Manipulability and tie-breaking in constrained school choice

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Abstract

In school choice problems, we provide an in-depth analysis of the manipulability of the constrained *deferred acceptance* (DA) and *Boston* (BOS) mechanisms. We characterize dominant strategies in both mechanisms and show that constrained DA is less manipulable than constrained BOS in the sense of Arribillaga and Massó (2015). We argue that, from a manipulability perspective, tie-breakers should be revealed before preferences are reported. When this is the case, we are able to compare the manipulability of DA for different tie-breaking rules. We show that single tie-breaking (STB) outperforms multiple tie-breaking in terms of manipulability. We also show that other tie-breaking rules share the desirable manipulability properties of STB while improving on STB's ex-post fairness, an important concern for practitioners.

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1 Introduction

In 2005, the Boston School Committee replaced its school choice mechanism known as the Boston mechanism (BOS) by a deferred acceptance mechanism (DA). An important motivation for this reform was that, when students can rank all the schools they could potentially attend, DA is non-manipulable whereas BOS is not (Abdulkadiroğlu et al., 2006; Pathak and Sönmez, 2008).[1] However, it is rare for mechanisms used in practice to let students rank all the schools in a district. Instead, school choice mechanisms are

[1] Schools districts in Chicago and England abandoned mechanisms similar to BOS in 2007 and 2009 out of analogous manipulability concerns (Pathak and Sönmez, 2013).

typically *constrained* (Haeringer and Klijn, 2009), with students allowed to report preferences on a limited number of schools only.[2] Under such constraints, DA is manipulable and it is unclear whether replacing BOS by DA actually reduces manipulability.

Constraints on the number of schools students can report are pervasive in practice (Pathak and Sönmez, 2013). Such constraints may be viewed as a way to keep mechanisms simple to interact with and operate. Recent results also suggest that constraints may have a positive effect on efficiency in the case of DA, either by alleviating stability requirements (Dur and Morrill, 2016), or by pushing students to reveal information about their cardinal preferences (Van der Linden, 2017).

In spite of their prevalence, little is known about the manipulability of the constrained versions of DA and BOS, henceforth denoted by DA^k and BOS^k, where k is the maximum number of schools students can rank. This paper contributes to filling this gap in the literature. We provide an in-depth study of the manipulability of DA^k and BOS^k, focusing on dominant strategies. For DA^k and BOS^k, we characterize preferences and priorities for which a student has a dominant strategy. Although both mechanisms sometimes fail to be strategy-proof, we show that the mechanisms can be ordered according to the "amount" of dominant strategies they provide in the sense of Arribillaga and Massó (2015). According to the same criterion, manipulability also decreases as k increases in DA^k, but not in BOS^k. These results confirm parallel results from Pathak and Sönmez (2013) and provide further justifications for recent reforms where a number of districts were observed switching from BOS^k to DA^k, or increasing the number of schools student can report in DA^k (Pathak and Sönmez, 2013, Table 1).

Our manipulability comparisons are *not* the mere consequence of some students being able to report all their acceptable schools.[3] Instead, our comparisons relate to correlations in students' priorities at different schools. In DA^k, correlations in priorities provide students with what we call *safe sets* of school: sets of school which, if they are reported, protect a student from ending up unassigned. We show that students who have safe sets can have dominant strategies even when they are unable to report all their acceptable schools.

The above results assumed that priorities are strict and known to students before they report their preferences. When priorities are coarse, school districts must rely on tie-breakers before DA^k or BOS^k can be applied. When a tie-breaking rule is used, the dominant practice seems to consist in breaking ties *after* preferences are reported (Abdulkadiroglu et al., 2009; Calsamiglia et al., 2014), forcing students to report their preference with only partial information on priorities. We show that unless the profile of cardinal utilities is extreme, this practice can decrease the number of students with dominant strategies in constrained mechanisms.

Even when post tie-breaking priorities are revealed before students report their preferences, the selection of the tie-breaking rule remains an important decision for district officials. The two most common tie-breaking rules are the *single* tie-breaking rule (STB)

[2] For example, at the time Haeringer and Klijn (2009) was written, the authors reported that the New York City school district allowed students to report only 12 programs, while the district had more than 500 different programs available.

[3] A school is acceptable for a student if the student prefers this school to being unassigned. Students who can report all their acceptable schools have a dominant strategy in DA^k, but not necessarily in BOS^k.

and the *multiple* tie-breaking rule (MTB), with STB breaking ties in the same way at all schools whereas MTB draws a different tie-breaker for each school. The literature suggests that DA is more *efficient* when used with STB than with MTB.[4] In contrast, we provide the first analysis of the *manipulability* of tie-breaking rules when used with DA^k. In a special case (one seat per school and $k \leq 2$), we show that priority profiles in the support of STB give dominant strategies to more students than profiles in the support of MTB. We also provide results from simulations suggesting that this result generalizes beyond the special case for which we have analytical results.

Our simulations reveal that the incentive advantage of STB over MTB can be sizable for some value of the parameters.[5] However, this advantage is limited for other values of the parameters. This is troubling because ex-post, STB selects priority profiles in which the same tie-breaker is given to every student at every school, which officials and parents view as unfair (Pathak, 2011). A natural question is therefore whether the incentive properties of STB can be harnessed while avoiding its fairness costs.

To answer this question, we develop a new metric of the fairness of priority profiles. In a strict priority profile, students are given a priority rank at each school. The sum of a student's priority ranks reflects how high a priority the student is given across the different schools (e.g., a student who occupies the first rank at each of the m schools has a sum of ranks of m). We compare priority profiles based on the distribution across students of these sums of rank. We say that a profile F is more fair than another profile F' if the distribution induced by F can be obtained from the distribution induced by F' through a series of progressive transfers. A tie-breaking rule g is more ex-post fair than another tie-breaking rule g' if *any* profile in the support of g is more fair than *any* profile in the support of g'. We show that it is possible to construct tie-breaking rules that significantly improve upon STB in terms of fairness while preserving STB's incentive properties.

Related Literature. The closest paper to ours is Pathak and Sönmez (2013) who compare the manipulability of constrained school choice mechanism from the perspective of truthful Nash equilibria. In contrast, we focus on dominant strategies and rely on a comparison criterion introduced by Arribillaga and Massó (2015). As we explain on pg. 10, most comparison criteria have limitations and our analysis should be viewed as complementary to that of Pathak and Sönmez (2013), who use a criterion independent from that of Arribillaga and Massó (2015). Our paper also provides a number of newer insights that have no counterpart in Pathak and Sönmez (2013), including our study of the manipulability of tie-breaking rules in DA^k.[6]

Our paper contributes to the relatively thin literature on constrained school choice mechanisms. The constrained school choice problem was introduced by Haeringer and Klijn (2009), who studied the efficiency and stability properties of the Nash equilibria of constrained school choice mechanisms. Calsamiglia et al. (2010) studied constrained

[4] See Abdulkadiroglu et al. (2009) and De Haan et al. (2015) for simulations based on field data, and Ashlagi et al. (2015), Ashlagi and Nikzad (2015) and Arnosti (2015) for theoretical results in the large.

[5] The parameters are the number of schools, number of students, number of seats per schools, and number of schools students can report.

[6] We also show that, according to Arribillaga and Massó's criterion, DA^k is less manipulable than BOS^ℓ *even* when $\ell > k$, whereas Pathak and Sönmez (2013) only prove a parallel result using their criterion for $\ell = k$.

mechanisms experimentally replicating the design of Chen and Sönmez (2004) while adding a treatment where mechanisms are constrained. More recently, Dur and Morrill (2016) and Van der Linden (2017) have showed how constraints on the number of schools students can report can make DA^k more efficient than the unconstrained DA.

In this paper, our focus is on the manipulability of constrained school choice mechanisms from the perspective of dominant strategies. In a companion paper (Decerf and Van der Linden, 2017), we study the strategies of students who do *not* have dominant strategies. We show that DA^k outperforms BOS^k in terms of the ability of students to eliminate a large set of strategies using weak dominance, and in terms of students' maximin assignments.

This paper also contributes to the growing literature comparing the manipulability of pairs of mechanisms that both fail to be strategy proof. Recent work in this area using manipulability criteria from Pathak and Sönmez (2013) and Arribillaga and Massó (2015) include Chen et al. (2016), Van der Linden (2016), Harless (2017), Chen and Kesten (2017) and Turhan (2017).[7]

Whereas we argue that revealing information on priorities may have a positive impact on manipulability, Li (2017) finds that when priorities depend on exam scores, asking students to report their preferences *before* scores are known may have a positive effect on ex-ante utility.

To our knowledge, our paper is the first to formally analyze the ex-post fairness of tie-breaking rules, and to propose a criterion for ex-post fairness comparisons.

2 The school choice model and constrained school choice mechanisms

The model is similar to Haeringer and Klijn (2009). There is a finite set of schools $S := \{s_1, \ldots, s_m\}$ with $m \geq 2$, and a finite set of students $T := \{t_1, \ldots, t_n\}$. As this assumption is always satisfied in practice and simplifies some of our results, we impose that $n \geq m$.

A typical school is denoted by s_j, or sometimes s. Every school $s_j \in S$ has a capacity q_j and a priority profile F_j. Capacity q_j represents the number of seats available at school s_j. A set of schools $\bar{S} \subseteq S$ is in **oversupply** if together, the schools in $\bar{S}$ can accept all the students, i.e., $\sum_{s_j \in \bar{S}} q_j \geq n$. A set of schools is in **short-supply** otherwise. Throughout this paper, we assume that no pair of schools is in oversupply.[8] Priorities F_j are linear orderings of the students in T. A profile of priorities $F := (F_1, \ldots, F_m)$ is a list containing the priorities of every $s_j \in S$ and the domain of all priority profiles is $\mathcal{F}$.

A typical student is denoted by t_i, or sometimes t. Every student t_i has a preference R_i. Preference R_i is a linear ordering on $S \cup \{t_i\}$. The domain of all preferences for t_i is $\mathcal{R}_i$. A **preference profile** $R := (R_1, \ldots, R_n)$ is a list containing the preference of every

[7] Other manipulability comparison criteria have been suggested by Aleskerov and Kurbanov (1999), Parkes et al. (2002), Maus et al. (2007), Fujinaka and Wakayama (2015), Mennle and Seuken (2014), Barberà and Gerber (2017) and Andersson et al. (2014), among others.

[8] Chen (2014) shows that if every pair of schools is in oversupply, BOS^m is strategy-proof. In this case, DA^k is also strategy-proof for any $k \in \{2, \ldots, m\}$ (see Proposition 2). Even if every pair of schools is in oversupply, it is not hard to show that BOS^k may still be manipulable when $k < m$.

$t_i \in T$. For a given preference profile R, the list containing the preferences of everyone but t_i is R_{-i}.

A strict preference of t_i for school s over school s' is denoted by $s\ P_i\ s'$, while $s\ R_i\ s'$ denotes a weak preference, allowing for $s = s'$. A school $s \in S$ is **acceptable** for t_i if $s\ R_i\ t_i$. To avoid trivialities, we assume that every student has at least one acceptable school. For simplicity, we abuse the notation and write $s \in R_i$ when s is acceptable given R_i, and $\#R_i$ for the number of acceptable schools in R_i. By the same token, $S' \subseteq R_i$ indicates that all schools in S' are acceptable for t_i given R_i.

An **assignment** is a function $\mu : T \to S \cup T$ that matches every student with a school or with herself ($\mu(t) \in S \cup \{t\}$ for any $t \in T$). If $\mu(t) = t$, we say that t is **unassigned** in μ. An assignment is **feasible** if no school exceeds its capacity, i.e., for any $s_j \in S$, we have $\#\{t \in T \mid \mu(t) = s_j\} \leq q_j$, where for any set A, $\#A$ denotes the cardinality of A.

A (school choice) mechanism M associates every profile of reported preferences $Q := (Q_1, \ldots, Q_n)$ in some domain $\mathcal{Q} := \times_{t_i \in T} \mathcal{Q}_i$ with a feasible assignment μ.[9] The notation and terminology for preferences extend to reported preferences : (a) $s\ Q_i\ s'$ means that t_i reports s weakly before s' in Q_i (where possibly $s = s'$), (b) school $s \in S$ is **reported** by t_i in Q_i if $s\ Q_i\ t_i$, (c) $s \in Q_i$ indicates that s is reported in Q_i, (d) $\#Q_i$ is the number of reported schools in Q_i, (e) $S \subseteq Q_i$ indicates that all schools in S are reported in Q_i (f) a typical profile of reported preferences is $Q := (Q_1, \ldots, Q_n)$, and (g) given reported profile Q, the list of reported preferences of every student but t_i is Q_{-i}.

For a domain of reported profiles $\mathcal{Q}$ and a student $t_i \in T$, the set of possible subprofiles for all $t_j \in T \backslash \{t_i\}$ is $\mathcal{Q}_{-i}$ (i.e., $\mathcal{Q}_{-i} := \{Q_{-i} \in \times_{t_j \in T \backslash \{t_i\}} \mathcal{Q}_j \mid (Q_i, Q_{-i}) \in \mathcal{Q}$ for some $Q_i \in \mathcal{Q}_i\}$). In a **constrained mechanism** M^k, the domain is $\mathcal{Q}^k := \times_{t_i \in T} \mathcal{Q}_i^k$, where for every $t_i \in T$, $\mathcal{Q}_i^k$ is the set of all reported preferences in which t_i reports no more than $k \leq m$ schools.

For any reported profile Q and any student t_i, the school t_i is assigned to in $M(Q)$ is $M_i(Q)$. Student t_i is **assigned** in M given Q if $M_i(Q) \neq t_i$ and **unassigned** if $M_i(Q) = t_i$.

A pair (M, R) defines a strategic form game known as a game of school choice (Ergin and Sönmez, 2006). As a consequence, we sometimes refer to a reported preference Q_i as a *strategy*. Given mechanism M, Q_i is a **dominant strategy** if

$$M_i(Q_i, Q_{-i})\ R_i\ M_i(Q_i', Q_{-i}), \qquad \text{for any } Q_{-i} \in \mathcal{Q}_{-i} \text{ and any } Q_i' \in \mathcal{Q}_i.$$

The two classes of mechanisms we focus on correspond to constrained versions of BOS and DA identified by Haeringer and Klijn (2009). We first describe the well-known *un*constrained BOS.

Round 1: Students apply to the school they reported as their most-preferred acceptable school (if any). Every school that receives more applications than its capacity starts rejecting the lowest applicants in its priority ranking, up to the point where it meets its capacity. All other applicants are *definitively accepted* at the schools they applied to and capacities are adjusted accordingly.

⋮

[9] As is common in school choice, we assume that schools are non-strategic players and that the priority profile is known to the social decision maker.

Round ℓ : Students who are not yet assigned apply to the school they reported as their ℓth acceptable school (if any). Every school that receives more *new* applications in round ℓ than its *remaining* capacity starts rejecting the lowest *new* applicants in its priority ranking, up to the point where it meets its capacity. All other applicants are definitively accepted at the schools they applied to and capacities are adjusted accordingly.

The algorithm terminates when all reported schools have been considered, or when every student is assigned to a school. The constrained versions of BOS which we denote by BOS^k are identical to BOS except that no student is allowed to report more than k schools.

We now turn to DA. Again, we first describe the famous *un*constrained version of DA.

Round 1: Students apply to the school they reported as their most-preferred acceptable school (if any). Every school that receives more applications than its capacity *definitively rejects* the lowest applicants in its priority ranking, up to the point where it meets its capacity. All other applicants are *temporarily* accepted at the schools they applied to (this means they could still be rejected in a later round).

⋮

Round ℓ : Students who were rejected in round $\ell - 1$ apply to their next acceptable school (if any). Every school considers the new applicants of round ℓ *together with* the students it temporarily accepted. If needed, each school *definitely rejects* the lowest students in its priority ranking, up to the point where it meets its capacity. All other applicants are *temporarily accepted* at the schools they applied to (this means they could still be rejected in a later round).

The algorithm terminates when all reported schools have been considered, or when every student is assigned to a school. The constrained versions of DA which we denote by DA^k are identical to DA except that no student is allowed to report more than k schools.

3 Safe sets and safe strategies

When students cannot report all available schools, they face the risk of "running out" of reported schools and being unassigned. Experimental evidence shows that students understand this risk. When students cannot report all available schools, students are more likely to report a school they dislike but at which they have a high priority in order to protect themselves from being unassigned (Calsamiglia et al., 2010).

In DA^k, if student t reports a school s_j where t is among the q_j students with highest priority, t cannot be assigned to a school that she reported lower than s_j. In particular, t cannot be unassigned. We call such a school a **top-priority school** for t. Formally, s_j is a top-priority school for t if no more than $q_j - 1$ students have a higher priority at s_j than t.

Interestingly, reporting a top-priority school is *not* the only way for a student to protect herself from being unassigned in DA^k. Students with *no* top-priority school can often guarantee they will be assigned by reporting an appropriate *set* of schools. Consider the following profile, where the left panel represents students' preferences and the right panel represents schools' priorities. Each school has one seat and "$\vdots$" indicates that the rest of the ordering is arbitrary.

$$\begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \\ \hline \vdots & \vdots & \vdots & \boldsymbol{s_1} \\ & & & \boldsymbol{s_2} \\ & & & \boldsymbol{s_3} \\ & & & s_4 \end{array} \qquad \begin{array}{cccc} \boldsymbol{F_1} & \boldsymbol{F_2} & \boldsymbol{F_3} & F_4 \\ \hline t_1 & t_1 & t_2 & t_1 \\ t_2 & t_2 & t_1 & t_2 \\ t_4 & t_4 & t_4 & t_3 \\ \vdots & \vdots & \vdots & \vdots \end{array} \tag{1}$$

Student t_4 does not have a top-priority school. However, only students t_1 and t_2 have a higher priority than t_4 at schools s_1, s_2 and s_3. As a consequence, if t_4 reports s_1, s_2 and s_3, she is guaranteed to be assigned to one of these schools. Given (1), any strategy $Q_i^* \in \mathcal{Q}_i^k$ with $\{s_1, s_2, s_3\} \subseteq Q_i^*$ is what we call a safe strategy for t_i in DA^k. In general, given mechanism M, Q_i is a **safe strategy** for t_i if playing Q_i protects t_i from being unassigned, i.e., $M_i(Q_i, Q_{-i}) \neq t_i$ for any $Q_{-i} \in \mathcal{Q}_{-i}$. Given mechanism M and priority profile F, a set of schools $S^* \subseteq S$ is a **safe set** for t_i if any of t_i's strategies in which S^* is reported is safe, i.e., for any $Q_i \in \mathcal{Q}_i$, $S^* \subseteq Q_i$ implies that Q_i is safe. For example, $\{s_1, s_2, s_3\}$ is a safe set for t_4 in DA^k when the priority profile is (1). In general, any set of schools containing a top-priority school is also a safe set in DA^k.

