considering fairness as a setwise solution concept (where justified envy is eliminated only with legal assignments), there exists a unique legal and efficient assignment. Independently in which order we regard efficiency and legality (vNM-stability), we obtain the same outcome, namely the student-optimal legal assignment. It is fair in a meaningful way, and it Pareto dominates any other fair or legal assignment. Combined, our results offer a foundation of the generalization of the assignment made by Kesten's EADA from responsive choice priorities to our general framework. Our contribution is the first one to propose a setwise solution concept when choice functions are not necessarily responsive. One may see this as the ideal school assignment.

Most importantly, we generalize all our results to the framework of assignment with contracts (or matching with contracts): any contract is associated with one student and one school. School choice is the special case where for any student-school pair there exists exactly contract which is associated with this pair. The general framework allows to capture many other applications where the terms of the match can vary. The Appendix shows that all our conclusions continue to hold for this important framework, and therefore, the student-optimal legal assignment provides an efficient and vNM-stable solution for these applications. In particular, we provide an algorithm (which one could call the efficiency-adjusted cumulative offer process) for calculating the student-optimal legal assignment in the contracts framework.

APPENDIX: ASSIGNMENT WITH CONTRACTS

Below we generalize all our results from school choice to matching with contracts. For the Appendix, we use Lemma N' to denote Lemma N from the main text translated to the setting of assignment with contracts (and similarly, for Corollary N'). The structure of the Appendix is parallel to the one in the main text, we follow the same order as for school choice.

A Model

Recall that A denotes the set of students and O denotes the set of schools. Let $\mathcal{X}$ denote the set of all contracts. Each contract $x \in \mathcal{X}$ is associated with one student $x_A \in A$ and one school $x_O \in O$. Given $Y \subseteq \mathcal{X}$, let Y_i denote the set of contracts associated with student i and Y_a denote the set of contracts associated with school a. In school choice, we simply have $\mathcal{X}_i = O$ for all $i \in A$ (i.e. there is exactly one contract associated with any school).

Each student i has a strict preference P_i over $\mathcal{X}_i \cup \{i\}$. Let C_i denote the choice function induced by P_i: for any $Y \subseteq \mathcal{X}$, let $C_i(Y) = \max_{P_i} Y_i \cup \{i\}$.

Any school a has a choice function $C_a : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$ such that for any $Y \subseteq \mathcal{X}$ we have $C_a(Y) \subseteq Y_a$. Substitutability and LAD are straightforward to adapt to the setup with contracts: Let $a \in A$ and $C_a : 2^A \to 2^A$ be a choice function.

(a) The choice function C_a is **substitutable** if for all $X \subseteq Y \subseteq A$ we have $C_a(Y) \cap X \subseteq C_a(X)$.

(b) The choice function C_a satisfies the **law of aggregate demand (LAD)** if for all $X \subseteq Y \subseteq A$ we have $|C_a(X)| \leq |C_a(Y)|$.

Any $\mu \subseteq \mathcal{X}$ is an assignment. An assignment μ is **individually rational** if for all $i \in A$, $\mu_i = C_i(\mu)$ and for all $a \in O$, $C_a(\mu) = \mu_a$. Let $\mathcal{IR}$ denote the set of all individually rational assignments. Again, throughout we consider only individually rational assignments. An assignment μ is **efficient** (among all individually rational assignments) if there exists no $\nu \in \mathcal{IR}$ such that $\nu_i R_i \mu_i$ for all $i \in A$ and $\nu_j P_j \mu_j$ for some $j \in A$.

Given assignment μ, student i and school a **block** μ via contract x if $x P_i \mu_i$ and $x \in C_a(\mu \cup \{x\})$ (where this implies $x_A = i$ and $x_O = a$). An assignment μ is **non-wasteful** if there do not exist i and a and a contract x such that $x P_i \mu_i$ and $C_a(\mu_a \cup \{x\}) = \mu_a \cup \{x\}$. An assignment is **fair** if there do not exist i and a and a contract x such that $x P_i \mu_i$ and $x \in C_a(\mu_a \cup \{x\}) \neq \mu_a \cup \{x\}$. An assignment is **stable** if it is individually rational, non-wasteful and fair.

B Legal Assignments

Now blocking among assignments carries over in a straightforward fashion: i blocks μ with ν if for some $x \in \mathcal{X}_i$,

(1) $xP_i\mu_i$,

(2) $x \in C_a(\mu_a \cup \{x\})$ and

(3) $\nu_i = x$.

Then μ blocks ν if there exists a student i who blocks μ with ν.

Now $L \subseteq \mathcal{IR}$ is a **legal** set of assignments if and only if

(i) for all $\nu \in \mathcal{IR} \backslash L$ there exists $\mu \in L$ such that μ blocks ν and

(ii) for all $\mu, \nu \in L$, μ does not block ν.

B.1 Pointing, Decomposition and Rural Hospitals Theorem

Regarding pointing, we let students and schools point to contracts instead of pointing to schools and students. Given two assignments μ and ν, student i points to μ_i (ν_i) if $\mu_i R_i \nu_i$ ($\nu_i R_i \mu_i$) and school a points to $x \in \mathcal{X}$ if $x \in C_a(\mu_a \cup \nu_a)$. Then Lemma 3 (Weak Pointing Lemma) carries over in the following way: let μ and ν be two individually rational assignments which do not block each other. Then (i) no student and school point to the same contract unless the contract belongs to μ and ν and (ii) no two schools point to two contracts which are associated with the same student.

Lemma 3'. *(Weak Pointing Lemma) Let μ and ν be two individually rational assignments which do not block each other. Then:*

(i) no student and school point to the same contract unless the contract belongs to both μ and ν, and

(ii) no two schools point to two contracts which are associated with the same student.

Proof. Consider any student i such that $\mu_i \neq \nu_i$. Without loss of generality, assume $\mu_i P_i \nu_i$. By individual rationality of μ and ν, we have $\mu_i \neq i$. Let $(\mu_i)_O = a$. Then i points to μ_i. By substitutability of C_a and $\mu_i \in \mu_a$, if $\mu_i \in C_a(\mu_a \cup \nu_a)$, then $\mu_i \in C_a(\nu_a \cup \{\mu_i\})$. Therefore, if a pointed to μ_i (meaning $\mu_i \in C_a(\mu_a \cup \nu_a)$), then i would block ν with μ (because $\mu_i \in \mu_a$), a contradiction. For any student i such that $\mu_i \neq \nu_i$, by $\mu_i R_i i$ and $\nu_i R_i i$, i must point to a contract. Therefore, if two schools point to two contracts associated with the same student, there must be a student and a school pointing to the same contract which would be a contradiction to the above. □

Definition 5'. Given assignments μ and ν, define $\mu \wedge \nu$ by $\mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$ for all $a \in O$.

Our main focus is on any two individually rational assignments μ and ν which do not block each other. The following lemma demonstrates that $\mu \wedge \nu$ yields a well-defined assignment. Note that receiving a contract corresponds to receiving a student in the school choice model.

Lemma 4'. *Let μ and ν be two individually rational assignments which do not block each other. Then:*

(i) $\mu \wedge \nu$ is an individually rational assignment;

(ii) if $\mu_i \neq i$, then $\mu \wedge \nu_i \neq i$; and

(iii) every school receives the same number of contracts under μ and $\mu \wedge \nu$, i.e. $|\mu_a| = |\mu \wedge \nu_a|$.

Proof. (i): Suppose for contradiction that there are two contracts $x \neq y$ with $x_A = y_A = i$ and $a, b \in O$ such that both $x \in \mu \wedge \nu_a$ and $y \in \mu \wedge \nu_b$. Then $x \in C_a(\mu_a \cup \nu_a)$ and $y \in C_b(\mu_b \cup \nu_b)$. Then a points to x and b points to y. Then ($x \in \mu_a$ and $y \in \nu_b$) or ($x \in \mu_b$ and $y \in \nu_a$), and i must point to either x or y. Therefore, there is a student and a school pointing to the same contract which contradicts the Pointing Lemma. In showing that $\mu \wedge \nu$ is individually rational, we have by definition $C_a(\mu \wedge \nu_a) = \mu \wedge \nu_a$.[26] Furthermore, $\mu_i R_i i$ and $\nu_i R_i i$ imply $\mu \wedge \nu_i R_i i$. Hence, $\mu \wedge \nu \in \mathcal{IR}$.
(ii) and (iii): For counting purposes, in this proof we use the convention $|\mu_i| = 1$ if $\mu_i \neq i$ and $|\mu_i| = 0$ if $\mu_i = i$. First note that if $\mu \wedge \nu_i = x$ but $\mu_i = i$, then i blocks μ with ν: by individual rationality, $\nu_i = x P_i i$; and $x \in C_a(\mu_a \cup \nu_a)$ and substitutability of C_a imply $x \in C_a(\mu_a \cup \{x\})$. Therefore, $|\mu \wedge \nu_i| = 1$ implies that $|\mu_i| = 1$ and $|\nu_i| = 1$. Hence,