As we show below, safe sets are tightly related to dominant strategies in DA^k. In particular, if a student has a safe set that covers her most-preferred schools, she has a dominant strategy in DA^k consisting in reporting the schools in this safe set in the order of her preference. The same is not true in BOS^k: even in the presence of a safe set S^*, there is no optimal way for a student to report the schools in S^* when the mechanism is BOS^k (different ordering of the schools in S^* are best-responses to different reports from other students).

Safe sets are also more rare in BOS^k than in DA^k. Even if s is a top-priority school for t, student t must report s *first* in BOS^k to guarantee herself an assignment at s. Thus, unlike in DA^k, sets of schools containing a top-priority school are not necessarily safe in BOS^k. In general, the existence of safe sets in BOS^k requires that a group of no more than k schools be in oversupply, which is relatively rare in practice. The next proposition characterizes safe sets in DA^k and BOS^k (all the proofs can be found in the Appendix).

Proposition 1. *(i) For any $k \in \{1, \ldots, m\}$, the set of schools $\tilde{S} \subseteq S$ with $\#\tilde{S} \leq k$ is safe for student t in DA^k if and only if there exists a subset $\hat{S} \subseteq \tilde{S}$ such that no more than $\sum_{s \in \hat{S}} q_s$ students have a higher priority at some $s \in \hat{S}$ than t, i.e.,*

$$\#\{t' \in T \backslash \{t\} \mid t' \ F_s \ t \textit{ for some } s \in \hat{S}\} < \sum_{s \in \hat{S}} q_s. \tag{2}$$

(ii) For any $k \in \{2, \ldots, m\}$, the set of schools $\tilde{S} \subseteq S$ with $\#\tilde{S} = k$ is safe for student t in BOS^k if and only if $\tilde{S}$ is in oversupply or t has top-priority at all schools in $\tilde{S}$.[10]

4 Dominant strategies in DA^k and BOS^k

One way a student can have a dominant strategy in DA^k is if she is able to report all her acceptable school. But there are more subtle ways for a student to have a dominant strategy in DA^k. For example, a student who reports her preference truthfully up to one of her top-priority schools is playing a dominant strategy in DA^k. More generally, student t has a dominant strategy if her k most-preferred schools form a safe set. When she truthfully reports the safe set made of her k most-preferred schools, t does not face the risk of running out of reported schools and being unassigned. Also, t does not have to "skip" any of her k most-preferred schools in her report. In this case, it is as if the constraint was not binding for t, and her dominant strategies in DA^k are essentially the same as in DA^m. (Recall that m is the total number of schools and DA^m is therefore the unconstrained version of DA)

The next proposition shows that having less than k acceptable schools or having a safe set covering one's k most-preferred schools are the only cases in which a student has a dominant strategy in DA^k.[11]

Proposition 2 (Dominant strategies in DA^k)**.** *For any $k \in \{1, \ldots, m\}$, a student has a dominant strategy if and only if (i) she has no more than k acceptable schools or (ii) her k most-preferred schools are all acceptable and form a safe set.*

As we show in Section 5.2, priority profiles and preferences for which the k most-preferred schools of a student form a safe set are not uncommon. As a consequence, many students can have dominant strategy in DA^k even when no student is able to report all her acceptable schools.[12]

Dominant strategies are much more rare in BOS^k. The reason for this scarcity of dominant strategies are the same as in BOS^m: In BOS^k as in BOS^m, there often exist reports of the other students for which a student would benefit from misreporting her first school, and other reports for which she would benefit from reporting her first school truthfully. This is true even if the student has less than k acceptable schools (provided she has at least two acceptable schools).

Proposition 3 (Dominant strategies in BOS^k)**.** *For any $k \in \{1, \ldots, m\}$, a student has a dominant strategy in BOS^k if and only if (i) she has only one acceptable school or (ii) she has a top-priority at her most-preferred school.*

[10] If $\#\tilde{S} < k$, $\tilde{S}$ can also be safe in BOS^k if *all* schools are in oversupply. For $k = 1$, BOS^1 is strategically equivalent to DA^1 and the characterization in (i) applies.

[11] Every other strategy $Q_i \in \mathcal{Q}_i^k$ is either (a) unsafe and does not report all of t_i's acceptable schools, or (b) safe but fails to report at least one school s^* that t_i prefers to a school she could be assigned to when reporting Q_i. In case (a), t_i could end up unassigned whereas she would have been assigned to one of her unreported acceptable schools had she reported that school. In case (b), t_i could be assigned to a school she likes less than s^* whereas she would have been assigned to s^* had reported s^*.

[12] In the 2003-2004 NYC match, Abdulkadiroğlu et al. (2005) report that about 22,000 of the almost 100,000 students reported the maximum number of 12 schools, suggesting that these 22,000 students may have had more than 12 acceptable schools. According to Abdulkadiroglu et al. (2009), the number of students reporting 12 schools remained similar in later years (between 28% and 20%).

5 Comparing the occurrence of dominant strategies across mechanisms

Based on our characterizations of dominant strategies, it is possible to compare the incentive properties of DA^k and BOS^k.

5.1 Inclusion comparisons

We first follow Arribillaga and Massó (2015) in saying that mechanism B is at least as manipulable as mechanism A if for any t_i, whenever t_i has a truthful dominant strategy given R_i in B, t_i also has a truthful dominant strategy given R_i in A. Formally, mechanism B is **at least as manipulable as** mechanism A if for any profile of priorities $F \in \mathcal{F}$, for any profile of capacities $(q_1, \ldots, q_m)$, and for any $t_i \in T$,

$$\begin{aligned} &\{R_i \in \mathcal{R}_i \mid t_i \text{ has a truthful dominant strategy in } B\} \\ \subseteq &\{R_i \in \mathcal{R}_i \mid t_i \text{ has a truthful dominant strategy in } A\}. \end{aligned} \tag{3}$$

In the context of a constrained school choice mechanism M^k, we consider that any strategy in which t_i reports her $\min\{k, \#R_i\}$ most-preferred schools truthfully is truthful.

Mechanism B is **more manipulable than** mechanism A if B is at least as manipulable as B, but the converse is not true. In the context of constrained school choice, the latter means that *there exists* $F^* \in \mathcal{F}$, $(q_1^*, \ldots, q_m^*)$, and $t_i \in T$ such that $\subset$ replaces $\subseteq$ in (3). Mechanism B is **equally manipulable as** A if A is at least as manipulable as B and B is at least as manipulable as A, i.e., for *any* $F \in \mathcal{F}$, any $(q_1, \ldots, q_m)$, and any $t_i \in T$, $=$ replaces $\subseteq$ in (3).

The three next corollaries follow from our characterizations of dominant strategies in DA^k and BOS^k and are summarized in Figure 1. From our characterizations it is easy to see that, for any $k \in \{2, \ldots, m\}$ and any $\ell \in \{2, \ldots, m\}$, students who have dominant strategies in BOS^ℓ also have dominant strategies in DA^k, which yields the following result.

Corollary 1. *For any $k \in \{2, \ldots, m\}$ and any $\ell \in \{2, \ldots, m\}$, BOS^ℓ is more manipulable than DA^k.*[13]

In particular, even the heavily constrained DA^2 is less manipulable than the unconstrained BOS^m.

Proposition 1 implies that the collection of a student's safe sets can only grow with k in DA^k, which together with Propositions 2 yields the following corollary.

Corollary 2. *For any $k \in \{1, \ldots, m-1)\}$, DA^{k+1} is less manipulable than DA^k.*

Finally, because dominant strategies in BOS^k result from students having a single acceptable school or having a top-priority at their most-preferred school, dominant strategies in BOS^k are insensitive to changes in k.

Corollary 3. *For any $k \in \{1, \ldots, m-1\}$, BOS^{k+1} is equally manipulable as BOS^k.*

[13] For $k = 1$, DA^k and BOS^k are strategically equivalent and the two mechanisms are therefore equally manipulable.

$$DA^m > \cdots > DA^1 = BOS^m = \cdots = BOS^1$$

Figure 1: Manipulability comparisons of BOS^k and DA^k in the sense of Arribillaga and Massó (2015), where $A > B$ indicates that A is less manipulable than B and $A = B$ indicates that A and B are equally manipulable.

Corollaries 1 to 3 parallel and confirm Proposition 2 and Corollary 2 in Pathak and Sönmez (2013). As Pathak and Sönmez (2013), we find that the manipulability advantage of DA^m over BOS^m carries over to DA^k and BOS^k. Also, like Pathak and Sönmez (2013), we find that increasing the number of schools student can report reduces manipulability in DA^k. Our results provide further incentive justifications for reforms identified in Pathak and Sönmez (2013, Table 1) where a number of districts were observed switching from BOS^k to DA^k or increasing the number of schools student can report in DA^k.

Pathak and Sönmez (2013, Proposition 2 and Corollary 2) use a manipulability partial order which is not related to the partial order from Arribillaga and Massó (2015) we use here. Corollaries 1 to 3 are therefore independent from results in Pathak and Sönmez (2013). As noted above, we also show that for $k \geq 2$, DA^k is less manipulable than BOS^ℓ even if $\ell > k$, a result that has no counterpart in Pathak and Sönmez (2013).

Most manipulability partial orders have limitations and our analysis should be viewed as complementary to that of Pathak and Sönmez (2013). By relying on the partial order developed by Arribillaga and Massó (2015), we focus on dominant strategies. An advantage of focusing on dominant strategies is they provide students with clearcut incentives that are independent of any beliefs about other students' reported preferences. The cost of focusing on dominant strategies is our disregard for the effect of choosing one mechanism over another on the incentives of students who have dominant strategies in *neither* mechanisms.[14] Differently, Pathak and Sönmez (2013) compare mechanisms based on the existence of truthful Nash equilibria, which includes cases where students do not have dominant strategies, but disregards the well-documented difficulty for students to coordinate on equilibria.[15]

5.2 Quantitative comparisons for random priority profiles

In most school districts, priorities are based on a few criteria (such as living within walking distance of a school) which do not enable a strict ordering of all students. When this is the case, ties in priorities must be broken before standard school choice mechanisms — such as DA^k and BOS^k — can be applied. We call **pre-existing priorities** the (possibly weak) profile of priorities determined by these criteria. The (strict) priority profile F is then selected according to some distribution g among the set of profiles that respect pre-existing priorities. We call the realization of g the **ex-post priority profile**.

[14] In a companion paper (Decerf and Van der Linden, 2017), we show that even for students who do not have a dominant strategy, DA^k outperforms BOS^k in terms of the ability of students to eliminate a large set of strategies using weak dominance.

[15] Especially in the presence of multiple equilibria, as can be the case in DA^k and BOS^k (Haeringer and Klijn, 2009).

The distribution g typically follows from the application of a tie-breaking rule to the profile of pre-existing priorities. The two most common tie-breaking rules are the *single* tie-breaking rule (STB) and the *multiple* tie-breaking rule (MTB). In STB, a *unique* ordering of the students is drawn uniformly at random. At *every* school, ties in pre-existing priorities are then broken according to this unique ordering, which induces high levels of correlation between ex-post priorities. MTB breaks ties in pre-existing priorities at each school according to a *new* ordering of students, drawn uniformly at random *specifically for this school*. On average, MTB therefore induces much less correlation between ex-post priorities than STB.

Recall that, by definition, mechanism B is more manipulable than mechanism A if (3) is true for *all* $F \in \mathcal{F}$ (and for all $(q_1, \ldots, q_m)$ and $t_i \in T$). Therefore, assuming that the realization of g is revealed to students before preferences are submitted — an issue we get back to in the next section, we know that DA^k provides more students with a dominant strategy than BOS^k *regardless* of g (Corollary 1). We also know that, irrespective of g, the number of students with a dominant strategy increases with k in DA^k (Corollary 2). These are strong qualitative comparisons, but they lack quantitative content.

To obtain quantitative estimates, we focus on STB.[16] We also focus on the commonly studied case of no pre-existing priorities (priorities are exclusively determined by the tie-breaker) and homogeneous quotas, i.e., $q_s = q$ for every $s \in S$. Besides making the model analytically tractable (Miralles, 2009; Abdulkadiroğlu et al., 2011, 2015), the absence of pre-existing priorities can be viewed as an approximation of the common real world scenario where pre-existing priorities are rare.[17] Homogeneous quotas are another common assumption that is required to obtain clear analytical results.[18] These two assumptions are maintained throughout the rest of this paper.

As Propositions 2 and 3 make clear, any student with less than k acceptable schools (weakly) adds to the difference between the number of students with a dominant strategy in DA^k and BOS^k. Also, the number of students with at most k acceptable schools increases with k. This increase adds to the difference between the number of students with a dominant strategy in DA^{k+1} and DA^k. Since these effects are well-understood, we focus on quantifying other sources of dominant strategies that do *not* stem from students having less than k acceptable schools. To do so while keeping the results simple, we assume that all students have more than k acceptable schools. For a student $t_i \in T$, we denote by $\bar{\mathcal{R}}_i$ the set of t_i's preferences in which more than k schools are acceptable.

To assess the magnitude of the difference in incentives between mechanisms, we compute the expected number of students with a dominant strategy in DA^k and BOS^k. Formally, for any g and any $k \in \{1, \ldots, m\}$, the expected number of students with a dominant strategy under mechanism M^k is

$$\mathbb{E}^{dom}_{M_k}(g) := \frac{1}{\#\bar{\mathcal{R}}_1} \sum_{t_i \in T} \sum_{R_i \in \bar{\mathcal{R}}_i} \sum_{F \in \mathcal{F}} \mathbb{1}(R_i, F) g(F), \tag{4}$$

[16] STB has been shown to have better efficiency properties than MTB (see footnote 4). In Section 7, we show that STB also has better *incentive* properties than MTB.

[17] Some real world school choice problems do exclude pre-existing priorities (Abdulkadiroğlu et al., 2011).

[18] In fact, the stronger assumption $q_s = 1$ for every $s \in S$ is often used, for example in Immorlica and Mahdian (2005) and Abdulkadiroglu et al. (2017).

where $\mathbb{1}(R_i, F)$ is an indicator function which takes value 1 when t_i has a dominant strategy in M^k given preference R_i and priorities F, and zero otherwise. In the above definition of $\mathbb{E}^{dom}_{M_k}(g)$, preferences are implicitly drawn uniformly at random. We slightly abuse the notation and denote by $\tilde{F}$ the degenerate distribution $\tilde{g}$ for which $\tilde{g}(\tilde{F}) = 1$. In particular, $\mathbb{E}^{dom}_{M_k}(\tilde{F}) = \frac{1}{\#\bar{\mathcal{R}}_1} \sum_{t_i \in T} \sum_{R_i \in \bar{\mathcal{R}}_i} \mathbb{1}(R_i, \tilde{F})$.

When DA^k is used with STB, students who are given priorities 1 to q have a top-priority at every school and clearly have a dominant strategy. Students who are given priorities $q+1$ to $2q$ do not have top-priorities. However, they know that if they are rejected from a school, the school must be filled with q students whose priority is higher than theirs. Therefore, whichever school a student with priority $q+1$ to $2q$ ranks second is a school at which she has a top-priority *among* the students who are not assigned to her most-preferred school. As a consequence, any pair of schools is a safe set for such a student, and by Proposition 2 this student has a dominant strategy in DA^2. More generally, the following result is a corollary of Proposition 2.

Corollary 4. *For every $k \in \{1, \ldots, m\}$, $\mathbb{E}^{dom}_{DA^k}(g^{STB}) = \min\{kq, n\}$.*

Corollary 4 is illustrated in Figure 2 for the cases $n = 100$, $m = 10$, and $q \in \{3, 5, 8, 10\}$. As Corollary 4 shows, the expected number of students with a dominant strategy increases with k by a factor of q. In contrast, the expected number of students with a top-priority at their most-preferred school is independent of k, and the following is therefore a corollary of Proposition 3.

Corollary 5. *For every $k \in \{1, \ldots, m\}$ and every distribution g, the expected number of students with a dominant strategy in BOS^k is $\mathbb{E}^{dom}_{BOS^k}(g) = q$.*

As Corollaries 4 and 5 show, $\mathbb{E}^{dom}_{DA^k}(g^{STB}) - \mathbb{E}^{dom}_{BOS^k}(g)$ is of the order of $q(k-1)$, regardless of g. Besides being robust in the sense of Corollary 1, the incentive advantage of DA^k over BOS^k can therefore be sizable. The same is true of the incentive advantage of DA^x over DA^y when $x > y$. For example, when $n = 100$, $m = 10$ and $q = 8$, DA^6 on average provides 48% of the students with a dominant strategy whereas DA^2 only gives a dominant strategy to 16% of the students, and BOS^k to 8% of the students (regardless of k).

In contrast, when the total number of seats is in drastic short-supply, even DA^{m-1} provides only a limited improvement over BOS^{m-1}. For example, when $n = 100$, $m = 10$, and $q = 1$, DA^9 on average provides 9% of the students with a dominant strategies, versus 1% for BOS^9. The same is true when k is small compared to m. Assuming that there are as many seats as students ($mq = n$) and considering the parameters of the NYC match (2003-2004) $m = 500$, $n = 100,000$, and $k = 12$, we find that, on average, only 2.4% of the students have a dominant strategy in DA^{12}.

Recall that Corollaries 4 and 5 only consider students with *more* than k acceptable schools. As explained above, the fact that some students are able to report all their acceptable schools further favors DA^k over BOS^k, and DA^{k+1} over DA^k. Therefore, in spite of Corollaries 4 and 5, DA^k can have a sizable incentive advantage over BOS^k (and DA^k a sizable incentive advantage over DA^{k+1}) *even* when q is small or k is small compared to m. However, in these cases, incentive improvements predominantly come

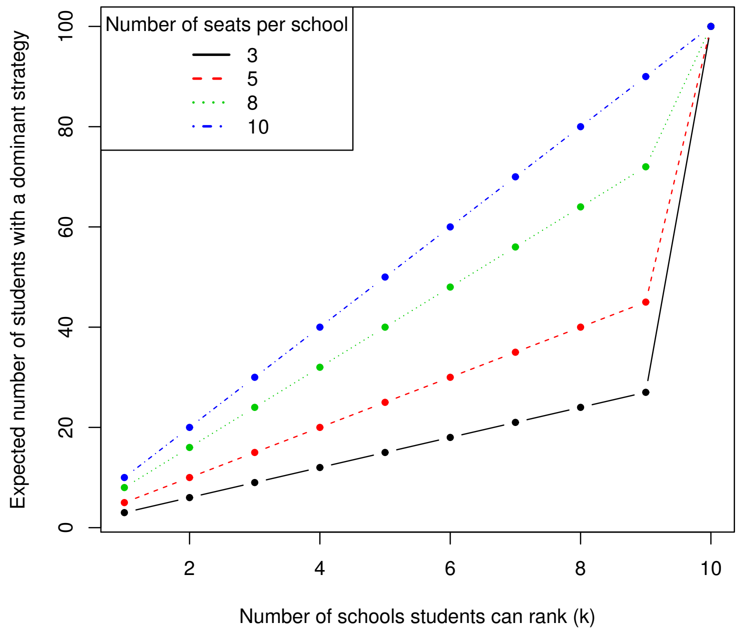

Figure 2: Expected number of students with a dominant strategy in DA^k from Corollary 4 ($n = 100$, $m = 10$ and $q \in \{3, 5, 8, 10\}$).

from students being able to report all their acceptable schools, and not from safe sets induced by correlations in priorities.[19]

6 Timing of tie-breaking and information on priorities

All previous results assumed that priorities are known to the students when they report their preferences. In DA^k, knowledge of the priorities allows students to identify their safe sets and determine whether they have a dominant strategy. Even when a student does not have a dominant strategy, knowing her safe sets can help the student rule out dominated strategies and identify maximin strategies (Decerf and Van der Linden, 2017). In BOS^k, knowledge of the priorities can help the chosen few who have a top-priority at their most-preferred school

Revealing the priorities before preferences are reported is compatible with priorities being drawn at random. Officials simply have to announce the realization of the draw before students report their preferences. In most school districts, however, the dominant practice seems to consist in breaking ties in priorities *after* preferences are reported

[19] As explained in footnote 12, about 80% of the students who participated in the NYC match (2003-2004) reported less than 12 schools. These students are likely to have had less than 12 acceptable schools, and therefore had a dominant strategy in DA^{12} regardless of the priorities. Unless some had a single acceptable school, only a few of these students would have had a dominant strategy in BOS^{12}.

(Abdulkadiroglu et al., 2009; Calsamiglia et al., 2014). Students must then choose their reported preference under partial information on priorities.

Under partial information (on priorities), students only know the distribution of priority profiles before reporting their preferences. In this extended model, every student $t_i \in T_i$ has a cardinal preference on the set of schools represented by u_i. However, students can only report ordinal preferences. Let $M(Q;F)$ be the outcome of M when the priority profile is F and the profile of (ordinal) reported preferences is Q. The expected utility of student t_i in mechanism M when reported preferences are Q and priorities are drawn according to g is

$$\mathbb{E}_i(Q;M) \coloneqq \sum_{F\in\mathcal{F}} u_i(M(Q;F))g(F).$$

Given M and g, Q_i is an ***expected* dominant strategy for t_i** if

$$\mathbb{E}_i(Q_i, Q_{-i}; M) \geq \mathbb{E}_i(Q'_i, Q_{-i}; M) \qquad \text{for any } Q'_i \in \mathcal{Q}_i \text{ and any } Q_{-i} \in \mathcal{Q}_{-i}. \tag{5}$$

Observe that unlike dominant strategies, *expected* dominant strategies need not be safe strategies in DA^k (see Example 2).

We are interested in comparing the occurrence of dominant strategies when ex-post priorities are concealed and when they are revealed before preferences are reported. In DA^k, students who have less than k acceptable schools have an expected dominant strategy regardless of whether priorities are revealed or concealed, and these students do not impact the comparison. Similarly, students who have a single acceptable school have an expected dominant strategy in BOS^k regardless of information on priorities, and the number of acceptable schools does not otherwise impact expected dominant strategies in BOS^k. Therefore, as in the previous section, we focus on students with more than k acceptable schools. To keep the results simple, we again assume that all students have more than k acceptable schools.

In many cases, a lack of information on ex-post priorities leads to a significant decrease in the number of students with an expected dominant strategy in DA^k.

Example 1 (Revealing priorities and dominant strategies)**.** Consider DA^1 with two schools, each having a single seat. Suppose that there are three students with identical preferences, where $u_i(s_1) = 2$, $u_i(s_2) = 1$ and $u_i(t_i) = 0$ for $i \in \{1, 2, 3\}$. Further suppose that STB is used, i.e., the priority profile is drawn uniformly at random among the six following profiles

F_1^1	F_2^1	F_1^2	F_2^2	F_1^3	F_2^3	F_1^4	F_2^4	F_1^5	F_2^5	F_1^6	F_2^6
t_1	t_1	t_1	t_1	t_2	t_2	t_2	t_2	t_3	t_3	t_3	t_3
t_2	t_2	t_3	t_3	t_1	t_1	t_3	t_3	t_1	t_1	t_2	t_2
t_3	t_3	t_2	t_2	t_3	t_3	t_1	t_1	t_2	t_2	t_1	t_1

If students only know that STB is used but do not know which of the six profiles will be drawn, every student would prefer to report s_1 if no other student reported s_1. But every student would prefer to report s_2 if both other students reported s_1. Hence, no student has an expected dominant strategy.

In contrast, if the realization of STB is revealed before students report their preference, then one of the three students systematically has a top-priority at her most-preferred school, and therefore has a dominant strategy (Proposition 2).

Given Proposition 2, it may seem that revealing more information on priorities can only increase the number of students with an expected dominant strategy. Although we show below that this is true in most cases, there are cardinal utility profiles for which maintaining uncertainty can provide more students with an expected dominant strategy than if the ex-post priority profile is revealed.

Example 2. This example is identical to Example 1 but with $u_i(s_1) = 10$ instead of 1 for $i \in \{1, 2, 3\}$. Under this more extreme priority profile, reporting s_1 is always a best response. Reporting s_1 yields an expected utility of at least 10/3 (when both other students report s_1), which is higher than 1, the maximum expected utility any of student can secure when reporting s_2 instead of s_1. Hence, any student has an expected dominant strategy under partial information, whereas only one of the three students has a dominant strategy if the ex-post priorities are revealed.

As Example 1 suggests, in many cases, no student has an expected dominant strategy under partial information, whereas a number of students would have a dominant strategy if ex-post priorities were revealed (see the simulations in Section 7 for quantitative estimates of the latter). However, as Example 2 illustrates, when the difference between schools' cardinal utilities is particularly large, partial information can yield a larger number of dominant strategies. The next proposition makes these observations more general and more precise.

Recall that we focus on the case of no pre-existing priorities and homogeneous quotas. In this context, distribution g is **ex-ante fair** if students are given any priority rank at any school with the same probability. Formally, for $p \in \{1, \ldots, n\}$, let $F_s(t) = p$ indicate that student t has priority rank p at school s. Then, g is ex-ante fair if

$$\sum_{\{F \in \mathcal{F} \mid F_s(t)=p\}} g(F) = \frac{1}{n} \qquad \text{for any } p \in \{1, \ldots, n\}, \text{ any } s \in S, \text{ and any } t \in T.$$

Observe that ex-ante fair distributions allow for correlation between priorities ex-post. For example, STB is ex-ante fair although it features high levels of correlations between priorities ex-post. MTB is also ex-ante fair but on average features lower levels of correlations between priorities than STB. For any utility function over schools u_i, let $u_i(x)$ be the utility of the x-th highest ranked school (i.e., $u_i(1)$ is the utility of the most-preferred school, $u_i(2)$ is the utility of the second most-preferred school, and so on). To simplify the statement of our results, we normalize $u_i(t_i) = 0$, where $u_i(t_i)$ is the utility of being unassigned.[20]

Proposition 4 (Expected dom. strat. require extreme preference in DA^k)**.** *For any $k \in \{1, \ldots, m-1\}$ and any ex-ante fair tie-breaking rule g, if the k most-preferred schools of t_i are in short-supply and if*

$$u_i(k+1)\left(\frac{n}{q} - (k-1)\right) > u_i(k), \tag{6}$$

then student t_i does not *have an expected dominant strategy in DA^k.*

[20] Without normalization, inequality (6) would become $[u_i(k+1) - u_i(t_i)]\left(\frac{n}{q} - (k-1)\right) > u_i(k) - u_i(t_i)$.

When there are as many students as seats, which implies $m = \frac{n}{q}$, condition (6) simplifies to

$$u_i(k+1)(m-(k-1)) > u_i(k) \tag{7}$$

Consider parameters $m = 500$ and $k = 12$ from the NYC match (2003-2004). Condition (7) then says that, under partial information, the (normalized) utility a student attaches to her 12th most-preferred school must be over 489 times higher than the utility she attaches to her 13th most-preferred school for her to have an expected dominant strategy in DA^k, which seems unrealistic. In this case, under partial information, most students likely fail to have an expected dominant strategy in DA^k. Although when $m = 500$ and $k = 12$ the number of students with a dominant strategies under full information is also small (Corollary 4), the advantage of revealing ex-post priorities can be much larger for other larger values of the parameters.[21]

Full information can be preferred to partial information even when there are less (expected) dominant strategies under full information than under partial information. Dominant strategies are cognitively less demanding than expected dominant strategies. First, identifying expected dominant strategies requires students to form cardinal preferences over schools, which takes more effort than forming ordinal preferences (a student who forms cardinal preference necessarily has ordinal preference, whereas the converse is not true). Significant effort is also required to perform expected utility computations, and district officials may want to spare parents the costs associated with these efforts.

In light of Proposition 4 and the lower cognitive load required by dominant strategies, there appears to be an incentive justification for revealing ex-post priorities *before* students report their preferences, which is not the current dominant practice. Districts may be reluctant to revealing the full ex-post priority profile out of practical and privacy concerns. Revealing ex-post priorities may also exacerbate feelings of unfairness. Although tie-breaking rules are typically fair *ex-ante*, a student may *ex-post* feel that she has been treated unfairly if she learns that she has received a low tie-breaker at all schools (see Section 7). Finally, the full priority profile may be too much information for a student to process. As condition (2) shows, computing a student's safe sets is relatively straightforward computationally, but it can be tedious.

For these reasons, instead of revealing the raw priorities and quotas, we recommend that districts set up decision-support platforms through which students can learn about their safe sets and dominant strategies. A proof of concept of such a platform can be found at https://martinvanderlinden.shinyapps.io/Decision_support_DAk. Based on a student's preference input, the platform uses its knowledge of the ex-post priority profile to compute the student's safe sets. The platform then provides the student with a recommendation on the list of schools to report. If a student has a dominant strategy, she is advised to report her k most-preferred schools truthfully. If she does not have a dominant strategy and has no safe set, the student is advised to include a school she knows is in low-demand as a protection against being unassigned. Even if a student does

[21] Suppose that k increases to 100. Then, 20% of the students have a dominant strategy in DA^{100} when ex-post priorities are revealed. However, for a student to have an *expected* dominant strategy in DA^{100} under partial information, condition (6) requires that the utility the student attaches to her 12th most-preferred school be over 400 times larger than the utility she attaches to her 13th most-preferred school, which remains unlikely.

not have a dominant strategy, the platform informs her of *some* of her safe sets, which can help her rule out dominated strategies and identify maximin strategies (Decerf and Van der Linden, 2017). Complete examples of advice and screen-shots of the platform can be found in Appendix B.

Somewhat extreme cardinal preferences are also required for students to have expected dominant strategies in BOS^k.

Proposition 5 (Expected dom. strat. require extreme preference in BOS^k)**.** *For any $k \in \{2, \ldots, m\}$ and any ex-ante fair tie-breaking rule g, if*

$$u_i(1)\frac{q}{n-q} > u_i(2) - u_i(3), \tag{8}$$

then student t_i does not *have an expected dominant strategy in BOS^k.*

When there are as many students as seats, which implies $m = \frac{n}{q}$, condition (8) simplifies to

$$u_i(1)\frac{1}{m-1} > u_i(2) - u_i(3). \tag{9}$$

Consider again parameters $m = 500$ and $k = 12$ from the NYC match (2003-2004). Condition (9) then says that, under partial information, a student must have an extreme preference for her first school over her second school (or be almost indifferent between her second and third schools) to have an expected dominant strategy in BOS^k ($u_i(1)\frac{1}{500-1} > u_i(2) - u_i(3)$), In this case, expected dominant strategies should therefore be rare. However, unlike in DA^k, dominant strategies are also rare in BOS^k when priorities are revealed before preferences are reported (Corollary 5). Gains or losses associated with revealing or concealing priorities in BOS^k are therefore likely to be marginal.

7 Comparing tie-breaking rules

Although STB is usually considered more efficient than MTB (see footnote 4), the issue of tie-breaking selection remains vividly debated. Of particular concern for school board officials is the parents' perception of STB's *un*fairness. Pathak (2011) reports that during the reform of the NYC assignment mechanism, an official remarked:

> "I believe that the equitable approach is for a child to have a new chance with each [...] program. [i.e, use MTB] [...] If we use only one random number [i.e, use STB], and I had the bad luck to be the last student in the line this would be repeated 12 times and I would never get a chance. I do not know how we could explain this to a parent." (Pathak, 2011)

As the quote suggests, although STB and MTB are both *ex-ante* fair, STB is perceived as unfair *ex-post* because it gives the same tie-breaking order to every student at every school.

To determine whether STB is worth its apparent ex-post fairness cost, it is useful to get a more complete picture of STB's advantages over MTB. Previous research has focused on efficiency. In this section, we provide new insights into STB's *incentive* performances. We show that, in a special case, STB dominates *any* tie-breaking rule (including MTB) in terms of the expected number of students it provides with a dominant strategy. For more general cases, we provide quantitative estimates of STB's incentive advantage over MTB through simulations. Our simulations suggest that STB's incentive advantage over MTB is not specific to the special case for which we have analytical results. Finally, we show that the incentive performances of STB can be preserved while improving on STB's ex-post fairness.

Because dominant strategies are not affected by tie-breaking rules in BOS^k (Corollary 5), we only study DA^k, the mechanism for which the use of STB and MTB is usually debated. Motivated by the results of Section 6, we assume that the realization of tie-breaking rules is disclosed before preferences are reported. Also, we again assume that all students have more than k acceptable schools (as students with less than k acceptable schools have dominant strategies in DA^k regardless of priorities, these students do not impact comparisons between tie-breaking rules).

7.1 STB v. MTB: analytical results for $k \leq 2$ and $q = 1$

For the degenerate distribution F, the probability that student t_i has a dominant strategy depends on the probability that t_i has a safe set covering her k most-preferred schools (Proposition 2). If R_i is drawn uniformly at random, this probability is a linear function of the number of t_i's safe sets of size k given F, and is independent across students. Thus, the expected number of students with a dominant strategy $\mathbb{E}^{dom}_{DA^k}(F)$ is directly proportional to the number of safe sets of size k generated by F (see Lemma 5 in the Appendix for a formal proof). Tie-breaking rules that, on average, select priorities generating more safe sets therefore provide more students with dominant strategies.

The safe sets generated by a priority profile F can be decomposed into two categories, depending on the properties of F from which these safe sets originate:

(a) Students having top-priorities: When a student has a top-priority at a school s^*, any set of schools containing s^* is a safe set which we call a **first-order safe set**.

(b) Correlations in priorities: A student can also have **higher-order safe sets** (i.e., safe sets that are not first-order) if she is among the kq highest-priority students at a number of schools and students who have higher priorities than her are correlated across schools, as illustrated in (1).

Because STB produces maximal correlations between priorities, STB generates numerous higher-order safe sets. However, STB performs poorly when it comes to first-order safe sets because of a form of "decreasing return" to correlations in top-priorities.

Example 3 (Decreasing return to correlations in top-priorities)**.** Suppose that there are four schools, each with one seat, and four students $\{t_1, \ldots, t_4\}$. Consider DA^2 and the first-order safe sets of size $k = 2$ (2-FSS) induced by different distribution of the top-priorities.

We focus on student t_1. Suppose that t_1 is awarded the top-priority at school s_1. Then $\{s_1, s_2\}, \{s_1, s_3\}$, and $\{s_1, s_4\}$ are 2-FSS for student t_1.

Now, consider the increase in the number of t_1's 2-FSS if she is also awarded the top-priority at school s_2. Alone, a top-priority at s_2 gives t_1 three 2-FSS, namely $\{s_2, s_1\}, \{s_2, s_3\}$, and $\{s_2, s_4\}$. However, $\{s_2, s_1\}$ is already a 2-FSS for t_1 because of her top-priority at s_1. Hence, awarding t_1 the top-priority at school s_2 only gives her two *additional* 2-FSS, namely $\{s_2, s_3\}$ and $\{s_2, s_4\}$.

In terms of the total number of 2-FSS, it is better to award the top-priority at s_2 to another student, say t_2, than to give it to t_1. Assuming t_2 does not yet have any 2-FSS, awarding the top-priority at s_2 to t_2 gives her three additional 2-FSS ($\{s_2, s_1\}, \{s_2, s_3\}$, and $\{s_2, s_4\}$), instead of the two additional 2-FSS given to t_1 if she is awarded the top-priority at s_2.

Based on Example 3, it is not hard to see that STB is among the least effective tie-breaking rules at generating first-order safe sets. Surprisingly, for $k \in \{1, 2\}$ and $q = 1$, STB is nevertheless optimal in the following sense.

Proposition 6. *For $k \in \{1, 2\}$ and $q = 1$, for any tie-breaking rule B, any $F^B \in supp(B)$, and any $F^{STB} \in supp(STB)$, we have $\mathbb{E}^{dom}_{DA^k}(F^{STB}) \geq \mathbb{E}^{dom}_{DA^k}(F^B)$.*

As Proposition 6 shows, when $k \in \{1, 2\}$ and $q = 1$, any gains in terms of first-order safe sets that may follows from spreading top-priorities in some $F^B \notin supp(STB)$ is *at least* compensated by the larger number of higher-order safe sets generated by any $F^{STB} \in supp(STB)$. For a number of profiles outside of $supp(STB)$ (including some profiles in $supp(MTB)$), the gains from spreading top-priorities are, in fact, *more* than compensated by the larger number of higher-order safe sets generated by profiles in $supp(STB)$.

Example 4 (First-order gains do not compensate higher-order losses)**.** Consider DA^2 and suppose that there are four schools, each with one seat. Let $\tilde{F}$ be any priority profile in $supp(MTB)$ with the following two top-priorities[22]

$$\begin{array}{cccc} \tilde{F}_1 & \tilde{F}_2 & \tilde{F}_3 & \tilde{F}_4 \\ \hline t_1 & t_1 & t_1 & t_2 \\ t_3 & t_4 & t_5 & t_6 \\ \vdots & \vdots & \vdots & \vdots \end{array} \tag{10}$$

Observe that because the second-highest priority is never given to the same student twice in $\tilde{F}$, the profile does not take full advantage of opportunities to generate higher-order safe sets of size $k = 2$. As a consequence, all safe sets of size 2 in $\tilde{F}$ are first-order safe sets, with t_1 having safe sets $\{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_2, s_3\}, \{s_2, s_4\}$, and $\{s_3, s_4\}$, and t_2 having safe sets $\{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}$. Given this collection of safe sets, t_1 always has a dominant strategy whereas t_2 has a dominant strategy with probability one-half (specifically, when her two most-preferred schools are $\{s_1, s_2\}, \{s_1, s_3\}$, or $\{s_1, s_4\}$). Hence, given $\tilde{F}$, the expected number of students with a dominant strategy is 1.5, which is lower than 2, the expected number of students with a dominant strategy given any $F^{STB} \in supp(STB)$.