$$\sum_{i \in A} |\mu \wedge \nu_i| \leq \sum_{i \in A} |\mu_i|. \tag{3}$$

By the Law of Aggregate Demand and $\mu_a \cup \nu_a \supseteq \mu_a$, $|C_a(\mu_a \cup \nu_a)| \geq |C_a(\mu_a)|$. Therefore,

$$\sum_{a \in O} |\mu \wedge \nu_a| \geq \sum_{a \in O} |\mu_a| \tag{4}$$

Note that for any assignment λ we have

$$\sum_{i \in A} |\lambda_i| = \sum_{a \in O} |\lambda_a|. \tag{5}$$

Combining the three equations yields that $\sum_{i \in A} |\mu \wedge \nu_i| = \sum_{i \in A} |\mu_i|$. Since $|\mu \wedge \nu_i| = 1$ implies that $|\mu_i| = 1$, it must also be that $|\mu_i| = 1$ implies that $|\mu \wedge \nu_i| = 1$. Similarly, since $|\mu \wedge \nu_a| \geq |\mu_a|$ for every school a and $\sum_{a \in O} |\mu \wedge \nu_a| = \sum_{a \in O} |\mu_a|$, it must be that for every school a, $|\mu_a| = |\mu \wedge \nu_a|$. □

[26] Note that substitutability and LAD of C_a imply IRC: for all $X \subseteq Y$, if $C_a(Y) \subseteq X$, then $C_a(X) = C_a(Y)$.

An immediate corollary of Lemma 4' is our version of the Rural Hospitals Theorem for the assignment with contracts setting.

Corollary 1'. *(Rural Hospitals Theorem) Let μ and ν be two individually rational assignments which do not block each other. Then*

(i) for any school a, $|\mu_a| = |\nu_a|$; and

(ii) for any student i, $\mu_i = i$ if and only if $\nu_i = i$.

Proof. By Lemma 4', $|\mu_a| = |\mu \wedge \nu_a| = |\nu_a|$ (which implies (i)), and if $\mu_i \neq i$, then $\mu \wedge \nu_i \neq i$. Let $(\mu_i)_O = a$. If $\nu_i = i$, then by individual rationality of μ and ν, we have $\mu_i P_i i$, and by Lemma 4', $\mu \wedge \nu_i = \mu_i$. Thus, $\mu_i \in C_a(\mu_a \cup \nu_a)$ and by substitutability of C_a, $\mu_i \in C_a(\nu_a \cup \{\mu_i\})$, which implies that i blocks ν with μ, a contradiction. □

Lemma 4' allows us to strengthen the Pointing Lemma.

Corollary 2'.*(Strong Pointing Lemma) Let μ and ν be two individually rational assignments which do not block each other.*

(i) If a student is assigned a contract under either μ or ν, then she points to a contract and one school points to a contract which is associated with her.

(ii) For any school a, a points to $|\mu_a| = |\nu_a|$ contracts and $|\mu_a| = |\nu_a|$ students point to contracts associated with a.

Proof. (i): Consider a student i who is assigned a contract under either μ or ν. By $\mu, \nu \in \mathcal{IR}$, i points to one contract by strict preferences. By (ii) of Lemma 4', $\mu \wedge \nu_i \neq i$. Without loss of generality, $\mu \wedge \nu_i = \mu_i$ and $(\mu_i)_O = a$. Since $\mu_i \in C_a(\mu_a \cup \nu_a)$, a points to μ_i. Two schools cannot point to two contracts associated with i, or else we would violate the Pointing Lemma.
(ii): This follows from the same counting exercise as in the proof of Lemma 4'. If some school a had fewer than $|\mu_a|$ students pointing to contracts associated with a, then some school b would have to have more than $|\mu_b|$ students pointing to contracts associated with b. Then b would have to point to one of these contracts which would contradict the Pointing Lemma. □

We have already established that if we reassign each contract to the school which is pointing to it that this results in a well-defined assignment. We now show that reassigning each student to the contract she is pointing to is also a well-defined assignment. We refer to this assignment as $\mu \vee \nu$.

Definition 6'. *Let μ and ν be two individually rational assignments which do not block each other. Define the assignment $\mu \vee \nu$ as follows: for all $i \in A$, $\mu \vee \nu_i = \max_{P_i} \{\mu_i, \nu_i\}$.*

Lemma 5'. *Let μ and ν be two individually rational assignments which do not block each other. Then $\mu \vee \nu$ is an individually rational assignment.*

Proof. First, we show that for every school a, $C_a(\mu \vee \nu_a \cup \mu_a) = \mu_a$ (and symmetrically that $C_a(\mu \vee \nu_a \cup \nu_a) = \nu_a$). Suppose for contradiction that $C_a(\mu \vee \nu_a \cup \mu_a) \neq \mu_a$. Since μ is individually rational, we have $C_a(\mu_a) = \mu_a$. By the Law of Aggregate Demand, $|C_a(\mu \vee \nu_a \cup \mu_a)| \geq |\mu_a|$, so if $C_a(\mu \vee \nu_a \cup \mu_a) \neq \mu_a$, there must exist $x \in C_a(\mu \vee \nu_a \cup \mu_a)$ such that $x \notin \mu_a$. Let $x_A = i$. Therefore, $\mu \vee \nu_i = x$ and $\nu_i = x$. In words, since $\mu \vee \nu_i = x$, i prefers $\nu_i = x$ to μ_i. Since $x \in C_a(\mu \vee \nu_a \cup \mu_a)$, by substitutability of C_a, $x \in C_a(\mu_a \cup \{x\})$. Therefore, i blocks μ with ν which is a contradiction.

Second, we prove the lemma. By construction, each student is assigned only one contract, and by individual rationality of μ and ν we have $\mu \vee \nu_i R_i i$. We must show that for every school a, $C_a(\mu \vee \nu_a) = \mu \vee \nu_a$. By definition, $C_a(\mu \vee \nu_a) \subseteq \mu \vee \nu_a$. Suppose $\mu \vee \nu_i = x$ and $x_O = a$. Assume without loss of generality that $\mu_i = x$. We have already shown that $C_a(\mu \vee \nu_a \cup \mu_a) = \mu_a$. Since $x \in \mu_a$, $x \in C_a(\mu \vee \nu_a \cup \mu_a)$. Therefore, by substitutability of C_a and $x \in \mu \vee \nu_a$, $x \in C_a(\mu \vee \nu_a)$. □

Lemma 6'. *(Generalized Decomposition Lemma) Let μ and ν be two individually rational assignments which do not block each other, and let i be a student such that $\mu_i \neq \nu_i$. Student i chooses contract x if and only if school $a = x_O$ rejects x. Formally, $\mu \vee \nu_i = x$ if and only if $x \notin \mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$.*

Proof. Suppose that $\mu_i \neq \nu_i$ and without loss of generality assume that i points to $\mu_i = x$, and $x_O = a$. If x is not rejected by a ($x \in \mu \wedge \nu_a$), then a points to x. This contradicts the Weak Pointing Lemma which says that a student and a school cannot point to the same contract. Similarly, suppose that $\mu_i = x$ but that school a rejects x ($x \notin \mu \wedge \nu_a$). Then school a does not point to x. By the Strong Pointing Lemma and since a school and a student cannot point to the same contract, it follows that i points to $\mu_i = x$. □

B.2 Lattice Theorem

The operator $\pi : 2^{\mathcal{IR}} \to 2^{\mathcal{IR}}$ is defined in the same way as in the main text, and its properties carry over without change, namely Lemma 7 and Lemma 8, and that there exists n such that (1) $S^n \subseteq \pi(S^n)$ and (2) $S^n = \pi^2(S^n)$.

As a reminder, we defined $S^0 = \emptyset$ (and thus, $\pi(\emptyset) = \mathcal{IR}$), and in general let $S^k = \pi^2(S^{k-1})$ and $B^k = \pi(S^k)$. Since π^2 is increasing, eventually $S^n = S^{n+1}$ for some n. The two key properties of S^n are (1) $S^n \subseteq \pi(S^n)$ (for any two assignments $\mu, \nu \in S^n$, μ and ν do not block each other); and (2) $S^n = \pi^2(S^n)$ (if $\mu \notin S^n$, then μ is blocked by an assignment in $\pi(S^n)$).

The following result from Blair (1988) will be useful.[27]

Lemma 15 (Blair 1988, Proposition 2.3)**.** *For all $X, Y \in 2^{\mathcal{X}}$ and all $a \in O$, $C_a(X \cup Y) = C_a(C_a(X) \cup Y)$.*

[27]For completeness, we include its proof.