Profiles like (10) can be found for any $k \geq 2$, which yields the following proposition

Proposition 7. *For every $k \in \{2, \ldots, m\}$ and every $q \in \mathbb{N}$, either*

[22] Because $supp(MTB) = \mathcal{F}$, $supp(MTB)$ contains many profiles satisfying (10).

(a) every set of k schools is in oversupply, and hence, $\mathbb{E}^{dom}_{DA^k}(F) = n$ for any $F \in \mathcal{F}$, or

(b) there exists $\tilde{F}^{MTB} \in supp(MTB)$ such that $\mathbb{E}^{dom}_{DA^k}(F^{STB}) > \mathbb{E}^{dom}_{DA^k}(\tilde{F}^{MTB})$ for any $F^{STB} \in supp(STB)$.

In practice, most districts allow students to report more than two schools, with DA^3 being an especially common mechanism (see Pathak and Sönmez, 2013, Table 1, for data on England). Even for DA^2, Propositions 6 and 7 show that STB clearly outperforms MTB for the case $q = 1$ only, which is again uncommon.

As Corollary 4 shows, when STB is used, it is possible to compute the expected number of students with a dominant strategy for a variety of parameters. In general, the same expected number can be hard to compute for MTB, and direct analytical comparisons remain out of reach.[23] To compare STB with MTB when $k > 2$ or $q > 1$, we therefore resort to simulations which we present in the next section. Although a formal proof is left as an open question, our simulations suggest that Proposition 7 generalizes beyond $k \leq 2$ and $q = 1$.

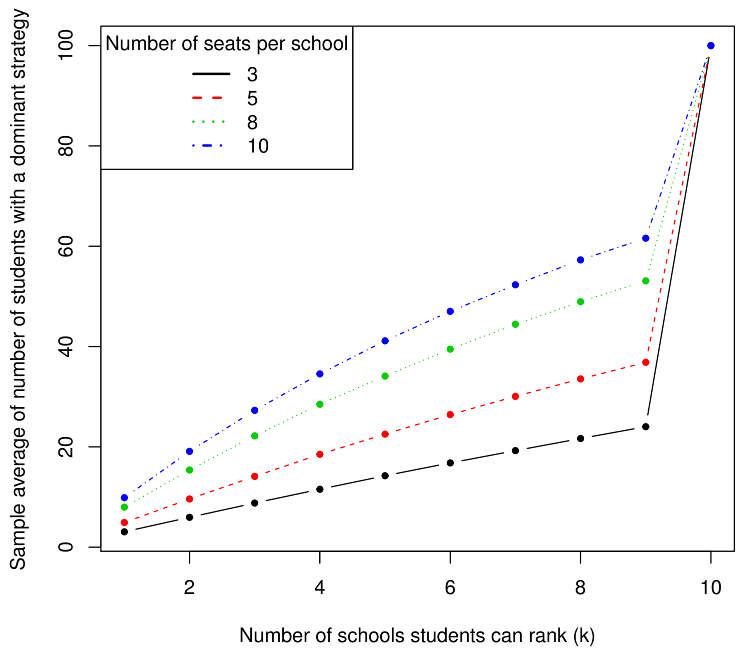

Figure 3: Sample average (1000 observations) of the number of students with a dominant strategy in DA^k with MTB as a function of the number of schools that students can report ($n = 100$, $m = 10$, and $q \in \{3, 5, 8, 10\}$).

7.2 STB v. MTB: simulations

In order to compare STB with MTB when $k > 2$ or $q > 1$ (to quantify the differences in dominant strategies between STB and MT), we compute the average number of students

[23] Unlike the expected number under STB which is linear in k, the same expected number can be a potentially complex *concave* function of k under MTB (see the simulations in Figure 3).

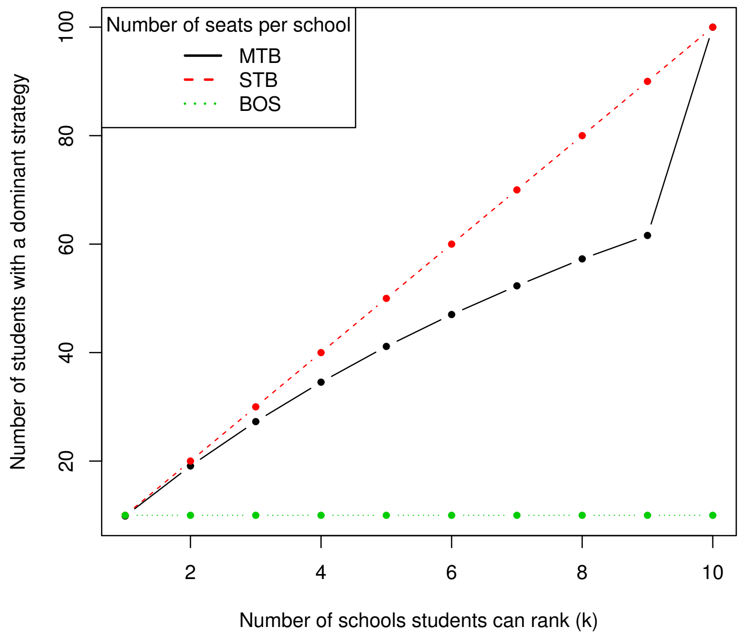

Figure 4: Expected number of students with a dominant strategy in DA^k with STB (Corollary 4) and in BOS^k with any tie-breaking rule g (Corollary 5), as well as sample average (1000 observations) of the number of students with a dominant strategy in DA^k with MTB ($n = 100$, $m = 10$, and $q = 10$).

with a dominant strategy under each tie-breaking rule for a sample of priority and preference profiles. We report the results of our computational experiment in Figures 3 to 4. In our experiment, there are 10 schools ($m = 10$) and 100 students ($n = 100$). Averages are over 1000 random profiles of preferences and priorities. We report results for each $k \in \{1, \dots, 10\}$ and each $q \in \{3, 5, 8, 10\}$.

Our experiment satisfies the conditions of Corollary 4. In particular, the profiles of preferences and priorities are drawn *uniformly* at random with no pre-existing priorities, and each school has the same number of seats q. Therefore, the expected number of dominant strategies under STB is given by Corollary 4 and illustrated in Figure 2, and we only report the sample average for MTB in Figure 3. Differences between the sample average of MTB and the theoretical average of STB are reported in Figure 5. The results in Figure 5 suggest that Proposition 6 generalizes to the parameters used in our experiment. Under these values of the parameters, the expected number of students with a dominant strategy under STB is higher than the sample average under MTB even when $k > 2$ (and although $q > 1$).

Observe also that, although MTB does not equate the incentive properties of STB, DA^k with MTB remains preferable to BOS^k, regardless of the tie-breaking rule used in BOS^k. This is illustrated in Figure 4 for $q = 10$. This turns out to be a general result. Based on Corollary 5, it is not hard to see that DA^k outperforms BOS^ℓ regardless of the tie-breaking rules used in either of the mechanisms.

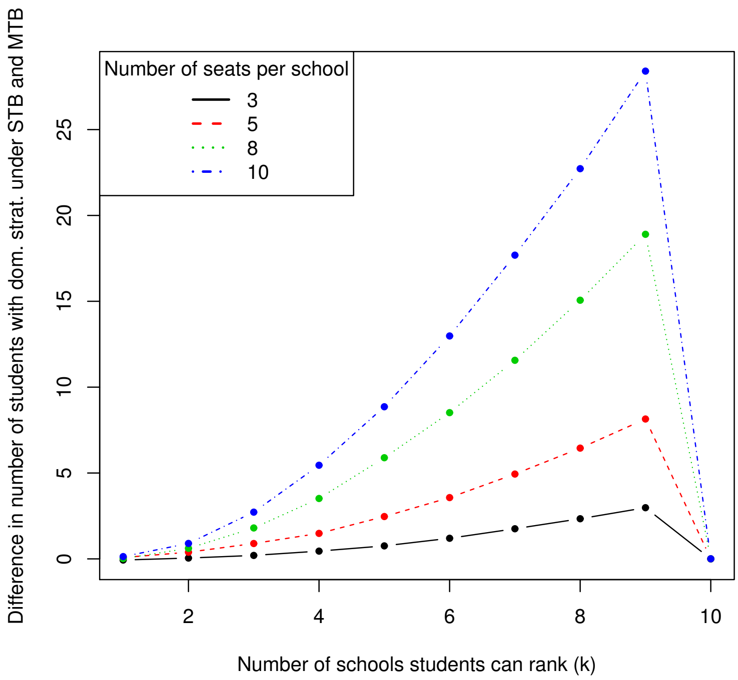

Figure 5: Difference in DA^k between the expected number of students with a dominant strategy with STB (Corollary 4) and the sample average (1000 observations) of the number of students who have a dominant strategy with MTB ($n = 100$, $m = 10$, and $q \in \{3, 5, 8, 10\}$).

Proposition 8. *For every $k \in \{1, \dots, m\}$ and every tie-breaking rules g^{DA} and g^{BOS}, $\mathbb{E}^{dom}_{DA^k}(g^{DA}) \geq \mathbb{E}^{dom}_{BOS^\ell}(g^{BOS})$ for any $\ell \in \{1, \dots, m\}$.*

The fact that STB provides more students with dominant strategies than MTB in our experiment is robust to other combinations of values for n, m, and q that we have investigated. This result is also robust to allowing for heterogeneous quotas and pre-existing priorities.

7.3 Equating the performances of STB while improving on ex-post fairness

As the simulations reveal, STB can provide sizable incentive improvements over MTB, but these improvements are limited for some values of the parameters. When improvements are limited, the incentive advantage of STB over MTB may not outweigh the perceived ex-post fairness cost of using STB instead of MTB (see the quote from Pathak, 2011, on page 17). A natural question is therefore whether the incentive advantage of STB can be harnessed while avoiding its ex-post fairness costs.

To answer this question, we develop what we believe to be the first metric of the fairness of (strict) priority profiles. In a priority profile, students are given a priority rank at each school. The sum of a student's priority ranks reflects how high a priority the student is given across the different schools (e.g., a student who occupies the first rank at each of the m schools has a sum of ranks of m). We propose to compare priority profiles based on the distribution across students of these sums of rank. We then show how STB can be improved upon in terms of ex-post fairness without jeopardizing STB's incentive properties (and while maintaining ex-ante fairness).

Consider the following priority profile in the support of STB when $n = 8$ and $m = 3$.

$$\begin{array}{ccc} F_1^{STB} & F_2^{STB} & F_3^{STB} \\ \hline t_1 & t_1 & t_1 \\ t_2 & t_2 & t_2 \\ t_3 & t_3 & t_3 \\ t_4 & t_4 & t_4 \\ t_5 & t_5 & t_5 \\ t_6 & t_6 & t_6 \\ t_7 & t_7 & t_7 \\ t_8 & t_8 & t_8 \end{array} \tag{11}$$

The sums of students' priority ranks in F are $(3, 6, 9, \ldots, 24)$, where 3 is the sum of t_1's priority ranks, 6 the sum of t_2's priority ranks, and so on. The remark from the NYC official on pg. 17 suggests that perceptions of unfairness are linked to the spread in this distribution. In the words of the official, given F^{STB} in (11), t_8 has a low chance to match with a school she likes because she has the "bad luck" of being the "last student in the line" at each and every school. Being the "last student in the line" at every school gives t_8 a sum of priority ranks of 24. Student t_1 on the other hand is guaranteed a seat at her most-preferred school as she has been given the top-priority at each of the three schools. This gives t_1 a sum of priority ranks of 3.

Among other changes, one would expect that reversing the order of priorities at one of the schools, as in F' below, should alleviate feelings of ex-post unfairness:

$$\begin{array}{ccc} F'_1 & F'_2 & F'_3 \\ \hline t_1 & t_8 & t_1 \\ t_2 & t_7 & t_2 \\ t_3 & t_6 & t_3 \\ t_4 & t_5 & t_4 \\ t_5 & t_4 & t_5 \\ t_6 & t_3 & t_6 \\ t_7 & t_2 & t_7 \\ t_8 & t_1 & t_8 \end{array} \tag{12}$$

The sums of students' priority ranks given F' are $(10, 11, 12, \ldots, 17)$. This distribution features a tighter spread than $(3, 6, 9, \ldots, 24)$. Specifically,

$$(10, 11, \ldots, 16, 17) = (3, 6, \ldots, 21, 24) + (7, 5, 3, 1, -1, -3, -5, -7),$$

where $(7, 5, 3, 1, -1, -3, -5, -7)$ corresponds to a series of "progressive transfers" from students with higher sums of ranks to students with lower sums of ranks.

Formally, for any priority profile F, let r_i^F be the sum over all schools of t_i's priority ranks. We call r_i^F the **total priority rank** of t_i given F. For example, for F^{STB} in (11), we have $r_3^{F^{STB}} = 9$ and $r_6^{F^{STB}} = 18$. The vector of total priority ranks given F is

$r^F := (r_1^F, \ldots, r_n^F)$. Vector r^F can be **obtained from** $r^{F'}$ **by a progressive transfer** if for some positive integer d and for two different students $t_i, t_j \in T$,

$$r_i^F = r_i^{F'} + d \leq r_j^{F'} - d = r_j^F, \qquad \text{while } r_k^F = r_k^{F'} \text{ for any } t_k \in T \backslash \{t_i, t_j\}.$$

We say that F is **more ex-post fair than** F' if r^F can be obtained from $r^{F'}$ by a sequence of progressive transfers. Similarly, *tie-breaking rule* g is **more ex-post fair than** tie-breaking rule g' if *any* profile in $supp(g)$ is more ex-post fair than *any* profile in $supp(g')$.

It is not hard to find tie-breaking rules that improve upon STB in terms of ex-post fairness. Our goal here is (a) to improve upon STB in terms of ex-post fairness while (b) preserving STB's incentive properties, and (c) maintaining ex-ante fairness.

Ex-ante fairness turns out to be easy to satisfy. Suppose that we find a priority profile F^* that improves upon all profiles in $supp(STB)$ in terms of ex-post fairness and incentive. Then, drawing a priority profile uniformly at random from the set of all permutations of F^* guarantees ex-ante fairness (by a permutation of F^*, we mean a profile that is obtained from F^* by permuting the identities of a number of students). By symmetry, such a tie-breaking rule also preserves the incentive properties of F^*.

Hence, to satisfy (a), (b), and (c), it is sufficient to identify a *single* priority profile that satisfies (a) and (b). We call **(tie-breaking) pattern** a profile F^* from which an ex-ante fair tie-breaking rule is generated by permutations. The associated ex-ante fair tie-breaking rule is denoted by $\bar{g}^{F^*}$.

Proposition 9. *For any $F \in \mathcal{F}$, any tie-breaking rule $\bar{g}^F$ obtained by drawing uniformly at random among the permutations of F is ex-ante fair.*

Recall that under STB, any student t who is among the kq highest-priority students has a dominant strategy because it is impossible for more than $kq - 1$ students to have a higher priority than t at all of her k most-preferred schools (hence, t's k preferred schools form a safe set). The same is true if priorities are redistributed among the kq highest-priority students. For example, suppose that $n = 8$, $m = 3$, $q = 2$ and $k = 2$, and consider the following profile

F_1^{FTB}	F_2^{FTB}	F_3^{FTB}
t_1	t_4	t_3
t_2	t_1	t_4
t_3	t_2	t_1
t_4	t_3	t_2
t_5	t_8	t_7
t_6	t_5	t_8
t_7	t_6	t_5
t_8	t_7	t_6

(13)

Clearly, students t_1 to t_4 have a dominant strategy, just as they would in F^{STB}. At the same time, priority profile F^{FTB} improves upon F^{STB} in terms of ex-post fairness, with $r^{F^{FTB}} = (6, 9, 8, 7, 18, 21, 20, 19)$ as opposed to $r^{F^{STB}} = (3, 6, 9, 12, 15, 18, 21, 24)$.[24]

[24] Sorting the two vectors in increasing order, one finds that $(6, 7, 8, 9, 18, 19, 20, 21) = (3, 6, 9, 12, 15, 18, 21, 24) + (3, 1, -1, -3, 3, 1, -1, -3)$.

The improvement in ex-post fairness brought about by F^{FTB} is more than cosmetic. In particular, four *different* students are given a top-priority at at least one school in F^{FTB}, as opposed to two under F^{STB}. Also, unlike in F^{STB}, no student is last at every school in F^{FTB}.

In general, the pattern in (13) can be constructed by the following series of translations. Fix an arbitrary priority order F_1^{STB} and focus first on priority ranks 1 to kq. For any $a \in \{2, \ldots, m\}$, the highest-kq priorities in F_a are obtained by translating the highest-kq priorities in F_{a-1} by $\lfloor kq/m \rfloor$.[25] That is, the rank of a generic "high-priority" student in F_a is equal to her rank in F_{a-1} plus $\lfloor kq/m \rfloor$ (modulo kq). Priority ranks $kq+1$ to n are constructed in a similar fashion: For any $a \in \{2, \ldots, m\}$, the $kq+1$ to n priorities in F_a are obtained by translating the same priorities in F_{a-1} by $\lfloor (n-kq)/m \rfloor$.

The next proposition shows that $\bar{g}^{F^{FTB}}$ improves upon g^{STB} in terms of ex-post fairness while maintaining the incentive properties of g^{STB}.

Proposition 10. *For any $k \in \{1, \ldots, m\}$, (i) $\mathbb{E}^{dom}_{DA^k}(\bar{g}^{F^{FTB}}) = \mathbb{E}^{dom}_{DA^k}(g^{STB}) = \min\{kq, n\}$. (ii) If $\lfloor kq/m \rfloor \geq 1$, then $\bar{g}^{F^{FTB}}$ is more ex-post fair than g^{STB}.*

An obvious pitfall of F^{FTB} is that it still divides students between lucky (top-kq priorities) and unlucky students (priorities $kq+1$ to n). Although F^{FTB} does a better job than F^{STB} at distributing priorities fairly among lucky and among unlucky students, unlucky students have a much lower chance of accessing the school of their choice than lucky ones.

In general, further improving upon the ex-post fairness of F^{FTB} while maintaining the incentive properties of STB can be complicated. As Proposition 7 shows, when $k \leq 2$ and $q = 1$, STB is optimal in terms of incentives. At least under such parameters, any alternative profile F' must therefore be itself optimal if it is to equate the incentive properties of STB. In particular, all opportunities for generating higher-order safe sets must be taken advantage of in the construction of F', which poses optimization and combinatorial challenges.

In one important case, however, it is possible to improve upon F^{FTB} while equating the incentive properties of STB. When seats are in short-supply ($mq \leq n$), it is possible to distribute top-priorities between students without awarding any student more than one top-priority. In this case, the decreasing return to correlations in top-priorities described in Example 3 turns out to exactly equate the benefits of correlations in STB, as illustrated in Example 5. This, in turn, allows any priority profile that evenly spreads top-priorities to match STB's incentives properties while improving upon F^{FTB} in terms of ex-post fairness.

Example 5 (Decreasing returns to correlations equate benefits of correlations when $mq \leq n$)**.** Suppose that $m = 3$, $q = 3$, and $n = 9$. For any k, STB provides a dominant strategy to $3k$ students. Consider the following "spread" priority profile that evenly

[25] For every real number $x \in \mathbb{R}$, the floor function $\lfloor . \rfloor$ returns the largest integer $\lfloor x \rfloor$ that is not larger than x.

distributes the 9 top-priorities among the 9 students:

F_1^{SPR}	F_2^{SPR}	F_3^{SPR}
t_1	t_4	t_7
t_2	t_5	t_8
t_3	t_6	t_9
$\vdots$	$\vdots$	$\vdots$

Under F^{SPR}, students have a dominant strategy if and only if their k most-preferred schools include the school at which they have a top-priority. For each student, this occurs with probability k/m, Thus, the expected number of students with a dominant strategy under F^{SPR} is nk/m, which by $mq = 9 = n$ is equal to kq, the expected number of students with a dominant strategy under F^{STB}.

Again, pattern F^{SPR} can in general be obtained by a series of translations. Fix an arbitrary F_1^{SPR}. For any $a \in \{2, \ldots, m\}$, the priority order F_a is obtained by translating priority order F_{a-1} by $\lfloor n/m \rfloor$. That is, the rank of a generic student in F_a is equal to her rank in F_{a-1} plus $\lfloor n/m \rfloor$ (modulo n). Observe that $\lfloor n/m \rfloor \geq q$ when $mq \leq n$, which implies that, as in Example 5, all students have at most one top-priority in F^{SPR}.

A complete example of F^{SPR} is illustrated bellow for the case $n = 8$ and $m = 3$ (with $\lfloor 8/3 \rfloor = 2$).

F_1^{SPR}	F_2^{SPR}	F_3^{SPR}
t_1	t_7	t_5
t_2	t_8	t_6
t_3	t_1	t_7
t_4	t_2	t_8
t_5	t_3	t_1
t_6	t_4	t_2
t_7	t_5	t_3
t_8	t_6	t_4

One can verify that $r^{F^{SPR}} = (9, 12, 15, 18, 13, 16, 11, 14)$ which represents an ex-post fairness improvement over $r^{F^{STB}} = (3, 6, 9, 12, 15, 18, 21, 24)$.[26] When $kq < n$ (which is, e.g., implied by the short-supply assumption together with $k < m$), pattern F^{SPR} also improves upon F^{FTB} in terms of ex-post fairness.[27] The next proposition generalizes these examples.

Proposition 11. *For any $k \in \{1, \ldots, m\}$, (i) if $mq \leq n$, $\mathbb{E}^{dom}_{DA^k}(\bar{g}^{F^{SPR}}) = \mathbb{E}^{dom}_{DA^k}(g^{STB}) = \min\{kq, n\}$. (ii) if $\lfloor n/m \rfloor \geq 1$, $\bar{g}^{F^{SPR}}$ is more ex-post fair than g^{STB}, and if in addition $kq < n$ then $\bar{g}^{F^{SPR}}$ is more ex-post fair than $\bar{g}^{F^{FTB}}$.*

In the short-supply case, it is sometimes possible to find ex-post fairness *optimal*

[26] Sorting the two vectors in increasing order, one finds that $(9, 11, 12, 13, 14, 15, 16, 18) = (3, 6, 9, 12, 15, 18, 21, 24) + (6, 5, 3, 1, -1, -3, -5, -6)$.

[27] In particular, in example (13) with $q = 2$ and $k = 2$, we have $\hat{r}^{F^{SPR}} = r^{F^{FTB}} + (3, 4, 4, 4, -4, -4, -4, -3)$, where $\hat{r}^{F^{SPR}}$ is the increasing reordering of $r^{F^{SPR}}$.

patterns that perform as well as STB in terms of incentives. Consider for example

F_1^{OPT}	F_2^{OPT}	F_3^{OPT}
t_1	t_4	t_8
t_2	t_6	t_7
t_3	t_5	t_2
t_4	t_7	t_3
t_5	t_8	t_6
t_6	t_1	t_5
t_7	t_3	t_1
t_8	t_2	t_4

One can check that $r^{F^{OPT}} = (14, 13, 14, 13, 14, 13, 13, 14)$, which is optimal in terms of ex-post fairness (when $n = 8$ and $m = 3$, the mean total priority rank is always equal to 13.5). Although such patterns can be found (at least computationally), they are harder to generate and describe. The tie-breaking rule often plays an important role in school choice assignment and crystallizes a lot of the parents' concerns. It is therefore important that the tie-breaking rule be transparent and easy to navigate, and official may prefer the suboptimal F^{SPR} out of simplicity concerns.

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Appendix

A Proofs

Additional notation. For any $x \in \{1, \ldots, m+1\}$, school $R_i(x)$ is the school ranked in position x in R_i. Similarly, for any $x \in \{1, \ldots, m\}$, $Q_i(x)$ is the school reported in position x in Q_i.

Proof of Proposition 1. We first prove part (i). Consider any $k \in \{1, \ldots, m\}$, any $t_i \in T$ and any $\tilde{S} \subseteq S$ with $\#\tilde{S} \leq k$.

(i) **Sufficiency**: If there exists a set of schools $\hat{S} \subseteq \tilde{S}$ such that

$$\#\{t' \in T \backslash \{t\} \mid t' \; F_s \; t \text{ for some } s \in \hat{S}\} < \sum_{s \in \hat{S}} q_s,$$

then set $\tilde{S}$ is safe for student t_i in DA^k.

We prove the contrapositive: if the set $\tilde{S}$ is *not* safe for t_i in DA^k, then there exists no subset $\hat{S} \subseteq \tilde{S}$ such that

$$\#\{t' \in T \backslash \{t\} \mid t' \; F_s \; t \text{ for some } s \in \hat{S}\} < \sum_{s \in \hat{S}} q_s.$$

By definition, if the set $\tilde{S}$ is not safe for t_i in DA^k, then there exists a strategy $Q'_i \in \mathcal{Q}_i^k$ reporting all schools in $\tilde{S}$ and some $Q'_{-i} \in \mathcal{Q}_{-i}^k$ such that t_i is unassigned in $DA^k(Q'_i, Q'_{-i})$. This implies that, over the course of mechanism DA^k applied to profile Q', student t_i has been rejected from all schools in $\tilde{S}$.

In DA^k, in order for t_i to be rejected from any school $s \in S$, at least q_s students with higher priority at s than t_i must be temporarily assigned to s. In turn, these students can only be rejected from s by other students with higher priority at s than themselves. Altogether, t_i having been rejected from all schools in $\tilde{S}$ implies that, in the list of assignments $DA^k(Q'_i, Q'_{-i})$, each school $s \in \tilde{S}$ is filled with students having a higher priority at s than t_i. Therefore, there are at least $\sum_{s \in \tilde{S}} q_s$ students with higher priority than t_i at some school in $\tilde{S}$. Formally, this means that

$$\#\{t' \in T \backslash \{t\} \mid t' \; F_s \; t \text{ for some } s \in \tilde{S}\} \geq \sum_{s \in \tilde{S}} q_s.$$

Given that student t_i has been rejected from all schools in $\tilde{S}$, the same reasoning leads to the same conclusion for any subset $\hat{S} \subseteq \tilde{S}$. This shows sufficiency.

(i) **Necessity**: If set $\tilde{S}$ is safe for student t_i in DA^k, then there exists a set of schools $\hat{S} \subseteq \tilde{S}$ such

$$\#\{t' \in T \backslash \{t\} \mid t' \; F_s \; t \text{ for some } s \in \hat{S}\} < \sum_{s \in \hat{S}} q_s.$$

We prove the contrapositive: if for any $\hat{S} \subseteq \tilde{S}$

$$\#\{t' \in T \backslash \{t\} \mid t' \; F_s \; t \text{ for some } s \in \hat{S}\} \geq \sum_{s \in \hat{S}} q_s,$$

then the set $\tilde{S}$ is not safe, i.e., there exists a strategy $Q'_i \in \mathcal{Q}_i^k$ reporting all schools in $\tilde{S}$ and some $Q'_{-i} \in \mathcal{Q}_{-i}^k$ such that t_i is unassigned in $DA^k(Q'_i, Q'_{-i})$.

Let Q'_i be any strategy for which t_i reports all the schools in $\tilde{S}$ and no other school. We construct Q'_{-i} such that $DA^k_i(Q'_i, Q'_{-i}) = t_i$.

If we have

$$\#\{t' \in T\backslash\{t\} \mid t' \; F_s \; t \text{ for some } s \in \hat{S}\} \geq \sum_{s \in \hat{S}} q_s,$$

for any $\hat{S} \subseteq \tilde{S}$, then it is possible to partition the set of students

$$\{t_j \in T\backslash\{t_i\} \mid t_j \; F_s \; t_i \text{ for some } s \in \tilde{S}\}$$

into $\#\tilde{S}$ subgroups — one subgroup for each $s \in \tilde{S}$ — such that all students in the subgroup associated to s have higher priority at s than t_i. Construct Q'_{-i} such that, for each $s \in \tilde{S}$, each student t_k in the subgroup associated to s in the above partition reports $Q'_k : s \; t_k$. The strategies of other students can be picked arbitrarily.

By construction, $DA^k_i(Q'_i, Q'_{-i}) = t_i$ given that every school s reported in Q'_i has all its seats assigned in the first round of DA^k to students having higher priority at s than t_i. This shows that the condition is necessary for the set $\tilde{S}$ to be safe.

We now prove part (ii). Consider any $k \in \{2, \ldots, m\}$, any $t_i \in T$ and any $\tilde{S} \subseteq S$ with $\#\tilde{S} = k$.

(ii) **Sufficiency** ($\tilde{S}$ is in oversupply): If the set of schools $\tilde{S}$ is in oversupply, then $\tilde{S}$ is safe for student t_i in BOS^k.

We prove sufficiency by contraposition: if $\tilde{S}$ is not safe for student t_i in BOS^k, then $\tilde{S} \subseteq S$ is not in oversupply.

If $\tilde{S}$ is not safe for student t_i in BOS^k, then there exists a strategy $Q'_i \in \mathcal{Q}^k_i$ reporting all schools in $\tilde{S}$ and some $Q'_{-i} \in \mathcal{Q}^k_{-i}$ such that t_i is unassigned in $BOS^k(Q'_i, Q'_{-i})$. As $BOS^k(Q'_i, Q'_{-i}) = t_i$, over the course of BOS^k applied to profile Q', student t_i has been rejected from all schools in $\tilde{S}$. Thus, for each school $s \in \tilde{S}$, q_s students other than t_i are assigned to s in $BOS^k(Q'_i, Q'_{-i})$. Given that each student is assigned to at most to one school, the last statement implies that $n > \sum_{s \in \tilde{S}} q_s$, which, by definition, means that $\tilde{S}$ is in short-supply.

(ii) **Sufficiency** ($\tilde{S}$ is in short-supply): If t_i has top-priority at all schools in $\tilde{S}$, then $\tilde{S}$ is safe for student t_i in BOS^k.

Given that $\#\tilde{S} = k$, all strategies $Q_i \in \mathcal{Q}^k_i$ reporting all schools in $\tilde{S}$ only report schools in $\tilde{S}$. As t_i has top-priority at all schools in $\tilde{S}$, student t_i has top-priority at $Q_i(1)$. Therefore $BOS^k(Q_i, Q_{-i}) = Q_i(1)$ for any $Q_{-i} \in \mathcal{Q}^k_{-i}$. This shows that $\tilde{S}$ is safe for student t_i in BOS^k.

(ii) **Necessity**: If the set of schools $\tilde{S}$ is safe for student t_i in BOS^k, then $\tilde{S}$ is in oversupply or t_i has top-priority at all schools in $\tilde{S}$.

We prove sufficiency by contraposition: if $\tilde{S}$ is not in oversupply and there is $s^* \in \tilde{S}$ at which t_i does not have top-priority, then $\tilde{S}$ is not safe for student t_i in BOS^k, i.e. there exists a strategy $Q'_i \in \mathcal{Q}^k_i$ reporting all schools in $\tilde{S}$ and some $Q'_{-i} \in \mathcal{Q}^k_{-i}$ such that t_i is unassigned in $BOS^k(Q'_i, Q'_{-i})$.

Let Q'_i be any strategy for which t_i declares all the schools in $\tilde{S}$ and $Q'_i(1) = s^*$. We construct Q'_{-i} such that the q_{s^*} students t_j with top-priority at s^* declare $Q'_j : s^*\ t_j$. The remaining students $t_k \neq t_i$ are partitioned into $k-1$ subgroups, one for each $s' \in \tilde{S}\backslash\{s^*\}$. For each $s' \in \tilde{S}\backslash\{s^*\}$, the subgroup of students associated with s' has size at least $q_{s'}$. Such a partition is possible given that $\tilde{S}$ is not in oversupply. Any student t_k in the subgroup associated with s' reports $Q'_j : s'\ t_j$.

By construction of Q'_i and Q'_{-i}, t_i is rejected from s^* in the first round of BOS^k and all seats at all schools in $\tilde{S}$ are occupied at the end of the first round. Therefore, we have $BOS^k_i(Q'_i, Q'_{-i}) = t_i$, the desired result.

Proof of Proposition 2. Consider any $k \in \{1, \dots, m\}$ and any $t_i \in T$.

Sufficiency: Sufficiency is directly implied by Lemma 1.

Lemma 1. *For any $k \in \{1, \dots, m\}$ and any student $t_i \in T$, (i) if t_i has no more than k acceptable schools, then the truthful strategy $Q^*_i :\ R_i(1)\ \dots\ R_i(\#R_i)\ t_i$ belongs to $\mathcal{Q}^k_i$ and is dominant in DA^k, (ii) if her k most-preferred schools are all acceptable and form a safe set, then the truthful truncated strategy $Q^{**}_i :\ R_i(1)\ \dots\ R_i(k)\ t_i$ belongs to $\mathcal{Q}^k_i$ and is dominant in DA^k.*

Proof. We prove both claims in turn.

(i) Let $R^{Q^*_i}$ be the preference relation over $S \cup \{t_i\}$ defined as

$$R^{Q^*_i} : Q^*_i\ t_i\ R^{S\backslash Q^*_i},$$

where $R^{S\backslash Q^*_i}$ is the sub-ordering of the schools in $S\backslash Q^*_i$ corresponding to that of preferences R_i.

Because DA^m is non-manipulable (Dubins and Freedman, 1981), we have

$$DA_i(Q^*_i, Q^m_{-i})\ R^{Q^*_i}\ DA_i(Q_i, Q^m_{-i}), \qquad \text{for any } Q^m_{-i} \in \mathcal{Q}_i \text{ and any } Q_i \in \mathcal{Q}_i.$$

In particular,

$$DA_i(Q^*_i, Q^k_{-i})\ R^{Q^*_i}\ DA_i(Q_i, Q^k_{-i}), \quad \text{for any } Q^k_{-i} \in \mathcal{Q}^k_{-i} \text{ and any } Q_i \in \mathcal{Q}^k_i.$$

But because DA^k is obtained from DA^m by considering only the profiles $Q^k \in \mathcal{Q}^k$, the last displayed relation implies

$$DA^k_i(Q^*_i, Q^k_{-i})\ R^{Q^*_i}\ DA^k_i(Q_i, Q^k_{-i}), \qquad \text{for any } Q^k_{-i} \in \mathcal{Q}^k_{-i} \text{ and any } Q_i \in \mathcal{Q}^k_i.$$

By construction, Q^*_i is without swap and reports all acceptable schools, and therefore, the last displayed relation implies

$$DA^k_i(Q^*_i, Q^k_{-i})\ R_i\ DA^k_i(Q_i, Q^k_{-i}), \quad \text{for any } Q^k_{-i} \in \mathcal{Q}^k_{-i} \text{ and any } Q_i \in \mathcal{Q}^k_i \text{ such that } DA^k_i(Q^*_i, Q^k_{-i}), DA^k_i(Q_i, Q^k_{-i}) \in Q^*_i \cup \{t_i\}. \tag{14}$$

Given that $DA_i^k(Q_i^*, Q_{-i}^k) \in Q_i^* \cup \{t_i\}$ for any $Q_{-i}^k \in \mathcal{Q}_{-i}^k$, (14) simplifies to

$$\begin{aligned} &DA_i^k(Q_i^*, Q_{-i}^k)\ R_i\ DA_i^k(Q_i, Q_{-i}^k), \\ &\qquad \text{for any } Q_{-i}^k \in \mathcal{Q}_{-i}^k \text{ and any } Q_i \in \mathcal{Q}_i^k \text{ such that } DA_i^k(Q_i, Q_{-i}^k) \in Q_i^* \cup \{t_i\}. \end{aligned} \tag{15}$$

By construction of Q_i^*, if $DA_i^k(Q_i, Q_{-i}^k) \notin Q_i^* \cup \{t_i\}$ then $t_i\ P_i\ DA_i^k(Q_i, Q_{-i}^k)$. Therefore, (14) further simplifies to

$$DA_i^k(Q_i^*, Q_{-i}^k)\ R_i\ DA_i^k(Q_i, Q_{-i}^k), \qquad \text{for any } Q_{-i}^k \in \mathcal{Q}_{-i}^k \text{ and any } Q_i \in \mathcal{Q}_i^k \tag{16}$$

the desired result.