Proof. Let $x \in C_a(X \cup Y)$. If $x \in C_a(X) \cup Y$, then by substitutability of C_a we have $x \in C_a(C_a(X) \cup Y)$. If $x \notin C_a(X) \cup Y$, then $x \in X \setminus C_a(X)$. But this contradicts substitutability of C_a as $x \in C_a(X \cup Y)$ and $x \in X$ imply $x \in C_a(X)$. Thus, $C_a(X \cup Y) \subseteq C_a(C_a(X) \cup Y)$.

Because $C_a(X) \subseteq X$, LAD implies $|C_a(X \cup Y)| \geq |C_a(C_a(X) \cup Y)|$. Hence, $C_a(X \cup Y) = C_a(C_a(X) \cup Y)$. □

We define the following partial ordering over assignments:

$$\mu \geq \nu \text{ if for every school } a \in O, C_a(\mu_a \cup \nu_a) = \nu_a \tag{6}$$

Lemma 9'. *Let μ and ν be two individually rational assignments which do not block each other. Then $\mu \vee \nu \geq \mu \geq \mu \wedge \nu$.*

Proof. Let $a \in O$. By definition, $\mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$. Therefore:

$$\begin{aligned} C_a(\mu_a \cup (\mu \wedge \nu_a)) &= C_a(\mu_a \cup C_a(\mu_a \cup \nu_a)) \\ &= C_a(\mu_a \cup \mu_a \cup \nu_a) \\ &= C_a(\mu_a \cup \nu_a) \\ &= \mu \wedge \nu_a \end{aligned}$$

where the second equality follows from Lemma 15. Therefore, $\mu \geq \mu \wedge \nu$ (and of course, by symmetry, $\nu \geq \mu \wedge \nu$).

In the proof of Lemma 5' we demonstrated that for every school a, $C_a(\mu \vee \nu_a \cup \mu_a)$. Therefore, by definition, $\mu \vee \nu \geq \mu$. □

Lemma 10'. *Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $\pi^2(S) = S$. For any $\mu, \nu \in S$, $\mu \vee \nu \in S$ and $\mu \wedge \nu \in S$. In particular, S with partial order $\geq$ is a lattice.*

Proof. Let $B = \pi(S)$. By assumption, $S \subseteq B$ and $S = \pi(B)$. Therefore, μ and ν are not blocked by any assignment in B, and in particular, μ and ν do not block each other. We have already shown that $\mu \vee \nu$ and $\mu \wedge \nu$ are well-defined assignments. Furthermore, by individual rationality of μ and ν and (ii) of Lemma 4', $\mu \wedge \nu_i R_i i$ for all $i \in A$, and by definition, $C_a(\mu \wedge \nu_a) = C_a(\mu_a \cup \nu_a) = \mu \wedge \nu_a$. Thus, $\mu \wedge \nu \in \mathcal{IR}$. By Lemma 5', $\mu \vee \nu \in \mathcal{IR}$. All that remains is to show that $\mu \vee \nu$ and $\mu \wedge \nu$ are not blocked by any assignment in B.

Suppose for contradiction that i blocks $\mu \wedge \nu$ with $\lambda \in B$. By individual rationality of $\mu \wedge \nu$, we have $\lambda_i \neq i$, say $\lambda_i = x$ and $x_O = b$. If $\mu \wedge \nu_i = i$, then by (ii) of Lemma 4' we have $\mu_i = i$ and $\nu_i = i$. But then by substitutability of C_b and $x \in C_b(\mu \wedge \nu_b \cup \{x\}) = C_b(\mu_b \cup \nu_b \cup \{x\})$, we have $x \in C_b(\mu_b \cup \{x\})$. Because $x \notin \mu_b$, now i blocks μ with λ, a contradiction to $\mu \in S$. Thus, $\mu \wedge \nu_i \neq i$, say $\mu \wedge \nu_i = y$ and without loss of generality, assume $\mu_i = y$ and $y_O = a$. Since i blocks $\mu \wedge \nu$ with x,

$$x \in C_b(\mu \wedge \nu_b \cup \{x\}). \tag{7}$$

Note that for any sets of contracts X and Y, $C_b(X \cup Y) = C_b(C_b(X) \cup Y)$ (Lemma 15). Therefore,

$$C_b(C_b(\mu_b \cup \nu_b) \cup \{x\}) = C_b(\mu_b \cup \nu_b \cup \{x\}). \tag{8}$$

By definition, $\mu \wedge \nu_b = C_b(\mu_b \cup \nu_b)$. Thus, by (7), $x \in C_b(C_b(\mu_b \cup \nu_b) \cup \{x\})$. By (8), $x \in C_b(\mu_b \cup \nu_b \cup \{x\})$. By substitutability of C_b, $x \in C_b(\mu_b \cup \{x\})$. Therefore, $xP_i\mu_i$, $x \in C_b(\mu_b \cup \{x\})$, and $\lambda_i = x$ where $\lambda \in B = \pi(S)$. Therefore, i blocks μ with λ implying that $\mu \notin \pi(B)$. This is a contradiction as $\mu \in S = \pi(B)$.

The proof for $\mu \vee \nu$ is similar. Suppose for contradiction that $\mu \vee \nu$ is blocked by student i with assignment $\lambda \in B$ where $\lambda_i = x$ and $x_O = a$. We first show that there exists a contract $y \in \mu \vee \nu_a$ which is rejected when a chooses from $\mu \vee \nu_a \cup \{x\}$, i.e. $y \notin C_a(\mu \vee \nu_a \cup \{x\})$. We have already shown that $\mu \wedge \nu$ is not blocked by i and λ (or by any other student); therefore, $x \notin C_a(\mu \wedge \nu_a \cup \{x\})$. Otherwise, i would block $\mu \wedge \nu$ since $\lambda_i P_i \mu \vee \nu_i$ implies $\lambda_i P_i \mu \wedge \nu_i$.

Because $\mu \wedge \nu \in \mathcal{IR}$, we have $C_a(\mu \wedge \nu_a) = \mu \wedge \nu_a$. Thus, by LAD and substitutability of C_a, we have $C_a(\mu \wedge \nu_a \cup \{x\}) = C_a(\mu_a \cup \nu_a \cup \{x\}) = \mu \wedge \nu_a$. As a reminder, $|\mu_a| = |\mu \wedge \nu_a| = |\nu_a| = |\mu \vee \nu_a|$. By the Law of Aggregate Demand and $\mu \vee \nu \in \mathcal{IR}$,

$$|\mu \vee \nu_a| = |C_a(\mu \vee \nu_a)| \leq |C_a(\mu \vee \nu_a \cup \{x\})| \leq |C_a(\mu_a \cup \nu_a \cup \{x\})| = |\mu \wedge \nu_a| = |\mu \vee \nu_a|.$$

Now all these inequalities become equalities. Because $x \in C_a(\mu \vee \nu_a \cup \{x\})$ and $x \notin C_a(\mu_a \cup \nu_a \cup \{x\})$, there must exist $y \in \mu \vee \nu_a \setminus C_a(\mu \vee \nu_a \cup \{x\})$. Without loss of generality, $y \in \mu_a$ and $y_A = j$. Then $x \notin C_a(\mu_a \cup \{x\})$ or else i would block μ with λ. Because μ is individually rational, $C_a(\mu_a) = \mu_a$. Therefore, by LAD and substitutability of C_a,

$$C_a(\mu_a \cup \{x\}) = \mu_a. \tag{9}$$

Note that

$$\begin{aligned} C_a(\mu_a \cup (\mu \vee \nu_a) \cup \{x\}) &= C_a(C_a(\mu_a \cup \mu \vee \nu_a) \cup \{x\}) \\ &= C_a(\mu_a \cup \{x\}) \\ &= \mu_a \end{aligned}$$

where the first equality follows from Lemma 15, the second equality follows from Lemma 2' ($\mu \vee \nu \geq \mu$ and therefore, $C_a(\mu_a \cup \mu \vee \nu_a) = \mu_a$), and the third inequality follows from (9). However, $y \in \mu_a$ and therefore $y \in C_a(\mu_a \cup (\mu \vee \nu_a) \cup \{x\})$. This contradicts substitutability of C_a as $y \notin C_a(\mu \vee \nu_a \cup \{x\})$ but $\mu \vee \nu_a \cup \{x\} \subseteq \mu_a \cup (\mu \vee \nu_a) \cup \{x\}$. □

Lemma 11'. *Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $\pi^2(S) = S$. Let μ^I be the student-optimal assignment in S and let μ^O be the school optimal assignment in S. For every $\lambda \in \pi(S)$ and every student i, $\mu^I_i \; R_i \; \lambda_i \; R_i \; \mu^O_i$.*

Proof. Let S be a set that satisfies (1) and (2) and let $B = \pi(S)$. We say that contract x is **possible** for i if there exists $\lambda \in B$ such that $\lambda_i = x$. Let

$$B(i) = \{x \in \mathcal{X}_i \mid \text{there exists } \lambda \in B \text{ such that } \lambda_i = x\}$$

denote the set of possible contracts for student i. Let $\hat{P}_i$ be defined as follows: (i) for all $x \in B(i)$ and $y \in \mathcal{X}_i \setminus B(i)$, $x\hat{P}_i i \hat{P}_i y$, (ii) for all $x, y \in B(i)$, $x\hat{P}_i y \Leftrightarrow xP_i y$ and (iii) for all