(ii) Let $R^{Q_i^{**}}$ be any preference relation over $S \cup \{t_i\}$ of the form

$$R^{Q_i^{**}} : Q_i^{**}\ t_i\ Q^{S \backslash Q_i^{**}},$$

where $Q^{S \backslash Q_i^{**}}$ is any sub-orderings of the schools in $S \backslash Q_i^{**}$. Because DA is non-manipulable (Dubins and Freedman, 1981), we have

$$DA_i(Q_i^{**}, Q_{-i}^m)\ R^{Q_i^{**}}\ DA_i(Q_i, Q_{-i}^m), \qquad \text{for any } Q_{-i}^m \in \mathcal{Q}_{-i} \text{ and any } Q_i \in \mathcal{Q}_i.$$

In particular,

$$\begin{aligned} &DA_i(Q_i^{**}, Q_{-i}^k)\ R^{Q_i^{**}}\ DA_i(Q_i, Q_{-i}^k), \\ &\qquad \text{for any } Q_{-i}^k \in \mathcal{Q}_{-i}^k \text{ and any } Q_i \in \mathcal{Q}_i^k. \end{aligned}$$

But because DA^k is obtained from DA by considering only the profiles $Q^k \in \mathcal{Q}^k$, the last displayed relation implies

$$DA_i^k(Q_i^{**}, Q_{-i}^k)\ R^{Q_i^{**}}\ DA_i^k(Q_i, Q_{-i}^k), \qquad \text{for any } Q_{-i}^k \in \mathcal{Q}_{-i}^k \text{ and any } Q_i \in \mathcal{Q}_i^k.$$

By construction, Q_i^{**} is without swap, and therefore, the last displayed relation implies

$$\begin{aligned} &DA_i^k(Q_i^{**}, Q_{-i}^k)\ R_i\ DA_i^k(Q_i, Q_{-i}^k), \\ &\qquad \text{for any } Q_{-i}^k \in \mathcal{Q}_{-i}^k \text{ and any } Q_i \in \mathcal{Q}_i^k \text{ such that } DA_i^k(Q_i^{**}, Q_{-i}^k), DA_i^k(Q_i, Q_{-i}^k) \in Q_i^{**}. \end{aligned} \tag{17}$$

Given that the k most-preferred schools of t_i form a safe set, we have that

$$DA_i^k(Q_i^{**}, Q_{-i}^k) \in Q_i^{**}, \qquad \text{for any } Q_{-i}^k \in \mathcal{Q}_{-i}^k,$$

and (17) simplifies to

$$\begin{aligned} &DA_i^k(Q_i^{**}, Q_{-i}^k)\ R_i\ DA_i^k(Q_i, Q_{-i}^k), \\ &\qquad \text{for any } Q_{-i}^k \in \mathcal{Q}_{-i}^k \text{ and any } Q_i \in \mathcal{Q}_i^k \text{ such that } DA_i^k(Q_i, Q_{-i}^k) \in Q_i^{**}. \end{aligned} \tag{18}$$

Finally, because only acceptable schools are reported in Q_i^{**} and because DA^k is individually rational, (18) is also true for Q_{-i}^k such that $DA_i^k(Q_i, Q_{-i}^k) \notin Q_i^{**}$, and we have

$$DA_i^k(Q_i^{**}, Q_{-i}^k)\ R_i\ DA_i^k(Q_i, Q_{-i}^k), \qquad \text{for any } Q_{-i}^k \in \mathcal{Q}_{-i}^k \text{ and any } Q_i \in \mathcal{Q}_i^k, \tag{19}$$

the desired result.

■

Necessity: If t_i has at least $k+1$ acceptable schools and the k most-preferred schools in R_i do not form a safe set, then t_i does not have a dominant strategy in DA^k.
The proof is by contradiction: assume that strategy $Q'_i \in \mathcal{Q}^k_i$ is dominant for t_i.

- **Case 1:** Q'_i reports the k most-preferred schools in R_i.

  As the k most-preferred schools in R_i do not form a safe set, there exists $Q'_{-i} \in \mathcal{Q}^k_{-i}$ such that

  $$DA^k_i(Q'_i, Q'_{-i}) = t_i.$$

  Let s^* denote the acceptable school $s^* = R_i(k+1)$ that is not reported in Q'_i. By assumption, t_i finds at least $k+1$ schools acceptable and hence $s^*\ P_i\ t_i$.

  We construct a profile $Q''_{-i} \in \mathcal{Q}^k_{-i}$ for which t_i ends up unassigned when playing Q'_i but for which she is assigned to s^* when playing an alternative strategy $Q''_i \in \mathcal{Q}^k_i$. Formally, Q''_{-i} such that

  $$DA^k_i(Q'_i, Q''_{-i}) = t_i \qquad \text{and} \qquad DA^k_i(Q''_i, Q''_{-i}) = s^*$$

  for $Q''_i : s^*\ t_i$. If such profile Q''_{-i} can be constructed, then Q'_i is not a dominant strategy.

  Profile Q''_{-i} is constructed as follows. In the course of DA^k applied to (Q'_i, Q'_{-i}), t_i is rejected from all the schools reported in Q'_i. Therefore, it must be that, in assignment $DA^k(Q'_i, Q'_{-i})$, there is another student assigned to each of the available seats in each of the schools reported in Q'_i. Let $A \subset T$ be the set of all these students who are assigned to a school t_i applied to but was rejected from in $DA^k_j(Q'_i, Q'_{-i})$.

  Now construct Q''_{-i} as follows :

  - For any $t_j \in A$, let Q''_j be the strategy in which t_j reports *only* $DA^k_j(Q'_i, Q'_{-i})$.
  - For any $t_h \in T \backslash \{A \cup \{t_i\}\}$, let Q''_h be any strategy in which t_h does not report s^*.

  By construction, for every school $s_j \in Q'_i$, there are at least q_j-students with higher priority at s_j than t_i who rank s_j first in Q''_{-i}. Thus, t_i will be rejected from any of these schools over the course of DA^k given that the reported profile is (Q'_i, Q''_{-i}). Therefore, we have

  $$DA^k(Q'_i, Q''_{-i}) = t_i.$$

  By construction again, no student reports s^* in Q''_{-i}. Therefore,

  $$DA^k_i(Q''_i, Q''_{-i}) = s^*.$$

- **Case 2:** Q'_i does not report all of the k most-preferred schools in R_i.

  This case is such that there is an acceptable school $s^* = R_i(r)$ for some $r \leq k$ such that $s^* \notin Q'_i$. As the k most-preferred schools in R_i do not form a safe set, there exists $Q'_{-i} \in \mathcal{Q}^k_{-i}$ such that

  $$s^*\ P_i\ DA^k_i(Q'_i, Q'_{-i}).$$

We show that there also exists $Q''_{-i} \in \mathcal{Q}^k_{-i}$ such that

$$s^* \; P_i \; DA^k_i(Q'_i, Q''_{-i}) \qquad \text{and} \qquad DA^k_i(Q''_i, Q''_{-i}) = s^*$$

for $Q''_i : s^* \; t_i$. If it is the case, then Q'_i is not a dominant strategy.
Let $S' = \{s \in Q'_i | s \; Q'_i \; DA^k_i(Q'_i, Q'_{-i})\}$ be the set of schools that t_i ranks in Q'_i above her assignement $DA^k_i(Q'_i, Q'_{-i})$. Let A be the subset of students who are assigned to a school in S' in the list of assignments $DA^k(Q'_i, Q'_{-i})$. Now, construct Q''_{-i} as follows :

- For any $t_j \in A$, let Q''_j be the strategy in which t_j reports *only* $DA^k_j(Q'_i, Q'_{-i})$.
- For any $t_h \in T \backslash \{A \cup \{t_i\}\}$, let Q''_h be any strategy in which t_h does not report s^*.

By construction, for every school $s_j \in S'$, there are at least q_j-students with higher priority at s_j than t_i who rank s_j first in Q''_{-i}. Therefore,

$$s^* \; P_i \; DA^k_i(Q'_i, Q''_{-i}).$$

Also, no student reports s^* in Q''_{-i}, which implies

$$DA^k_i(Q''_i, Q''_{-i}) = s^*.$$

Proof of Proposition 3. Consider any $k \in \{1, \dots, m\}$ and any $t_i \in T$.

Sufficiency is directly implied by Lemma 2.

Lemma 2. *For any $k \in \{1, \dots, m\}$ and any student $t_i \in T$, (i) if t_i has only one acceptable school, then the truthful strategy $Q^*_i : \; R_i(1) \; t_i$ belongs to $\mathcal{Q}^k_i$ and is dominant in BOS^k, (ii) if she has top-priority at her most-preferred school, then the truthful truncated strategy $Q^{**}_i : \; R_i(1) \; t_i$ belongs to $\mathcal{Q}^k_i$ and is dominant in BOS^k.*

Proof. We prove both claims in turn.
(i) We show that

$$BOS^k_i(Q^*_i, Q_{-i}) \; R_i \; BOS^k_i(Q_i, Q_{-i}), \qquad \text{for any } Q_{-i} \in \mathcal{Q}^k_{-i} \text{ and any } Q_i \in \mathcal{Q}^k_i. \quad (20)$$

Take any $Q_i \in \mathcal{Q}^k_i$ and any $Q_{-i} \in \mathcal{Q}^k_{-i}$. Two cases may arise:

- **Case 1:** $BOS^k_i(Q_i, Q_{-i}) = R_i(1)$. Given that school $R_i(1)$ is ranked first in Q^*_i, we have that

$$BOS^k_i(Q_i, Q_{-i}) = R_i(1) \qquad \text{implies that} \qquad BOS^k_i(Q^*_i, Q_{-i}) = R_i(1),$$

  and therefore (20) holds.

- **Case 2:** $BOS^k_i(Q_i, Q_{-i}) \neq R_i(1)$.

  As t_i only has one acceptable school, this case is such that $t_i \; R_i \; BOS^k_i(Q_i, Q_{-i})$. By definition of BOS^k, we have that $BOS^k_i(Q^*_i, Q_{-i}) \in \{R_i(1), t_i\}$ and therefore (20) holds.

(ii) Given that t_i has top-priority at $R_i(1)$, strategy Q_i^{**} is such that

$$BOS_i^k(Q_i^{**}, Q_{-i}) = R_i(1) \qquad \text{for any} \;\; Q_{-i} \in \mathcal{Q}_{-i}^k$$

and is therefore dominant. ■

Necessity: If t_i has at least 2 acceptable schools and does not have top-priority at her most-preferred school, then t_i does not have a dominant strategy in BOS^k.
Take any strategy $Q_i' \in \mathcal{Q}_i^k$. We show that Q_i' cannot be a dominant strategy for t_i in BOS^k.

- **Case 1:** Q_i' does not rank $R_i(1)$ first.

  This case is such that $Q_i' : s^* \;\ldots$ for some $s^* \neq R_i(1)$. Consider any $Q_{-i}' \in \mathcal{Q}_{-i}^k$ such that all students $t_j \neq t_i$ report neither s^* nor $R_i(1)$. By construction, we have

  $$BOS_i^k(Q_i', Q_{-i}') = s^* \qquad \text{and} \qquad BOS_i^k(Q_i^*, Q_{-i}') = R_i(1),$$

  which shows that Q_i' is not a dominant strategy.

- **Case 2:** Q_i' ranks $R_i(1)$ first.

  We show that there exists a profile $Q_{-i}' \in \mathcal{Q}_{-i}^k$ such that t_i is assigned to a school different from $R_i(1)$ and $R_i(2)$ when playing Q_i', but is assigned to $R_i(2)$ when playing $Q_i'' : R_i(2)\; t_i$.

  - **Subcase 2.1:** t_i has top-priority at $R_i(2)$.
    Profile Q_{-i}' is constructed as follows

    * Take the $q_{R_i(1)}$ students who have a top-priority at $R_i(1)$. These students only report $R_i(1)$. Observe that there are at least $q_{R_i(1)}$ such students given that t_i does not have to priority at this school.
    * All remaining students only report $R_i(2)$.

    The model assumes that no two schools have enough seats to jointly host all students. As a result, all the seats in schools $R_i(1)$ and $R_i(2)$ are filled in the first round of BOS^k applied to (Q_i', Q_{-i}'). Therefore, we have $R_i(2)\; P_i\; BOS_i^k(Q_i', Q_{-i}')$. Yet, given that t_i has a top-priority at $R_i(2)$, we have $BOS_i^k(Q_i'', Q_{-i}') = R_i(2)$. Together, we have

    $$BOS_i^k(Q_i'', Q_{-i}')\; P_i\; BOS_i^k(Q_i', Q_{-i}'),$$

    and Q_i' is not a dominant strategy.

  - **Subcase 2.2:** t_i does not have top-priority at $R_i(2)$.
    Profile Q_{-i}' is constructed as follows

    * Take the $q_{R_i(1)}$ students that have a top-priority at $R_i(1)$. These students only report $R_i(1)$.
    * If among the remaining students, there are at least $q_{R_i(2)}$ students that have higher priority than t_i at $R_i(2)$, then for any remaining student t_j let $Q_j' : R_i(1)\; R_i(2)\; t_j$, otherwise all remaining students only report $R_i(2)$.

Because there are not enough seats in $R_i(1)$ and $R_i(2)$ to jointly host all students, we have by construction that

$$R_i(2)\ P_i\ BOS_i^k(Q_i', Q_{-i}') \qquad \text{and} \qquad BOS_i^k(Q_i'', Q_{-i}') = R_i(2),$$

which shows that Q_i' is not a dominant strategy.

Proof of Corollary 1. Consider any $\ell \in \{2, \ldots, m\}$, any $k \in \{2, \ldots, m\}$ and any $t_i \in T$.

Step 1. For each R_i at which t_i has a dominant strategy in BOS^ℓ, t_i also has a dominant strategy in DA^k.

By Proposition 3, t_i has a dominant strategy in BOS^ℓ only in two cases

- **Case 1:** t_i has only one acceptable school.

  This case is such that t_i has no more than k acceptable schools. Then, by Proposition 2, t_i also has a dominant strategy in DA^k.
- **Case 2:** t_i has a top-priority at her most-preferred school.
  - Subcase 2.1: t_i has no more than k acceptable schools.
    By Proposition 2, t_i has a dominant strategy in DA^k.
  - Subcase 2.2: t_i has more than k acceptable schools.
    As t_i has top-priority at her most-preferred school, there are at most $q_{R_i(1)} - 1$ students with a higher priority at $R_i(1)$ than t_i. Then, by Proposition 1, t_i's k most-preferred schools form a safe set. By Proposition 2, t_i then has a dominant strategy in DA^k.

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Step 2. There exist R_i, F and q such that t_i has a dominant strategy in DA^k but not in BOS^ℓ.

Take any R_i, F and q such that t_i (a) finds all schools acceptable, (b) does not have top-priority at $R_i(1)$, but (c) has a top-priority at $R_i(2)$. By Proposition 3, t_i does not have a dominant strategy in BOS^ℓ. By Proposition 1, her 2 most-preferred schools form a safe set. By Proposition 1 again, her k most-preferred schools form a safe set. Then, by Proposition 2, t_i has a dominant strategy in DA^k.

Proof of Corollary 2. Consider any $k \in \{1, \ldots, \min(m-1, n-1)\}$ and any $t_i \in T$.

Step 1. For each R_i at which t_i has a truthful dominant strategy in DA^k, t_i has a truthful dominant strategy in DA^{k+1}.

By Proposition 2, t_i has a dominant strategy in DA^k only in two cases

- **Case 1:** t_i has no more than k acceptable schools.

  This case is such that t_i has no more than $k+1$ acceptable schools. By Proposition 2, t_i has a dominant strategy in DA^{k+1}.

- **Case 2:** the k most-preferred schools of t_i are all acceptable and form a safe set. If t_i has no more than $k+1$ acceptable schools, then t_i has a dominant strategy in DA^{k+1} by Proposition 2. Otherwise, t_i has more than $k+1$ acceptable schools. Given that her k most-preferred schools form a safe set, her $k+1$ most-preferred schools also form a safe set (Proposition 1). Then, by Proposition 2, t_i has a dominant strategy in DA^{k+1}.

Step 2. There exist R_i, F and q such that t_i has a dominant strategy in DA^{k+1} but not in DA^k.

Take any R_i with $k+1$ acceptable schools. Take q such that there is only one seat in each school. Take any F such that t_i has top-priority at $R_i(k+1)$ but has the lowest priority of all students at all other schools. As $k < n$, the k most-preferred schools of t_i are not oversupplied. By Proposition 1, the k most-preferred schools of t_i do not form a safe set. Thus, by Proposition 2, t_i does not have a dominant strategy in DA^k. By Proposition 1, the $k+1$ most-preferred schools of t_i form a safe set. Thus, by Proposition 2, t_i has a dominant strategy in DA^{k+1}.

Proof of Corollary 3. Consider any $k \in \{1, \dots, m-1\}$ and any $t_i \in T$. For each R_i at which t_i has a dominant strategy in BOS^k, t_i has a dominant strategy in BOS^r for any $r \in \{1, \dots, m\}$. By Proposition 3, t_i has a dominant strategy in BOS^k only in two cases

- **Case 1:** t_i has only one acceptable school.

  By Proposition 2, t_i has a dominant strategy in DA^r.

- **Case 2:** t_i has a top-priority at her most-preferred school.

  By Proposition 2, t_i has a dominant strategy in DA^r.

Proof of Corollary 4. There are two cases to consider: either sets of k schools are oversupplied, or not.

- **Case 1:** Over-supply: $kq \geq n$.

  If $kq \geq n$, then all sets of k schools are oversupplied. By Proposition 1, all sets of k schools are safe sets for any students. Hence, the k most-preferred schools of all students form a safe set. By Proposition 2, all students have a dominant strategy in DA^k. Hence we have $\mathbb{E}^{dom}_{DA^k}(g^{STB}) = n$. Given that $kq \geq n$, we have $\mathbb{E}^{dom}_{DA^k}(g^{STB}) = \min\{kq, n\}$, the desired result.

- **Case 2:** Short-supply: $kq < n$.

  STB creates perfect correlation across priorities at all schools. We show that all students whose priority rank is smaller or equal to kq have a dominant strategy, unlike students with a priority rank larger than kq.

  Take any t_i with priority rank smaller or equal to kq. Denote by S^k the set of her k most-preferred schools. Given the perfect correlation in priorities created by STB, we have that

$$\#\{t' \in T \backslash \{t\} \mid t' \ F_s \ t \text{ for some } s \in S^k\} < kq = \sum_{s \in S^k} q_s.$$

By Proposition 1, the set S^k is safe for t_i. By Proposition 2, t_i has a dominant strategy in DA^k.

Take any t_i with priority rank larger than kq. Denote by S^k the set of her k most-preferred schools. Given the perfect correlation in priorities created by STB, we have that

$$\#\{t' \in T \backslash \{t\} \mid t' \; F_s \; t \text{ for some } s \in S^k\} \geq kq = \sum_{s \in S^k} q_s.$$

By Proposition 1, the set S^k is not safe for t_i. We assumed that all students have more than k acceptable schools. Therefore, by Proposition 2, only students for whom the k most-preferred schools form a safe set have a dominant strategy in DA^k. Therefore, t_i does not have a dominant strategy in DA^k.

Given that there are kq students whose priority rank is smaller or equal to kq, we have that $\mathbb{E}^{dom}_{DA^k}(g^{STB}) = kq$. Given that $kq < n$, we have $\mathbb{E}^{dom}_{DA^k}(g^{STB}) = \min\{kq, n\}$, the desired result.

Proof of Corollary 5. Take any F in $supp(g)$. Given F, the expected number of students with a dominant strategy in BOS^k is given by (4):

$$\mathbb{E}^{dom}_{BOS^k}(F) := \frac{1}{\#\bar{\mathcal{R}}_1} \sum_{t_i \in T} \sum_{R_i \in \bar{\mathcal{R}}_i} \mathbb{1}(R_i, F),$$

where $\mathbb{1}(R_i, F)$ is an indicator function which takes value 1 when t_i has a dominant strategy in BOS^k given preferences R_i and priorities F, and zero otherwise. We assumed that all students have more than k acceptable schools. Therefore, by Proposition 3, only students who have top-priority at their most-preferred school have a dominant strategy in BOS^k.

For any $t_i \in T$, we have that $\mathbb{1}(R_i, F) = 1$ if t_i has top-priority at $R_i(1)$ in profile F and $\mathbb{1}(R_i, F) = 0$ otherwise. Letting $p_i(F) \in \{0, \ldots, m\}$ denote the number of schools at which t_i has top-priority for the priority profile F, we have that

$$\frac{1}{\#\bar{\mathcal{R}}_1} \sum_{R_i \in \bar{\mathcal{R}}_i} \mathbb{1}(R_i, F) = \frac{p_i(F)}{m}.$$

Aggregating over all students in T, we obtain

$$\mathbb{E}^{dom}_{BOS^k}(F) = \frac{1}{m} \sum_{t_i \in T} p_i(F).$$

Given that exactly q students receive top-priority at each school, we have that $\sum_{t_i \in T} p_i(F) = mq$. Therefore, the last equation becomes

$$\mathbb{E}^{dom}_{BOS^k}(F) = q,$$

where $q < n$ because no two schools have enough capacity to host all students. The last equation shows that $\mathbb{E}^{dom}_{BOS^k}(F)$ does not depend on the priority profile F. Hence, all distributions g on $\mathcal{F}$ are such that $\mathbb{E}^{dom}_{BOS^k}(g) = q$, the desired result.

Proof of Proposition 4. The following lemma shows that student t_i has an expected dominant strategy in DA^k only when the strategy obtained by truncating preferences R_i after $R_i(k)$ is expected dominant. Recall that we assume every student t_i to have at least $k+1$ acceptable schools.

Lemma 3. *For any $k \in \{1,\dots,m\}$, in the absence of pre-existing priorities and given an ex-ante fair tie-breaking rule g, a student t_i whose k most-preferred schools are in short-supply has an expected dominant strategy in DA^k only if the strategy $Q_i^* :\ R_i(1)\ R_i(2)\ \dots\ R_i(k)\ t_i$ is an expected dominant strategy.*

Proof. We show that no strategy $Q_i' \in \mathcal{Q}_i^k$ dominates Q_i^* (in expected terms) in DA^k. If this is the case, then it is only when Q_i^* is an expected dominant strategy that t_i has an expected dominant strategy. To do so, we identify for any strategy $Q_i' \in \mathcal{Q}_{-i}^k$ different from Q_i^* a profile $Q_{-i}' \in \mathcal{Q}_{-i}^k$ for which strategy Q_i is such that

$$\mathbb{E}_i(Q_i^*, Q_{-i}'; DA^k) > \mathbb{E}_i(Q_i', Q_{-i}'; DA^k).$$

Take any strategy $Q_i' \in \mathcal{Q}_i^k$ different from Q_i^*. Let $a \in \{1,\dots,k\}$ be the smallest rank for which $Q_i'(a) \neq R_i(a)$. That is, strategies Q_i^* and Q_i' coincide for ranks smaller than a: $Q_i'(1) = Q_i^*(1), \dots, Q_i'(a-1) = Q_i^*(a-1)$, but not at rank a for which $Q_i'(a) \neq Q_i^*(a) = R_i(a)$ (they need not coincide either at ranks larger than a).

Let Q_{-i}' be such that all other students report the $a-1$ most-preferred schools of t_i without swaps, and no other schools. That is, for any $t_j \neq t_i$, we have $Q_j'(1) = R_i(1)$, $\dots$, $Q_j'(a-1) = R_i(a-1)$, and $Q_j(a) = t_j$..

By construction, for any priority profiles, we have either that

$$DA_i^k(Q_i^*, Q_{-i}') = DA_i^k(Q_i', Q_{-i}'), \tag{21}$$

or that

$$DA_i^k(Q_i^*, Q_{-i}') = R_i(a) \qquad \text{and} \qquad DA_i^k(Q_i', Q_{-i}') = Q_i'(a). \tag{22}$$

As we assume that (i) the k most-preferred schools are in short-supply, (ii) there are no pre-existing priorities, and (iii) the tie breaking rule is ex-ante fair, there exist priority profiles in the support of g for which (22) holds. By the definition of a, we have that $R_i(a)\ P_i\ Q_i'(a)$, and hence $u_i(R_i(a)) > u_i(Q_i'(a))$. This shows that

$$\mathbb{E}_i(Q_i^*, Q_{-i}'; DA^k) > \mathbb{E}_i(Q_i', Q_{-i}'; DA^k),$$

the desired result.

■

Take any $k \in \{1,\dots,m-1\}$ and any ex-ante fair tie-breaking rule g. By Lemma 3, student t_i has a dominant strategy if and only if $Q_i^* :\ R_i(1)\ R_i(2)\ \dots\ R_i(k)\ t_i$ is such that

$$\mathbb{E}_i(Q_i^*, Q_{-i}; DA^k) \geq \mathbb{E}_i(Q_i', Q_{-i}; DA^k) \qquad \text{for any } Q_i' \in \mathcal{Q}_i^k \text{ and any } Q_{-i} \in \mathcal{Q}_{-i}^k.$$

The necessary condition stated in Proposition 4 is obtained by considering a particular strategy $Q_i'' \in \mathcal{Q}_i^k$ and a particular profile $Q_{-i}'' \in \mathcal{Q}_{-i}^k$ and checking whether

$$\mathbb{E}_i(Q_i^*, Q_{-i}''; DA^k) \geq \mathbb{E}_i(Q_i'', Q_{-i}''; DA^k).$$

Let the strategy Q_i'' be defined as

$$Q_i'':\ R_i(1)\ R_i(2)\ \ldots\ R_i(k-1)\ R_i(k+1)\ t_i$$

Strategy Q_i'' differs from Q_i^* only because $Q_i''(k) = R_i(k+1) \neq R_i(k) = Q_i^*(k)$.

Let the profile Q_{-i}'' be such that $Q_j'' = Q_i^*$ for any $t_j \neq t_i$. In profile Q_{-i}'', all other students report the same preferences as t_i when she reports Q_i^*.

Under strategy profile (Q_i^*, Q_{-i}''), the situation is ex-ante symmetric: all students report the same preferences, there are no pre-existing priorities and the tie-breaking rule is ex-ante fair. As a result, all students have the same probability of being assigned to any of the schools they report. Hence, for any school $s \in Q_i^*$, the probability that t_i be assigned to s is

$$Proba(t_i\ assigned\ to\ s|(Q_i^*, Q_{-i}'')) = \frac{q}{n}.$$

Under strategy profile (Q_i'', Q_{-i}''), the situation is also ex-ante symmetric for the $k-1$ most-preferred schools of t_i. Hence, for any school $s \in Q_i''$ such that $s\ R_i\ R_i(k-1)$, the probability that t_i be assigned to s is

$$Proba(t_i\ assigned\ to\ s|(Q_i'', Q_{-i}'')) = \frac{q}{n}.$$

In profile (Q_i'', Q_{-i}''), t_i is the only student who reports $R_i(k+1)$. Therefore, if t_i is not assigned to a school she ranks above $R_i(k)$, then she is assigned to $R_i(k+1)$ with probability one. Hence, the probability that t_i is assigned to $R_i(k+1)$ is

$$Proba(t_i\ assigned\ to\ R_i(k+1)|(Q_i'', Q_{-i}'')) = 1 - \frac{q}{n} * (k-1),$$

where $\frac{q}{n} * (k-1) < 1$ given that the k most-preferred schools are in short-supply.

When the sub-profile is Q_{-i}'', the difference in expected utility t_i associates with playing Q_i^* instead of Q_i'' is

$$\mathbb{E}_i(Q_i^*, Q_{-i}''; DA^k) - \mathbb{E}_i(Q_i'', Q_{-i}''; DA^k) = u_i(k)\frac{q}{n} + \left(1 - \frac{kq}{n}\right) u_i(t_i) - u_i(k+1)\left(1 - \frac{q}{n} * (k-1)\right),$$

(recall that $u_i(b)$ denotes the utility that t_i associates with school $R_i(b)$.).

Therefore, the necessary condition is obtained from

$$\begin{aligned}
\mathbb{E}_i(Q_i^*, Q_{-i}''; DA^k) \geq&\ \mathbb{E}_i(Q_i'', Q_{-i}''; DA^k) \\
u_i(k)\frac{q}{n} + \left(1 - \frac{kq}{n}\right) u_i(t_i) \geq&\ u_i(k+1)\left(1 - \frac{q}{n} * (k-1)\right), \\
(u_i(k) - u_i(t_i))\frac{q}{n} \geq&\ (u_i(k+1) - u_i(t_i))\left(1 - \frac{q}{n} * (k-1)\right).
\end{aligned}$$

Because we normalize $u_i(t_i) = 0$, this yields

$$\begin{aligned}
u_i(k)\frac{q}{n} \geq&\ u_i(k+1)\left(1 - \frac{q}{n} * (k-1)\right), \\
u_i(k) \geq&\ u_i(k+1)\left(\frac{n}{q} - (k-1)\right),
\end{aligned}$$

the desired result.

Proof of Proposition 5. The following lemma shows that student t_i has an expected dominant strategy in BOS^k only when the strategy obtained by truncating preferences R_i after $R_i(k)$ is expected dominant. Remember we assume that t_i has at least $k+1$ acceptable schools.

Lemma 4. *For any $k \in \{1, \ldots, m\}$, in the absence of pre-existing priorities and given an ex-ante fair tie-breaking rule g, a student t_i has an expected dominant strategy in BOS^k only if the strategy $Q_i^* : \ R_i(1)\ R_i(2)\ \ldots\ R_i(k)\ t_i$ is an expected dominant strategy.*

Proof. The proof uses the same argument as the proof of Lemma 3 and is therefore omitted. ■

Take any $k \in \{1, \ldots, m-1\}$ and any ex-ante fair tie-breaking rule g. By Lemma 4, student t_i has an expected dominant strategy in BOS^k only if strategy $Q_i^* : \ R_i(1)\ R_i(2)\ \ldots\ R_i(k)\ t_i$ is an expected dominant strategy. The necessary condition stated in Proposition 5 is obtained by considering the particular strategy $Q_i'' : \ R_i(2)\ t_i$ and a particular profile $Q_{-i}'' \in \mathcal{Q}_{-i}^k$ and checking whether

$$\mathbb{E}_i(Q_i^*, Q_{-i}''; BOS^k) \geq \mathbb{E}_i(Q_i'', Q_{-i}''; BOS^k).$$

Let the profile Q_{-i}'' be such that $Q_j'' = Q_i^*$ for any $t_j \neq t_i$.

Under strategy profile (Q_i^*, Q_{-i}''), the situation is ex-ante symmetric: all students report the same preferences, there are no pre-existing priorities and the tie-breaking rule is ex-ante fair. As a result, all students have the same probability of being assigned to any of the school they report. Hence, for any school $s \in Q_i^*$, the probability that t_i be assigned to s is

$$Proba(t_i \text{ assigned to } s | (Q_i^*, Q_{-i}'')) = \frac{q}{n}.$$

Under strategy profile (Q_i'', Q_{-i}''), student t_i is the only one to rank $R_i(2)$ as her favorite school and therefore $BOS_i^k(Q_i'', Q_{-i}'') = R_i(2)$.

When the profile is Q_{-i}'', a necessary condition for Q_i^* to be an expected dominant strategy is

$$\mathbb{E}_i(Q_i^*, Q_{-i}''; BOS^k) \geq \mathbb{E}_i(Q_i'', Q_{-i}''; BOS^k),$$
$$\sum_{\ell=1}^{k} u_i(\ell)\frac{q}{n} + \left(1 - \frac{kq}{n}\right) u_i(t_i) \geq u_i(2),$$

and because we normalize $u_i(t_i) = 0$, this is

$$\sum_{\ell=1}^{k} u_i(\ell)\frac{q}{n} \geq u_i(2),$$

and given that $(k-2)u_i(3) \geq \sum_{\ell=3}^{k} u_i(\ell)$, the following is also a (weaker) necessary condition

$$u_i(1)\frac{q}{n} + u_i(2)\frac{q}{n} + \left(1 - \frac{2q}{n}\right) u_i(3) \geq u_i(2),$$
$$u_i(1)\frac{q}{n} + \left(1 - \frac{2q}{n}\right) u_i(3) \geq \left(1 - \frac{q}{n}\right) u_i(2),$$

and given that $\left(1-\frac{q}{n}\right) \geq \left(1-\frac{2q}{n}\right)$, the following is also a (weaker) necessary condition

$$u_i(1)\frac{q}{n} + \left(1-\frac{q}{n}\right) u_i(3) \geq \left(1-\frac{q}{n}\right) u_i(2),$$
$$u_i(1)\frac{q}{n-q} \geq u_i(2) - u_i(3),$$

the desired result.

Proof of Proposition 6. We first present Lemmas 5 and 6 that provide new expressions for $\mathbb{E}^{dom}_{DA^k}(F)$. Observe that by (4)

$$\mathbb{E}^{dom}_{DA^k}(F) = \frac{1}{\#\bar{\mathcal{R}}_1} \sum_{R_i \in \bar{\mathcal{R}}_i} \sum_{t_i \in T} \mathbb{1}(R_i, F).$$

Recall that $\#\bar{\mathcal{R}}_1$ is the total number of preference profiles and $\mathbb{1}(R_i, F)$ is an indicator function equal to 1 when t_i has a dominant strategy in DA^k with preferences R_i and priorities F, and zero otherwise.

For any priority profile F, any $k \in \{1, \ldots, m\}$ and any $t_i \in T$, let $SS^k_i(F)$ be the number of sets of k schools that are safe for t_i given F.

Lemma 5 ($\mathbb{E}^{dom}_{DA^k}$ depends on the number of safe sets)**.** *For any priority profile F and any $k \in \{1, \ldots, m\}$,*

$$\frac{1}{\#\bar{\mathcal{R}}_1} \sum_{R_i \in \bar{\mathcal{R}}_i} \sum_{t_i \in T} \mathbb{1}(R_i, F) = c \sum_{t_i \in T} SS^k_i(F),$$

for some positive constant c.

Proof. For any priority profile F, any $k \in \{1, \ldots, m\}$ and any $t_i \in T$, let $\mathbb{1}^k(R_i, F)$ equal 1 if t_i has a safe set covering her k most-preferred schools given F *and* R_i. Because we assumed that all students find at least $k+1$ schools acceptable, we have by Proposition 2 that $\mathbb{1}(R_i, F) = \mathbb{1}^k(R_i, F)$. Slightly abusing the notation, for any set of k schools S^k, let $\#S^k_i$ be the number of preferences in $\bar{\mathcal{R}}_i$ for which t_i's k most-preferred schools are the schools in S^k. As $\bar{\mathcal{R}}_i$ contains all linear orders on S with at least $k+1$ acceptable schools, $\#S^k_i$ is identical for every S^k. Therefore, we can let $\tau := \#S^k_i$ and have $S^k_i = \tau$ for all $t_i \in T$, where τ is a positive constant. Also, let $\mathbb{1}_i(S^k, F)$ equal 1 if S^k is a safe set for t_i given F and zero otherwise. We then have

$$\begin{aligned}
\frac{1}{\#\bar{\mathcal{R}}_1} \sum_{R_i \in \bar{\mathcal{R}}_i} \sum_{t_i \in T} \mathbb{1}(R_i, F) &= \frac{1}{\#\bar{\mathcal{R}}_1} \sum_{R_i \in \bar{\mathcal{R}}_i} \sum_{t_i \in T} \mathbb{1}^k(R_i, F) \\
&= \frac{1}{\#\bar{\mathcal{R}}_1} \sum_{t_i \in T} \sum_{R_i \in \bar{\mathcal{R}}_i} \mathbb{1}^k(R_i, F) \\
&= \frac{1}{\#\bar{\mathcal{R}}_1} \sum_{t_i \in T} \sum_{\{S^k \subseteq S \mid \#S^k = k\}} \mathbb{1}_i(S^k, F)\#S^k_i \\
&= \frac{1}{\#\bar{\mathcal{R}}_1} \sum_{t_i \in T} \tau \sum_{\{S^k \subseteq S \mid \#S^k = k\}} \mathbb{1}_i(S^k, F) \\
&= \frac{\tau}{\#\bar{\mathcal{R}}_1} \sum_{t_i \in T} SS^k_i(F),
\end{aligned}$$

where $c := \frac{\tau}{\#\bar{\mathcal{R}}_1}$ is a positive constant. ■

For any $k \in \{1, \ldots, m\}$, any F, and any $t_i \in T$, a set of k schools S^k is a **safe set of order 1** if S^k contains a subset $\tilde{S} \subseteq S^k$ that is is safe for t_i in DA^1. A set S^k is a **safe set of order 2** if S^k does *not* contain a subset $\tilde{S} \subseteq S^k$ that is is safe for t_i in DA^1, but S^k contains a subset $\tilde{S} \subseteq S^k$ that is safe for t_i in DA^2. Similarly, S^k is **a safe set of order $\boldsymbol{\ell \leq k}$** if S^k does *not* contain a subset $\tilde{S} \subseteq S^k$ that is is safe for t_i in $DA^{\ell-h}$ for any $h \in \{1, \ldots, \ell-1\}$, but S^k contains a subset $\tilde{S} \subseteq S^k$ that is safe for t_i in DA^ℓ.

We let $\#S_i^k(F, \ell)$ denote the number of sets of k schools that are safe sets of order ℓ for student t_i in DA^k given that the priority profile is F.

Lemma 6 (Safe sets can be counted order by order)**.**

$$\sum_{t_i \in T} SS_i^k(F) = \sum_{\ell=1}^{k} \sum_{t_i \in T} \#S_i^k(F, \ell).$$

Proof. Simply observe that (i) every safe set of size k is a safe set of order ℓ for some $\ell \leq k$ and (ii) by definition, no safe set can be of two different orders. ■

Next, for $k = 2$, $q = 1$ and for an arbitrary F, we present Lemma 7 that computes the number of safe sets of order 1 generated by F. Lemma 7 relies the n dimensional vector $x(F)$ whose components represent the number of schools at which each student has a top-priority given F. For example, if $T = \{t_1, \ldots, t_6\}$ and there are 4 schools with $F_1 : t_1 \ldots$, $F_2 : t_1 \ldots$, $F_3 : t_2 \ldots$, $F_4 : t_6 \ldots$, then $x(F) = (2, 1, 0, 0, 0, 1)$. The value of $x(F)$ for student $t_i \in T$ is denoted by $x_i(F)$. Observe that when $q = 1$ we have for every F that

$$\sum_{t_i \in T} x_i(F) = m \tag{23}$$

Vector $x(F)$ turns out to summarize all the information on F that is necessary for our purpose.