$x, y \in \mathcal{X}_i \backslash B(i)$, $x\hat{P}_i y \Leftrightarrow xP_i y$. Now we introduce a natural modification of DA which we call rDA (restricted DA): when we run DA we only allow a student to propose possible contracts and we use the profile $(\hat{P}_i)_{i\in N}$ for students to propose contracts.[28] Formally, the rDA is defined as follows:

Step 1: Each student i proposes his most $\hat{P}_i$-preferred acceptable contract. Let X_a^1 denote the proposed contracts received by school a. Then school a tentatively accepts $C_a(X_a^1)$ and rejects $X_a^1 \backslash C_a(X_a^1)$.
Step t: Any student i rejected in Step $t-1$ proposes his most $\hat{P}_i$-preferred acceptable contract among the ones which were not yet rejected (if there is no acceptable contract left for i, then i does not make any proposal). Let X_a^t denote the set of proposed contracts received by school a and the ones tentatively accepted by a in the previous step. Then school a tentatively accepts $C_a(X_a^t)$ and rejects $X_a^t \backslash C_a(X_a^t)$.
Stop: There are no rejected contracts or all rejected students have applied to all acceptable contracts. Then the tentative acceptances become final assignments, which we denote by μ^I.

Note that μ^I is stable under $\hat{P}$, which implies $\mu^I \in S$.

We establish the result by showing that no contract is rejected under rDA. This implies that for each student i, μ_i^I is i's favorite possible contract (or equivalently, μ_i^I is i's most $\hat{P}_i$-preferred contract). If a contract was rejected, then there would have to be a last contract rejected. Call this contract x. Let $x_A = i$ and $x_O = a$, i.e. school a rejected x. Then x must be possible for i, so there exists a $\nu \in B$ such that $\nu_i = x$. Because $\nu \in B$ and $\mu^I \in S$, ν and μ^I do not block each other. Thus, by the Rural Hospitals Theorem, $\mu_i^I \neq i$. Let $\mu_i^I = y$ and $y_O = b$.

Let $Y = \{z \in \mathcal{X}_b |$ for $j = z_A, z\ \hat{R}_j\ \mu_j^I\}$ (in words, Y is the set of contracts with b which are possible for some student j and weakly preferred by j to her assignment under rDA). By construction and stability of μ^I under $\hat{P}$, $\mu_b^I = C_b(Y)$. When i proposes contract y to b, no contract is rejected (since x is the last contract rejected). Therefore, by substitutability of C_b,

$$C_b(Y \setminus \{y\}) = \mu_b^I \setminus \{y\}\,. \tag{10}$$

Since μ^I and ν do not block each other, by Lemma 6', $\nu' = \mu^I \vee \nu$ is an individually rational assignment. By the Strong Pointing Lemma', $|\nu_b'| = |\mu_b^I|$ (ν_b' is the set of students pointing to contracts associated with b). However, this leads us to our contradiction. By the definition of pointing, $\nu_b' \subseteq Y$. Since $\nu_i P_i \mu_i^I$, i points to x, not to any contract associated with b, i.e. $\mu_i^I \notin \nu_b'$. Therefore, $\nu_b' \subset Y \setminus \{\mu_i^I\}$; consequently, by the LAD and (10), $|C_b(\nu_b')| < |\mu_b^I|$. But ν' is an individually rational assignment meaning $C_b(\nu_b') = \nu_b'$. Since $|\nu_b'| = |\mu_b^I|$, $|C_b(\nu_b')| = |\mu_b^I|$ which is a contradiction.

Therefore, we conclude that no contract is rejected under rDA. Since for all $\lambda \in B$ and all $i \in A$, $\mu_i^I \hat{R}_i \lambda_i$ and $\lambda_i \hat{R}_i i$. It now follows that $\mu^I \in S$ and $\mu_i^I R_i \lambda_i$ for all $i \in A$.

[28]Since choice functions satisfy substitutability and LAD, the cumulative offer process and DA coincide.

Similarly, when under school proposing rDA, a school a can only propose contract x if x is possible for a, which we denote by $B(a) = \{x \in \mathcal{X}_a \,|\, x \in \mu_a \text{ for some } \mu \in B\}$. Then the school proposing rDA is defined as follows:

Step 1: Each school a proposes all contracts belonging to $C_a(B(a))$. Let X_i^1 denote the proposals received by student i. Then student i tentatively accepts the $\hat{P}_i$-preferred acceptable contract from X_i^1 and rejects the rest (and i rejects all contracts if all proposed contracts are unacceptable).

Step t: Let R_a^{t-1} denote the contracts associated with school a which were rejected in a step before Step t. Then school a proposes all contracts belonging to $C_a(B(a)\backslash R_a^{t-1})$. Let X_i^t denote the proposals received by student i. Then student i tentatively accepts the $\hat{P}_i$-preferred acceptable contract from X_i^t and rejects the rest (and i rejects all contracts if all proposed contracts are unacceptable).

Stop: There is no rejected contract. Then the tentative acceptances become final assignments, which we denote by μ^O.

Again, note that μ^O is stable under $\hat{P}$, which implies $\mu^O \in S$.

By an analogous argument, we show that no contract is rejected under the school-proposing rDA. Let μ^O be the outcome of school proposing rDA. Suppose for contradiction that some contract is rejected: let student i be rejecting contract x, $x_O = a$, and be this the last time that a student rejects a contract. Let student i reject x at Step t. Then $x \in C_a(B(a)\backslash R_a^{t-1})$. Since $\mu_a^O \subseteq B(a)\backslash R_a^{t-1}$, substitutability of C_a implies $x \in C_a(\mu_a^O \cup \{x\})$. We first show that a proposes another contract after i rejects x. Since a was allowed to propose x, there exists a $\nu \in B$ such that $\nu_i = x$. Since ν and μ^O do not block each other (because $\mu^O \in S$), by the Rural Hospitals Theorem $|\mu^O \wedge \nu_a| = |\mu_a^O|$. Since $x \in \nu_a \setminus \mu_a^O$ and $x \in C_a(\mu_a^O \cup \{x\})$, there must exist $y \in \mu_a^O \setminus C_a(\mu_a^O \cup \{x\})$. By substitutability of C_a and $\mu_a^O \subseteq B(a)\backslash R_a^{t-1}$, we have $y \notin C_a(B(a)\backslash R_a^{t-1})$. In words, a does not propose y until after i has rejected x. Note that if $y_A = j$ was holding onto a proposal then i's rejection of x would not be the last rejection. Therefore, no other school proposed a contract associated with j, and in particular, $z \notin C_b(\mu_b^O \cup \{z\})$ for any school $b \in O\backslash\{a\}$ and $z \in \mathcal{X}_j \cap \mathcal{X}_b$. Therefore, when we apply the pointing to μ^O and ν, school $(\nu_j)_O$ does not point to ν_j. However, we have already concluded that school a does not point to y (if $y \in C_a(\nu_a \cup \mu_a^O)$, then by $x \in \nu_a$ and substitutability of C_a, we have $y \in C_a(\mu_a^O \cup \{x\})$, a contradiction), so neither a nor $(\nu_j)_O$ point to a contract associated with j. This contradicts Corollary 2' which says that one school points to a contract associated with j.

Because for all $\lambda \in B$ and all $i \in A$, $\lambda_i \hat{R}_i \mu_i^O$ and $\lambda_i \hat{R}_i i$, now it follows that $\mu^O \in S$ and $\lambda_i R_i \mu_i^O$ for all $i \in A$. □

B.3 Existence and Uniqueness

We are now ready to prove the main theorem. As a reminder, we set $S^0 = \emptyset$, $S^1 = \pi^2(\emptyset)$, $S^k = \pi^2(S^{k-1})$ and $B^k = \pi(S^k)$. We defined S as the first fixed point of our construction, i.e. $S = \pi^2(S)$. Let $B = \pi(S)$. The following two facts will be useful for the proof of the uniqueness of a legal set (which is the main result).

Lemma 16. *(i) For an assignment μ and a school a, let*

$$V(\mu, a) = \{x \in \mathcal{X}_a| \text{ for some } i \in A, xR_i\mu_i \text{ and } \exists \nu \in B \text{ such that } \nu_i = x\}.$$

If $\mu \in S$, then $C_a(V(\mu, a)) = \mu_a$.

(ii) If $\mu \in S$, $\mu_j P_j x$ and x is possible for j (where $x_O = a$), then $x \in C_a(\mu_a \cup \{x\})$.

Proof. In showing (i), note that $S \subseteq \mathcal{IR}$ and $C_a(\mu_a) = \mu_a$. By $\mu_a \subseteq V(\mu, a)$ and LAD, $|C_a(V(\mu, a))| \geq |\mu_a|$. If $C_a(V(\mu, a)) \neq \mu_a$, then there exists $y \in C_a(V(\mu, a))\backslash\mu_a$. For student $y_A = i$ we have the following: $yP_i\mu_i$ and for some $\nu \in B$, $\nu_i = y$; if $y \in C_a(\mu_a \cup \{y\})$, then i blocks μ with ν, a contradiction; thus by LAD, $C_a(\mu_a \cup \{y\}) = \mu_a$. But $\mu_a \subseteq V(\mu, a)$ and $y \in C_a(V(\mu, a))$ would contradict substitutability of C_a.

In showing (ii), since x is possible, there exists $\lambda \in B$ such that $\lambda_j = x$. By construction, μ and λ do not block each other. Therefore, $\mu \wedge \lambda$ is well defined. Moreover, $\mu \wedge \lambda_j = x$ since $\mu_j P_j \lambda_j$. Therefore, $x \in C_a(\mu_a \cup \lambda_a)$. Thus, by substitutability of C_a, $x \in C_a(\mu_a \cup \{x\})$. □

Theorem 1'. *There exists a legal set of assignments.*

Proof. Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $S = \pi^2(S)$.[29] We show that $S = \pi(S) = B$. Then by Lemma 8, S is a legal set of assignments.

Suppose by contradiction that there exists an assignment $\nu \in B \setminus S$. Since $\nu \notin S$, ν is blocked by some student i with assignment $\mu \in B$. Let $x = \mu_i$. Note that there does not exist $\phi \in S$ such that $\phi_i = x$ as otherwise, i would block ν with ϕ in which case $\nu \notin B$.

Thus, by Lemma 11', $\mu_i^I \; P_i \; x \; P_i \; \mu_i^O$. For student i, define the "legal" contracts for i as

$$S(i) = \{z \in \mathcal{X}_i \,|\, \exists\, \phi \in S \text{ such that } \phi_i = z\}.$$

Among i's legal contracts that she prefers to x, let y be her least favorite, i.e. $y \in S(i)$, yP_ix, and there does not exist $z \in S(i)$ such that yP_izP_ix. Similarly, let u be i's favorite school among her legal contracts that she likes less than x, i.e. $u \in S(i)$, xP_iu, and there does not exist $z \in S(i)$ such that xP_izP_iu. By Lemma 11', y and u are well-defined. Let $\overline{X}^y = \{\phi \in S|\phi_i = y\}$. Note that if $\phi, \phi' \in X^y$, then $\phi \wedge \phi'_i = y$ and therefore $\phi \wedge \phi' \in X^y$. Thus, X^y has a well-defined minimum element (with respect to students' preferences). Let

$$\overline{\mu} := \min_{>} \overline{X}^y \tag{11}$$

Now we define the students' favorite assignment that is worse than $\overline{\mu}$. Let

$$\underline{X} = \{\phi \in S \,|\, \overline{\mu} \neq \phi \text{ and } \overline{\mu}_i R_i \phi_i \text{ for all } i \in A\}.$$

[29]Recall that the existence of S is assured because π^2 is an increasing function and for some n we have $S^n = \pi^2(S^n)$. As we have shown, $S^n \subseteq \pi(S^n)$ holds as well.

Note that by our choice of y and u we have for all $\phi \in \underline{X}$, $y = \phi_i$ or $uR_i\phi_i$. If $y = \phi_i$, then $\phi \in \overline{X}^y$, a contradiction to $\phi \neq \overline{\mu}$ and $\overline{\mu}_j R_j \phi_j$ for all $j \in A$. Thus, for all $\phi \in \underline{X}$, $uR_i\phi_i$. Now note that if $\phi, \phi' \in \underline{X}$, then $\phi \vee \phi' \in \underline{X}$ because $\overline{\mu}_i = yP_i\phi \vee \phi'_i$. Therefore, $\underline{X}$ has a well-defined maximum assignment. Let

$$\underline{\mu} := \max_{>} \underline{X} \tag{12}$$

As shown already above, we have $uR_i\underline{\mu}_i$. If $u \neq \underline{\mu}_i$, then by $u \in S(i)$, there exists $\phi \in S$ such that $\phi_i = u$. Because S is a lattice and yP_iu, we have $\phi \wedge \overline{\mu} \in S$ and $\phi \wedge \overline{\mu}_i = u$. Since $\overline{\mu}_j R_j \phi \wedge \overline{\mu}_j$ for all $j \in A$ and $\overline{\mu} \neq \phi \wedge \overline{\mu}$, we have $\phi \wedge \overline{\mu} \in \underline{X}$. Hence, we must have $\underline{\mu}_i = u$.

Let $x_O = a$, $y_O = b$ and $u_O = c$.

Claim 1: $\overline{\mu}_j R_j \underline{\mu}_j$ for all $j \in A$ and consequently for every school d, $V(\overline{\mu}, d) \subseteq V(\underline{\mu}, d)$.

Claim 1 follows from our construction of $\overline{\mu}$ and $\underline{\mu}$: we have $\overline{\mu}_j R_j \underline{\mu}_j$ for all $j \in A$. Thus, $V(\overline{\mu}, d) \subseteq V(\underline{\mu}, d)$ for all $d \in O$. In particular, $\underline{\mu} \in \underline{X}$ and for every $\phi \in \underline{X}$, $\overline{\mu}_j R_j \phi_j$ for all $j \in A$.

Since $\overline{\mu}_i P_i x = \mu_i$, we have $\overline{\mu} \wedge \mu_i = x$. In particular, $x \in C_a(\overline{\mu}_a \cup \mu_a)$ and by substitutability of C_a, $x \in C_a(\overline{\mu}_a \cup \{x\})$. By the Rural Hospitals Theorem, $|\overline{\mu}_a| = |C_a(\overline{\mu}_a \cup \mu_a)|$. Thus, by LAD, $|C_a(\overline{\mu}_a \cup \{x\})| = |\overline{\mu}_a|$, and there exists a unique contract $t_1 \in \overline{\mu}_a \setminus C_a(\overline{\mu}_a \cup \{x\})$. Let $(t_1)_A = r_1$.

We show $\overline{\mu}_{r_1} \neq \underline{\mu}_{r_1}$: otherwise by definition, $\overline{\mu}_{r_1} = \underline{\mu}_{r_1} = t_1$. But then $\overline{\mu} \wedge \underline{\mu}_{r_1} = t_1$ and $t_1 \in C_a(\overline{\mu}_a \cup \underline{\mu}_a)$. If $x \in C_a(\overline{\mu} \wedge \underline{\mu}_a \cup \{x\})$, then by $\overline{\mu} \wedge \underline{\mu} = \underline{\mu}$, we have that i blocks $\underline{\mu}$ with μ, a contradiction to $\underline{\mu} \in S$. Thus,

$$x \notin C_a(\overline{\mu} \wedge \underline{\mu}_a \cup \{x\}) = C_a(C_a(\overline{\mu}_a \cup \underline{\mu}_a) \cup \{x\}) = C_a(\overline{\mu}_a \cup \underline{\mu}_a \cup \{x\}),$$

where the first equality follows from the definition of $\overline{\mu} \wedge \underline{\mu}$ and the second one from Lemma 15. Thus, $x \notin C_a(\overline{\mu}_a \cup \underline{\mu}_a \cup \{x\})$ and $t_1 \in C_a(\overline{\mu}_a \cup \underline{\mu}_a)$. Now by substitutability of C_a and the LAD, we must have $t_1 \in C_a(\overline{\mu}_a \cup \underline{\mu}_a \cup \{x\})$. This is a contradiction to $t_1 \notin C_a(\overline{\mu}_a \cup \{x\})$ and substitutability of C_a. Thus, we must have $\overline{\mu}_{r_1} \neq \underline{\mu}_{r_1}$ and $\overline{\mu}_{r_1} P_{r_1} \underline{\mu}_{r_1}$.

Then $t_1 \in \overline{\mu}_a \setminus C_a(\overline{\mu}_a \cup \{x\})$ and in words, t_1 is a contract a would reject if $\overline{\mu}_a \cup \{x\}$ is proposed.

We define an iterative procedure that is a variation of the vacancy chains that is inherit in the Deferred Acceptance algorithm (when students propose sequentially à la McVitie and Wilson). For each student l, define all contracts that l strictly prefers to $\overline{\mu}_l$ to have been rejected. Formally, letting for student l,

$$\bar{O}(l) = \{z \in \mathcal{X}_l \mid z \in B(l) \text{ and } \overline{\mu}_l R_l z\}.$$

Then student l uses the preference $\bar{P}_l$ defined by (i) for all $v, w \in \bar{O}(l)$, $v\bar{P}_l w \Leftrightarrow vP_l w$ and (ii) for all $v \in \bar{O}(l)$ and all $w \in O \backslash \bar{O}(l)$, $v\bar{P}_l l \bar{P}_l w$. Let school a reject contract t_1. This starts a vacancy chain. We only allow student l to propose contracts which are possible for l. Whenever a student is rejected, she proposes her favorite contract that has not been rejected. In other words, we use the profile $(\bar{P}_l)_{l \in A}$ for the vacancy chain (starting first with rejecting t_1 by a). Each time a school receives a new application, it chooses among all the contracts that have ever applied to it.

Claim 2: In the vacancy chain, no student j proposes a contract worse than $\underline{\mu}_j$.

If not, then let l be the first student in the vacancy chain such that $\underline{\mu}_l$ is rejected. Let $d = (\underline{\mu}_l)_O$ and let Y be all contracts who have been proposed to d. For every $z \in Y$ with $z_A = j$, $zR_j\underline{\mu}_j$ since l is the first student rejected by her assignment under $\underline{\mu}$. Thus,

$$Y \subseteq V(\underline{\mu}, d).$$

By (i) of Lemma 16, $C_d(V(\underline{\mu}, d)) = \underline{\mu}_d$. Therefore, by $\underline{\mu}_l \in \underline{\mu}_d$ and substitutability of C_d, $\underline{\mu}_l$ cannot be rejected by d, a contradiction.

Note that Claim 2 also holds for student r_1 because $\overline{\mu}_{r_1} P_{r_1} \underline{\mu}_{r_1}$.

In the above definition of the vacancy chain, if a student j ever proposes a contract associated with school a, then we pause to make sure that school a is better off despite the fact that a did not voluntarily reject t_1. For now, assume that student j proposes z_j such that $(z_j)_O = a$ in the vacancy chain. By Claim 2, $z_j R_j \underline{\mu}_j$. By (i) of Lemma 16 and LAD, a chooses exactly $|\underline{\mu}_a| = |\overline{\mu}_a|$ contracts (because every student proposing a contract associated with a, the contract then belongs to $V(\underline{\mu}, a)$). Prior to j's proposal of z_j, a is holding onto $|\overline{\mu}_a| - 1$ proposals (because we rejected $t_1 \in \overline{\mu}_a$). If we allowed a to choose amongst t_1, z_j, and the $|\overline{\mu}_a| - 1$ proposals she is holding, then she would wish to hold onto $|\overline{\mu}_a|$ proposals and reject one contract. Call this contract t_2 and $(t_2)_A = r_2$. If $t_2 = t_1$ (the contract we already rejected), then we stop (because we rejected the "right" contract in first place). Otherwise, school a rejects t_2 and we continue. Note that in this case, the new proposed contract z_j did not come from r_1, or else (for $j = r_1$) we have $\overline{\mu}_j = t_1 P_j z_j$ and by (ii) of Lemma 16, $z_j \in C_a(\overline{\mu}_a \cup \{z_j\})$, and a would have wanted to reject t_1 by construction. Set $j = j_1$ and j_1 proposed z_{j_1}. Continue the vacancy chain with t_2 as the rejected contract. In general, whenever a student j_m proposes a contract z_{j_m} associated with a, we check to see if $t_1 \in C_a(\overline{\mu}_a \cup \{z_{j_1}, \ldots, z_{j_m}\})$. If $t_1 \notin C_a(\overline{\mu}_a \cup \{z_{j_1}, \ldots, z_{j_m}\})$, then we stop (because we rejected the "right" contract t_1 in first place). If $t_1 \in C_a(\overline{\mu}_a \cup \{z_{j_1}, \ldots, z_{j_m}\})$, then $j_m \neq r_1$ (as otherwise (ii) of Lemma 16 and substitutability would be violated by $t_1 \notin C_a(\overline{\mu}_a \cup \{x\})$), and a would prefer to reject one of her current proposals and keep t_1. We allow a to reject this contract and continue.

The process ends when $t_1 \notin C_a(\overline{\mu}_a \cup \{z_{j_1}, \ldots, z_{j_m}\})$ for some m or when a student's possible contracts all have been rejected or when a school accepts the application without rejecting one of its current contracts. Let ϕ be the assignment that results from this process. By Claim 2, we have for all $j \in A$, $\phi_j R_j \underline{\mu}_j$.

Claim 3: The vacancy chain ends with a proposed contract associated with a.

There are only three ways for the vacancy chain to end: (1) a student proposes a contract associated with a, (2) a student proposes a contract associated with school $b \neq a$ and b accepts the contract without rejecting any contract, and (3) a student's possible contracts are all rejected.

We show that (3) does not occur. If student l is part of the vacancy chain, then $\overline{\mu}_l \neq l$. Therefore, $\underline{\mu}_l \neq l$ by the Rural Hospitals Theorem. Since by Claim 2, $\phi_l R_l \underline{\mu}_l$, we have $\phi_l \neq l$. Therefore, the vacancy chain does not end with a student's possible contracts all having been rejected. Similarly, (2) does not occur: for every school $b \neq a$, $|\overline{\mu}_b| = |\underline{\mu}_b|$. Since $\phi_j R_j \underline{\mu}_j$ for all $j \in A$, we have $V(\phi, b) \subseteq V(\underline{\mu}, b)$ (meaning that b has more contracts to choose from under $\underline{\mu}$ as the students are less happy with their assignment). By (i) of Lemma 16, we have $C_b(V(\underline{\mu}, b)) = \underline{\mu}_b$. But then $|\phi_b| > |\underline{\mu}_b|$ would violate the Law of Aggregate Demand for b to accept a contract without rejecting another (because $\phi_b \subseteq V(\underline{\mu}, b)$). Therefore, (1) must occur and the vacancy chain can only conclude when a student l proposes a contract associated with a.

Claim 4: $\phi \in S$.

For any school $b \neq a$, school b receives a better set of contracts under ϕ than under $\overline{\mu}$ as it has weakly more contracts to choose from. Mathematically, $C_b(\phi_b \cup \overline{\mu}_b) = \phi_b$. School a is the only school which did not voluntarily reject all of its contracts as a did not voluntarily reject t_1. However, the key point is that the vacancy chain must stop with an application to a, and we only stop after an application to a if a now wants to reject t_1. Therefore, a is made strictly better off by the vacancy chain. Consider a student j, contract z_j and school $b = (z_j)_O$ such that z_j is possible for j and $z_j P_j \phi_j$. If $z_j P_j \overline{\mu}_j$, then $z_j \notin C_b(\overline{\mu}_b \cup \{z_j\})$ or else $\overline{\mu}$ would be blocked. Since b did not choose z_j before, b does not choose z_j now that she has weakly more contracts to choose from. If $\overline{\mu}_j R_j z_j$ then z_j was rejected by b during the vacancy chain and j is not able to block ϕ with b and contract z_j.

Claim 5: $x P_i \phi_i$

Since $\phi \in S$, we have $\phi_i \neq x$. Suppose by contradiction that $\phi_i P_i x$. By our choice of y and u and $\phi \in S$, this can only happen if y was never rejected by school b. Therefore, $y = \phi_i = \overline{\mu}_i$. Because the vacancy chain stops with an application to a where t_1 is rejected, we must have $t_1 = \overline{\mu}_{r_1} P_{r_1} \phi_{r_1}$. By Claim 4, $\phi \in S$ and thus, $\phi \in \overline{X}_b$. But now this is a contradiction as $\overline{\mu} = \min_{>} \overline{X}^b$.

Now Claim 5 yields the contradiction: student i proposed in the vacancy chain to x before proposing to ϕ_i (because $x \in \bar{O}(i)$). But when i proposed x, the vacancy chain must stop as $t_1 \in \overline{\mu}_a \backslash C_a(\overline{\mu}_a \cup \{x\})$, $x \in C_a(\overline{\mu}_a \cup \{x\})$, and thus when $j_m = i$, we must have $t_1 \notin C_a(\overline{\mu}_a \cup \{z_{j_1}, \ldots, z_{j_m}\})$ as otherwise substitutability of C_a is violated. But then we must have $x = \phi_i$ which contradicts Claim 5. □

The proof of Theorem 2 carries over unchanged to the assignment with contracts framework, i.e. there exists a unique legal set of assignments.

Theorem 2'. *There exists a unique legal set of assignments.*

Since this set is a lattice, there exists a student-optimal legal assignment. Using Lemma 11' and the same logic as in Proposition 1, again it follows that this assignment is efficient among all individually rational assignments.

Proposition 1'. *The student-optimal legal assignment μ^I is efficient.*

C General EADA

Below we provide an algorithm for calculating the student-optimal legal assignment. Note that this is the first formulation of Kesten's EADA for the framework of matching with contracts.

The following notion will turn out to be important.

Definition 7'. *Let $\mu \in \mathcal{IR}$ and $x \in \mu$. Then contract x is **irrelevant** for μ if for $x_O = a$ we have*

$$C_a(\{y \in \mathcal{X}_a | yR_j\mu_j \text{ where } y_A = j\} \backslash \{x\}) \subseteq \mu_a.$$

In words, contract x is irrelevant for μ if the school a associated with x, chooses from the set of contracts, which students weakly prefer to their assignment, excluding x, a subset of the contracts assigned to a under μ. Then it is irrelevant whether contract x is present, because from the set of contracts with a, which are weakly preferred by some students to their assignment, school a does not choose any new ones.

Given assignment μ and student i, we say that student i is **Pareto improvable** if there exists $\nu \in \mathcal{IR}$ such that $\nu_i P_i \mu_i$ and for all $j \in A$, $\nu_j R_j \mu_j$. This simply means that there exists a Pareto improvement over μ where i strictly prefers his assigned contract to the one from μ.

The following result show some basic facts of irrelevant contracts.

Lemma 12'. *Let μ be the DA assignment.*

(i) There always exists a contract which is irrelevant for μ.

(ii) If contract x is irrelevant for μ, then x_A is not Pareto improvable.

Proof. (i): If some student is unassigned, then the empty contract is irrelevant as $\mu \in \mathcal{IR}$ and only students weakly prefer being unassigned to their assignment if they are unassigned. Thus, suppose that $\mu_i \neq i$ for all $i \in A$. Let $x \in \mu$ be one of the last contracts assigned in DA, and let $x_A = i$ and $x_O = a$. Note that any contract belonging to

$$W_a = \{y \in \mathcal{X}_a | yR_j\mu_j \text{ where } y_A = j\}$$

must have been proposed in DA. If x is not irrelevant for μ, then $C_a(W_a \backslash \{x\}) \backslash \mu_a \neq \emptyset$. Let $z \in C_a(W_a \backslash \{x\}) \backslash \mu_a$. Then z was proposed at some point in DA, and was rejected at some later step. At the later step school a was facing a set of proposals $J_a \subseteq W_a \backslash \{x\}$ (because x is the last accepted contract and no contract is rejected when x is proposed), but then this is a contradiction to substitutability of C_a as $z \notin C_a(J_a)$, but $z \in C_a(W_a \backslash \{x\})$ and $J_a \subseteq W_a \backslash \{x\}$. Therefore, contract x is irrelevant for μ.
(ii): Suppose to the contrary, i.e. x_A is Pareto improvable. Let $\nu \in \mathcal{IR}$ Pareto improve μ, i.e. $\nu_i R_i \mu_i$ for all $i \in A$. We show that for all $a \in A$,

$$|\nu_a| = |\mu_a|. \tag{13}$$

For school a, let $W_a = \{y \in \mathcal{X}_a | yR_j\mu_j \text{ where } y_A = j\}$. Since μ is stable and C_a is substitutable, we have $C_a(W_a \cup \mu_a) = \mu_a$. Suppose there exists $b \in O$ such that $|\nu_b| > |\mu_b|$. Since $\nu_i R_i \mu_i$ for all $i \in A$, we have $\nu_b \subseteq W_b \cup \mu_b$. Because ν is individually rational, we have $C_b(\nu_b) = \nu_b$. But now this contradicts LAD as $|\nu_b| > |\mu_b|$, $\nu_b \subseteq W_b \cup \mu_b$, and $C_b(W_b \cup \mu_b) = \mu_b$. Now for all $a \in O$, $|\nu_a| \leq |\mu_a|$. Because for all $i \in A$, $\mu_i \neq i$ implies $\nu_i \neq i$, this yields (13).

Let $x_A = i$. Because i is Pareto improvable, say with $\nu \in \mathcal{IR}$, we have $\nu_i \neq \mu_i$. If $\mu_i = i$, then i is not Pareto improvable by (13) and the fact that for all $j \in A$, $\mu_j \neq j$ implies $\nu_j \neq j$. Thus, $\mu_i \neq i$, and say $x_O = a$. Then $x \in \mu_a \backslash \nu_a$ and by $|\mu_a| = |\nu_a|$, there exists $y \in \nu_a \backslash \mu_a$. Since ν is a Pareto improvement over μ, we have $\nu_a \subseteq W_a \cup \mu_a$. Since $x \notin \nu_a$, we have $\nu_a \subseteq (W_a \cup \mu_a) \backslash \{x\}$. Because x is irrelevant for μ and $x \in \mu_a$, we have

$$C_a((W_a \cup \mu_a) \backslash \{x\}) \subseteq \mu_a \backslash \{x\}.$$

By LAD and $C_a(\nu_a) = \nu_a$, $|\nu_a| \leq |C_a((W_a \cup \mu_a) \backslash \{x\})| \leq |\mu_a| - 1$, which is a contradiction to (13). □

We will show that the algorithm below works for any choice functions satisfying substitutability and LAD. One could call this alternatively the "general Efficiency Adjusted Cumulative Offer Process".

Given $Y \subseteq \mathcal{X}$, let $Y_A = \cup_{y \in Y} \{y_A\}$ denote the set of students associated with some contracts in the set Y.

The general Efficiency Adjusted Deferred Acceptance Mechanism (gEADA):
Round 0: Run DA for the problem P. Let μ^0 denote the DA assignment, $I^0 = \emptyset$ and $P^0 = P$.
Round k: This round consists of two steps.

1. Let $I^k = \{x \in \mu^{k-1} | x \text{ is irrelevant for } \mu^{k-1}\}$. If $x \in I^k$ and $x_A = i$, then let P_i^k be the preference for i where x is the only acceptable contract. If $i \notin (I^k)_A$, then let $P_i^k = P_i^{k-1}$, and let P^k denote the resulting profile.

2. Let μ^k denote the DA assignment obtained from P^k.

Stop when $(I^k)_A = A$.

We show that gEADA is well defined for assignment with contracts.

Lemma 13'.

(i) For all $k \geq 1$, $I^{k-1} \subseteq I^k$ and μ^k Pareto dominates μ^{k-1}.

(ii) If $A \backslash (I^{k-1})_A \neq \emptyset$, then $I^k \backslash I^{k-1} \neq \emptyset$.

Proof. (i): We proceed by induction. Obviously, $I^0 \subseteq I^1$. Then μ^0 is stable under P^1. Because choice functions satisfy substitutability and LAD, we have for all $i \in A$, $\mu_i^1 R_i^1 \mu_i^0$. Thus, by (ii) of Lemma 12', for all $x \in I^1$ with $x_A = i$, we have $\mu_i^1 = \mu_i^0 = x$. Thus, $\mu_i^1 R_i \mu_i^0$ for all $i \in A$.

Let $k > 1$. Then again, μ^{k-1} is stable under P^k. Because choice functions satisfy substitutability and LAD, we have for all $i \in A$, $\mu_i^k R_i^k \mu_i^{k-1}$. Thus, by construction, for all $i \in (I^{k-1})_A$, we have $\mu_i^k = \mu_i^{k-1}$. By (ii) of Lemma 12', for all $x \in I^k$ with $x_A = i$, we have $\mu_i^k = \mu_i^{k-1} = x$. Thus, $\mu_i^k R_i \mu_i^{k-1}$ for all $i \in A$, which is the desired conclusion.

It remains to show $I^{k-1} \subseteq I^k$: by $I^0 \subseteq I^1$ and by the induction hypothesis we have $I^{k-2} \subseteq I^{k-1}$. Let $x \in I^{k-1}$. Then (using again the induction hypothesis) x is irrelevant for μ^{k-2}. Thus, for $x_A = i$, for P^{k-1}, student i ranks x as the only acceptable contract. Because μ^{k-1} is a Pareto improvement of μ^{k-2}, we have for all $a \in O$,

$$W_a^{k-1} = \{y \in \mathcal{X}_a | y R_j^{k-1} \mu_j^{k-1} \text{ where } y_A = j\} \subseteq \{y \in \mathcal{X}_a | y R_j^{k-2} \mu_j^{k-2} \text{ where } y_A = j\} = W_a^{k-2}.$$

By (ii) of Lemma 12' and $x \in I^{k-1}$, x_A is not Pareto improvable and we have $x \in \mu^{k-1}$. Thus, $W_a^{k-1} \backslash \{x\} \subseteq W_a^{k-2} \backslash \{x\}$. By LAD,

$$|C_a(W_a^{k-1} \backslash \{x\})| \leq |C_a(W_a^{k-2} \backslash \{x\})|.$$

Because x is irrelevant under μ^{k-2}, we have

$$C_a(W_a^{k-2} \backslash \{x\}) \subseteq \mu_a^{k-2} \backslash \{x\}.$$

Because μ^{k-1} Pareto improves μ^{k-2}, we have by (13), $|\mu_a^{k-1}| = |\mu_a^{k-2}|$. But then by substitutability of C_a, we have $C_a(W_a^{k-1} \backslash \{x\}) \subseteq \mu_a^{k-1}$. Therefore, x is irrelevant for μ^{k-1} and $x \in I^k$.

(ii): Note that if all students are assigned their most preferred contract under μ^{k-1}, then $I^k = \mu^{k-1}$ and $(I^k)_A = A$. Otherwise, we have $(I^k)_A \neq A$. But then some student is not assigned his most preferred contract. Then the same argument as in the proof of (i) of Lemma 12' shows that the last accepted contract must be irrelevant (and this cannot be one of the contracts proposed in the first step.). □

The following captures the two key features of the gEADA algorithm: the output of gEADA is efficient and it coincides with the student-optimal legal assignment. Thus, the gEADA algorithm offers a polynomial algorithm to determine the student-optimal legal assignment.

Theorem 3'.

(i) The gEADA assignment is efficient.

(ii) The output of gEADA algorithm coincides with the student-optimal legal assignment.

Proof. Let η be the output of the gEADA algorithm.
(i): Suppose that η is not efficient, i.e. there exists $\nu \in \mathcal{IR}$ such that $\nu_i R_i \eta_i$ for all $i \in A$ and $\nu \neq \eta$. As in the gEADA algorithm, let $P^0 = P$ and μ^0 denote the DA assignment for P^0. As η Pareto improves μ^0, we have for all $i \in A$, $\nu_i R_i \eta_i R_i \mu_i^0$. Let I^0 denote the contracts which are irrelevant for μ^0. Then by (ii) of Lemma 12', for any $x \in I^0$, student x_A is not Pareto improvable. Thus, for all $x \in I^0$ and $x_A = i$, we have $\nu_i = \eta_i = \mu_i^0 = x$ and both $I^0 \subseteq \nu$ and $I^0 \subseteq \eta$. As in gEADA, let P^1 denote the profile such that for all $i \in A$, if $\mu_i^0 \in I^0$, then μ_i^0 is the unique acceptable contract under P_i^1 and otherwise $P_i^1 = P_i$.

Now by induction, let $k \geq 1$. Then we have both $I^{k-1} \subseteq \nu$ and $I^{k-1} \subseteq \eta$. As in gEADA, let P^{k-1} denote the profile such that for all $i \in A$, if $\mu_i^{k-1} \in I^{k-1}$, then μ_i^{k-1} is the unique acceptable contract under P_i^{k-1} and otherwise $P_i^{k-1} = P_i^{k-2}$. Let μ^{k-1} denote the DA assignment for P^{k-1} and I^k denote the set of contracts which are irrelevant for μ^{k-1}. By construction, we have for all $x \in I^{k-1}$ where $x_A = i$, $\nu_i = \eta_i = x$. Since ν Pareto improves η, η Pareto improves μ^{k-1} and $P_i^{k-1} = P_i^{k-2}$ for all $i \in A$ such that $\mu_i^{k-1} \notin I^{k-1}$, we obtain for all $i \in A$,

$$\nu_i R_i^{k-1} \eta_i \text{ and } \eta_i R_i^{k-1} \mu_i^{k-1}.$$

Then by (ii) of Lemma 12', for any $x \in I^k$, student x_A is not Pareto improvable. Thus, for all $x \in I^k$ and $x_A = i$, we have $\nu_i = \eta_i = \mu_i^{k-1} = x$ and both $I^k \subseteq \nu$ and $I^k \subseteq \eta$. Now by induction, we obtain $\nu = \mu$, which is a contradiction.
(ii): Because the student-optimal legal assignment is efficient, by (i) it suffices to show that η is legal. Suppose that η is not legal. Then there exists $\alpha \in L$ such that α blocks η. Let ν denote the student-optimal legal assignment in L. Because L is a lattice, for some $j \in A$, we have $\nu_j R_j \alpha_j P_i \eta_j$. Thus, $\nu_j P_j \eta_j$ As in the gEADA algorithm, let $P^0 = P$ and μ^0 denote the DA assignment for P^0. Obviously, $\mu^0 \in L$.

Let I^0 denote the contracts which are irrelevant for μ^0. As both ν and η Pareto improve μ^0, we have by (ii) of Lemma 12', for any $x \in I^0$, student x_A is not Pareto improvable. Thus, for all $x \in I^0$ and $x_A = i$, we have $\nu_i = \eta_i = \mu_i^0 = x$ and both $I^0 \subseteq \nu$ and $I^0 \subseteq \eta$. As in gEADA, let P^1 denote the profile such that for all $i \in A$, if $\mu_i^0 \in I^0$, then μ_i^0 is the unique acceptable contract under P_i^1 and otherwise $P_i^1 = P_i$. Let μ^1 denote the DA assignment for P^1. Then by construction, it follows that $\mu^1 \in L$: if $\alpha \in L$ blocks μ^1, then some student i blocks μ^1 with α. But then $i \notin (I^0)_A$ as $\nu_i = \alpha_i$ for all $i \in (I^0)_A$. Since μ^1 Pareto improves μ^0, then i blocks μ^0 with α, a contradiction to $\mu^0 \in L$.

Let $k \geq 1$. But then by induction we have both $I^{k-1} \subseteq \nu$ and $I^{k-1} \subseteq \eta$, and $\mu^{k-1} \in L$. Let I^k denote the contracts which are irrelevant for μ^{k-1}. As both ν and η Pareto improve μ^{k-1}, we have by (ii) of Lemma 12', for any $x \in I^k$, student x_A is not Pareto improvable. Thus, for all $x \in I^k$ and $x_A = i$, we have $\nu_i = \eta_i = \mu_i^{k-1} = x$ and both $I^k \subseteq \nu$ and $I^k \subseteq \eta$. Again, as above it follows $\mu^k \in L$.

Now by induction, we obtain $\nu = \mu$, which is a contradiction to the fact that there exists $j \in A$ such that $\nu_j P_j \eta_j$. □

References

Abdulkadiroglu, A., Y. Che, P. Pathak, A. Roth, and O. Tercieux (2017): "Minimizing Justified Envy in School Choice: The Design of New Orleans' OneApp," *mimeo.*

Abdulkadiroğlu, A., and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *The American Economic Review*, 93(3), 729–747.

——— (2013): "Matching Markets: Theory and Practice," *Advances in Economics and Econometrics*, 1, 3–47.

Alkan, A., and D. Gale (2003): "Stable schedule matching under revealed preference," *Journal of Economic Theory*, 112(2), 289 – 306.

Alva, S., and V. Manjunath (2016): "Strategy-proof Pareto-improvement," *Mimeo.*

Aygun, O., and B. Turhan (2016): "Dynamic Reserves in Matching Markets: Theory and Applications," *mimeo.*

Balinski, M., and T. Sönmez (1999): "A Tale of Two Mechanisms: Student Placement," *Journal of Economic Theory*, 84(1), 73–94.

Blair, C. (1988): "The lattice structure of the set of stable matchings with multiple partners," *Mathematics of operations research*, 13(4), 619–628.

Chen, Y., and O. Kesten (2016): "Chinese College Admissions and School Choice Reforms: A Theoretical Analysis," *Journal of Political Economy.*

Dur, U., R. Hammond, and T. Morrill (forthcoming): "Identifying the Harm of Manipulatable School-Choice Mechanisms," *American Economic Journal: Economic Policy.*

Fleiner, T. (2003): "A fixed-point approach to stable matchings and some applications," *Mathematics of Operations Research*, 28(1), 103–126.

Gale, D., and L. Shapley (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69(1), 9–15.

Gale, D., and M. Sotomayor (1985): "Some remarks on the stable matching problem," *Discrete Applied Mathematics*, 11(3), 223–232.

Hatfield, J. W., and P. R. Milgrom (2005): "Matching with contracts," *The American Economic Review*, 95(4), 913–935.

Kesten, O. (2010): "School choice with consent," *The Quarterly Journal of Economics*, 125(3), 1297–1348.

Klijn, F., and J. Masso (2003): "Weak stability and a bargaining set for the one-to-one matching model," *Games and Economic Behavior*, 42, 91 – 100.

Knuth, D. E. (1976): *Marriages Stables*, vol. 10. Les Presses de l'Universite de Montreal.

Kominers, S., and T. Sonmez (2016): "Matching with Slot-Specific Priorities: Theory," *Theoretical Economics*, 11(2), 683–710.

Martínez, R., J. Massó, A. Neme, and J. Oviedo (2001): "On the lattice structure of the set of stable matchings for a many-to-one model?," *Optimization*, 50(5-6), 439–457.

Morrill, T. (2017): "Petty Envy When Assigning Objects," *mimeo*.

Pathak, P. A. (2011): "The Mechanism Design Approach to Student Assignment," *Annual Review of Economics*, 3(1), 513–536.

Roth, A. (1982): "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, pp. 617–628.

Roth, A., and M. Sotomayor (1992): *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge University Press.

Roth, A. E. (1976): "Subsolutions and the supercore of cooperative games," *Mathematics of operations research*, 1(1), 43–49.

Sönmez, T., and U. Ünver (2010): "House Allocation with Existing Tenants: A Characterization," *Games and Economic Behavior*, 69(2), 425–445.

Wu, Q., and A. Roth (2016): "The lattice of envy-free matchings," *Mimeo*.