Lemma 7 (Computing $\#S_i^k(F, 1)$ based on $x(F)$ when $k = 2$)**.** *For $q = 1$,*

$$\sum_{t_i \in T} \#S_i^2(F, 1) = \sum_{t_i \in T} \frac{(m-1) + (m - x_i(F))}{2} x_i(F).$$

Proof. Without loss of generality, suppose that t_i has a top-priority at schools 1 to $x_i(F)$. Then every pair of schools $\{s_x, s_y\}$ with $s_x \in \{1, \ldots, x_i(F)\}$ is a safe sets of order 1 for t_i (and not other set is). These pairs are

$$\begin{array}{llll}
(s_1, s_2), & \ldots, & (s_1, s_m) & \rightarrow (m-1) \text{ pairs,} \\
(s_2, s_3), & \ldots, & (s_2, s_m) & \rightarrow (m-2) \text{ pairs,} \\
\vdots & & & \\
(s_{x_i(F)}, s_{x_i(F)+1}), & \ldots, & (s_{x_i(F)}, s_m) & \rightarrow (m - (x_i(F)) \text{ pairs.}
\end{array}$$

In total, there are $\frac{(m-1)+(m-x_i(F))}{2} x_i(F)$ such pairs. The desired result. ■

Next, for $k = 2$, $q = 1$ and for an arbitrary F, we present Lemma 8 that provides an upper-bound on the number of safe sets of order 2 generated by F.

Lemma 8 (Bounding $\#S_i^k(F, 2)$ based on $x(F)$ when $k = 2$)**.** *For $q = 1$,*

$$\sum_{t_i \in T} \#S_i^2(F, 2) \leq \sum_{t_i \in T} \frac{(x_i(F) - 1)x_i(F)}{2}.$$

Proof. Given the distribution of top-priorities, the maximum value of $\sum_{t_i \in T} \#S_i^2(F, 2)$ when $q = 1$ is obtained when a student t^* who does not have any top-priority at any school has the second highest priority at each and every school. In this case, for any student $t_j \neq t^*$, any pair (s_x, s_y) such that t_j has a top-priority at s_x *and* s_y is a safe set for t^*.

Again, assume without loss of generality that t_j has a top-priority at schools 1 to $x_j(F)$. These pairs are

$$\begin{array}{llll}
(s_1, s_2), & \ldots, & (s_1, s_{x_j(F)}) & \to x_j(F) - 1 \text{ pairs}, \\
(s_2, s_3), & \ldots, & (s_2, s_{x_j(F)}) & \to x_j(F) - 2 \text{ pairs}, \\
\vdots & & & \\
(s_{x_j(F)-2}, s_{x_j(F)-1}), & (s_{x_j(F)-2}, s_{x_j(F)}) & & \to 2 \text{ pairs}, \\
(s_{x_j(F)-1}, s_{x_j(F)}) & & & \to 1 \text{ pair}.
\end{array}$$

In total, there are $\frac{(x_j(F)-1)+1}{2}(x_j(F)-1) = \frac{(x_j(F)-1)x_j(F)}{2}$ such pairs for student t_j. Because t^* does not have top-priorities, all these safe sets are of order 2. The total number of safe sets for t^* is the sum over all students t_j of $\frac{(x_j(F)-1)x_j(F)}{2}$. But because the number of safe sets for t^* is the maximum of $\sum_{t_i \in T} \#S_i^2(F, 2)$, we have

$$\sum_{t_i \in T} \#S_i^2(F, 2) \leq \sum_{t_i \in T} \frac{(x_i(F) - 1)x_i(F)}{2}.$$

■

Put together, Lemmas 7 and 8 yield Lemma 9, which computes an upper-bound on the number of safe sets of size 2 created by F when $k = 2$ and $q = 1$.

Lemma 9 (Bounding $\sum_{\ell=1}^{2} \sum_{t_i \in T} \#S_i^k(F, \ell)$ based on $x(F)$ for $k = 2$)**.** *For $q = 1$,*

$$\sum_{\ell=1}^{2} \sum_{t_i \in T} \#S_i^2(F, \ell) \leq m(m-1)$$

Proof. By Lemmas 7 and 8

$$
\begin{aligned}
&\sum_{\ell=1}^{2}\sum_{t_i\in T}\#S_i^2(F,\ell)\\
\leq&\sum_{t_i\in T}\frac{(m-1)+(m-x_i(F))}{2}x_i(F)+\sum_{t_i\in T}\frac{(x_i(F)-1)x_i(F)}{2}\\
=&\sum_{t_i\in T}\frac{x_i(F)(m-1)+x_i(F)m-x_i(F)^2}{2}+\sum_{t_i\in T}\frac{x_i(F)^2-x_i(F)}{2}\\
=&\frac{(m-1)}{2}\sum_{t_i\in T}x_i(F)+\frac{m}{2}\sum_{t_i\in T}x_i(F)-\frac{1}{2}\sum_{t_i\in T}x_i(F)\\
=&\frac{(m-1)}{2}\sum_{t_i\in T}x_i(F)+\frac{(m-1)}{2}\sum_{t_i\in T}x_i(F)\\
=&(m-1)\sum_{t_i\in T}x_i(F)=(m-1)m,
\end{aligned}
$$

where the last equality follows from $\sum_{t_i\in T}x_i(F)=m$. ■

Proof of Proposition 6. Take any $F^B\in\mathcal{F}$ and any $F^{STB}\in supp(STB)$. We prove the proposition by considering in turn the cases $k=1$ and $k=2$.

- **Case 1**: $k=1$.
  When $k=1$, the number of safe sets of size 1 for t_i is simply equal to $x_i(F^B)$. Hence, the total number of safe sets generated is equal to $\sum_{t_i\in T}x_i(F^B)=m$. This value does not depend on F. Therefore, we have $\mathbb{E}^{dom}_{DA^k}(F^{STB})=\mathbb{E}^{dom}_{DA^k}(F^B)$.

- **Case 2**: $k=2$.
  The proof follows from the above lemmas. By Lemma 5, $\mathbb{E}^{dom}_{DA^k}(F^{STB})\geq\mathbb{E}^{dom}_{DA^k}(F^B)$ if and only if

  $$\sum_{t_i\in T}SS_i^2(F^{STB})\geq\sum_{t_i\in T}SS_i^2(F^B).$$

  By Lemma (6), this is equivalent to

  $$\sum_{\ell=1}^{2}\sum_{t_i\in T}\#S_i^2(F^{STB},\ell)\geq\sum_{\ell=1}^{2}\sum_{t_i\in T}\#S_i^2(F^B,\ell).\tag{24}$$

  By Lemma 8, we have that

  $$(m-1)m\geq\sum_{\ell=1}^{2}\sum_{t_i\in T}\#S_i^2(F^B,\ell).$$

  When F^{STB} is used, all sets of two schools are safe for the student who has a top-priority at all schools. These are all safe sets of order 1, and there are $\binom{m}{2}=\frac{(m-1)m}{2}$ of them. Similarly, all sets of two schools are safe for the student who has a top-priority at all schools. These are safe sets of order 2 and there are again $\binom{m}{2} = \frac{(m-1)m}{2}$ of them. Together, we have

$$\sum_{\ell=1}^{2} \sum_{t_i \in T} \#S_i^2(F^{STB}, \ell) = (m-1)m.$$

Therefore, inequality (24) holds, which completes the proof.

■

Proof of Proposition 7. Take any $k \in \{2, \dots, m\}$. There are two cases to consider: either sets of k schools are oversupplied, or sets of k schools are in short-supply.

- **Case 1:** Oversupply: $kq \geq n$.

  If $kq \geq n$, then all sets of k schools are oversupplied. By Proposition 1, all sets of k schools are therefore safe sets for any students. Hence, the k most-preferred schools of any student form a safe set. But then, by Proposition 2, all students have a dominant strategy in DA^k. Hence, we have $\mathbb{E}^{dom}_{DA^k}(F) = n$ for any $F \in \mathcal{F}$, which corresponds to case (ii.a) in Proposition 7.

- **Case 2:** Short-supply: $kq < n$.

  For a particular $\tilde{F}^{STB} \in supp(STB)$, we construct $\tilde{F}^{MTB} \in supp(MTB)$ such that $\mathbb{E}^{dom}_{DA^k}(\tilde{F}^{STB}) > \mathbb{E}^{dom}_{DA^k}(\tilde{F}^{MTB})$. The argument used for Case 2 in the proof of Corollary 4 shows that the expected number of dominant strategies is the same for any $F^{STB} \in supp(STB)$. As a result, if $\mathbb{E}^{dom}_{DA^k}(\tilde{F}^{STB}) > \mathbb{E}^{dom}_{DA^k}(\tilde{F}^{MTB})$, then $\mathbb{E}^{dom}_{DA^k}(F^{STB}) > \mathbb{E}^{dom}_{DA^k}(\tilde{F}^{MTB})$ for any $F^{STB} \in supp(STB)$.

  Let the priority profile $\tilde{F}^{STB} \in supp(STB)$ be such that the priority rank of t_1 is 1 at all schools, the priority rank of t_2 is 2 at all schools, and so on. This priority profile is as follows

$\tilde{F}_1^{STB}$	$\tilde{F}_2^{STB}$	$\dots$
t_1	t_1	$\dots$
t_2	t_2	$\dots$
$\vdots$	$\vdots$	
t_{kq}	t_{kq}	$\dots$
$\vdots$	$\vdots$	

  Given that $kq < n$, the priority rank of t_{kq} in $\tilde{F}_1^{STB}$ is lower than the priority rank of t_n in $\tilde{F}_1^{STB}$.

  Profile $\tilde{F}^{MTB}$ is constructed from $\tilde{F}^{STB}$ by swapping the priority ranks of students t_{kq} and student t_n at s_1. As a result, in $\tilde{F}_1^{MTB}$, the priority rank of t_n is kq and the priority rank of t_{kq} is n.

$\tilde{F}_1^{MTB}$	$\tilde{F}_2^{MTB}$	$\ldots$
t_1	t_1	$\ldots$
t_2	t_2	$\ldots$
$\vdots$	$\vdots$	
t_n	t_{kq}	$\ldots$
$\vdots$	$\vdots$	

We assumed that all students find at least $k+1$ schools acceptable. By Proposition 2, only students for whom the k most-preferred schools form a safe set have a dominant strategy in DA^k.

We show that all sets of k schools containing school s_1 are safe for t_{kq} in profile $\tilde{F}^{STB}$, whereas no set of k schools containing school s_1 is safe for t_{kq} in profile $\tilde{F}^{MTB}$. Let $\hat{S}_1^k$ be any set of k schools containing school s_1. For profile $\tilde{F}^{STB}$, we have that

$$\#\{t_j \in T \mid t_j \neq t_{kq} \text{ and } t_j \; \tilde{F}_s^{STB} \; t_{kq} \text{ for some } s \in \hat{S}_1^k\} = kq - 1.$$

Given that $\sum_{s \in \hat{S}_1^k} q_s = kq$, Proposition 1 implies that set $\hat{S}_1^k$ is safe for t_{kq} in profile $\tilde{F}^{STB}$. For profile $\tilde{F}^{MTB}$, we have that

$$\#\{t_j \in T \mid t_j \neq t_{kq} \text{ and } t_j \; \tilde{F}_s^{MTB} \; t_{kq} \text{ for some } s \in \hat{S}_1^k\} = kq.$$

By Proposition 1, set $\hat{S}_1^k$ is not safe for t_{kq} in profile $\tilde{F}^{MTB}$. This shows that t_{kq} has less safe sets of size k in $\tilde{F}^{MTB}$ than in $\tilde{F}^{STB}$. Also, no student has strictly more safe sets of size k in $\tilde{F}^{MTB}$ than in $\tilde{F}^{STB}$. In particular, if no set of k schools has enough seats to host all students, then t_n does not have any safe set of size k in $\tilde{F}^{MTB}$. Therefore, the number of safe sets of size k is strictly smaller in $\tilde{F}^{MTB}$ than in $\tilde{F}^{STB}$. By Lemma 5, this implies that $\mathbb{E}^{dom}_{DA^k}(\tilde{F}^{STB}) > \mathbb{E}^{dom}_{DA^k}(\tilde{F}^{MTB})$, the desired result.

Proof of Proposition 8. By Corollary 5, we have $\mathbb{E}^{dom}_{BOS^\ell}(g^{BOS}) = q$ for any $\ell \in \{1, \ldots, m\}$ and any distribution g^{BOS} on $\mathcal{F}$. We must show that we have $\mathbb{E}^{dom}_{DA^k}(g^{DA}) \geq q$ for any $k \in \{1, \ldots, m\}$ and any distribution g^{DA} on $\mathcal{F}$. The proof for the latter claim follows the same argument as the proof of Corollary 5 and is therefore omitted. We only observe that the inequality in $\mathbb{E}^{dom}_{DA^k}(g^{DA}) \geq q$ is strict for $k \geq 2$. The reason is that the necessary condition for having a dominant strategy in BOS^ℓ is sufficient but not necessary in DA^k. We assumed that all students have more than k acceptable schools. Therefore, by Proposition 3, only students who have top-priority at their most-preferred school have a dominant strategy in BOS^ℓ. Yet, by Proposition 2, students who have top-priority at any of their k most-preferred schools have a dominant strategy in DA^k.

Proof of Proposition 9. Take any $F^* \in \mathcal{F}$ and let the tie-breaking rule $\bar{g}^{F^*}$ be defined as the uniform distribution over the set of all permutations of F^*. Let the support of the tie-breaking rule $\bar{g}^{F^*}$ be denoted as $\bar{\mathcal{F}}^*$. We show that the rule $\bar{g}^{F^*}$ is ex-ante fair, i.e., for any $t_i \in T$, any $s \in S$, and any priority rank $p \in \{1, \ldots, n\}$ we have

$$Proba\left(F_s(t_i) = p \mid \bar{g}^{F^*}\right) = \sum_{\{F \in \bar{\mathcal{F}}^* \ \mid \ F_s(t_i)=p\}} \bar{g}^{F^*}(F) = \frac{1}{n}.$$

Take any $s \in S$ and any priority rank $p \in \{1, \ldots, n\}$. Given that rank p at school s is assigned to exactly one student in all $F \in \mathcal{F}$, we must have

$$\sum_{t_i \in T} \sum_{\{F \in \bar{\mathcal{F}}^* \ \mid \ F_s(t_i)=p\}} \bar{g}^{F^*}(F) = 1. \tag{25}$$

Given that $\bar{g}^{F^*}$ is a uniform distribution over $\bar{\mathcal{F}}^*$, we have for any $F \in \bar{\mathcal{F}}^*$ that

$$\bar{g}^{F^*}(F) = \frac{1}{\#\bar{\mathcal{F}}^*},$$

which implies that

$$\sum_{\{F \in \bar{\mathcal{F}}^* \ \mid \ F_s(t_i)=p\}} \bar{g}^{F^*}(F) = \frac{\#\{F \in \bar{\mathcal{F}}^* \mid F_s(t_i) = p\}}{\#\bar{\mathcal{F}}^*}. \tag{26}$$

By construction of $\bar{\mathcal{F}}^*$, we have that for any $t_j \neq t_i$,

$$\#\{F \in \bar{\mathcal{F}}^* \mid F_s(t_j) = p\} = \#\{F \in \bar{\mathcal{F}}^* \mid F_s(t_i) = p\}.$$

Therefore, we have that

$$\sum_{t_i \in T} \sum_{\{F \in \bar{\mathcal{F}}^* \ \mid \ F_s(t_i)=p\}} \bar{g}^{F^*}(F) = n * \frac{\#\{F \in \bar{\mathcal{F}}^* \mid F_s(t_i) = p\}}{\#\bar{\mathcal{F}}^*},$$

and (25) implies that

$$\frac{\#\{F \in \bar{\mathcal{F}}^* \mid F_s(t_i) = p\}}{\#\bar{\mathcal{F}}^*} = \frac{1}{n}.$$

The last equation together with (26), yields the desired result.

Proof of Proposition 10. (i) Take any $k \in \{1, \ldots, m\}$. By Corollary 4, we have that $\mathbb{E}^{dom}_{DA^k}(g^{STB}) = \min\{kq, n\}$. There remains to show that $\mathbb{E}^{dom}_{DA^k}(\bar{g}^{F^{FTB}}) = \min\{kq, n\}$. We consider two cases: either sets of k schools are oversupplied, or not.

- **Case 1:** Over-supply: $kq \geq n$.

  By the argument used in Case 1 of the proof of Corollary 4, we have $\mathbb{E}^{dom}_{DA^k}(g^{FTB}) = n$.

- **Case 2:** Short-supply: $kq < n$.

  We show that, unlike students whose priority ranks are never smaller or equal to kq, all students whose priority rank is smaller or equal to kq at some school have a dominant strategy in DA^k under F^{FTB}.

Take any t_i with priority rank smaller or equal to kq at some school. Denote by S^k the set of her k most-preferred schools. By construction of the pattern F^{FTB}, there are exactly $kq - 1$ students other than t_i who have at some school a priority rank no larger than kq. Therefore, we have that

$$\#\{t' \in T \backslash \{t\} \mid t' \; F_s \; t \text{ for some } s \in S^k\} < kq = \sum_{s \in S^k} q_s.$$

By Proposition 1, the set S^k is safe for t_i. Then, by Proposition 2, t_i has a dominant strategy in DA^k.

Take any t_i whose priority rank is never smaller or equal to kq at any school. Denote by S^k the set of her k most-preferred schools. This case is such that there are at least kq students other than t_i who receive at all schools a priority rank no larger than kq. These kq students have a higher priority than t_i at every school. Therefore, we have that

$$\#\{t' \in T \backslash \{t\} \mid t' \; F_s \; t \text{ for some } s \in S^k\} \geq kq = \sum_{s \in S^k} q_s.$$

By Proposition 1, the set S^k is not safe for t_i. We assumed that all students have more than k acceptable schools. Therefore, by Proposition 2, only students for whom the k most-preferred schools form a safe set have a dominant strategy in DA^k. Thus, t_i does not have a dominant strategy in DA^k.

Given that, under F^{FTB}, there are exactly kq students who receive at some school a priority rank no larger than kq, we have that $\mathbb{E}^{dom}_{DA^k}(g^{FTB}) = kq$. Given that $kq < n$, we have $\mathbb{E}^{dom}_{DA^k}(g^{STB}) = \min\{kq, n\}$, the desired result.

(ii) The proof is based on Lemma 10, which characterizes the ex-post fairness partial ordering of priority profiles.

Lemma 10 (Second-order dominance)**.** *Take any two F and $F' \in \mathcal{F}$. Let $\hat{r}^F$ and $\hat{r}^{F'}$ denote the vectors obtained when sorting in increasing order the vectors of total priority ranks r^F and $\mathrm{r}^{F'}$ associated to F and F'. F is* ***at least as ex-post fair as*** *F' if and only if*

$$\sum_{i=1}^{b} \hat{r}_i^F \geq \sum_{i=1}^{b} \hat{r}_i^{F'} \qquad \text{for any } b \in \{1, \dots, n\}. \tag{27}$$

Moreover, F is ***more ex-post fair than*** *F' if the inequality is strict for some b^*.*

Proof. This lemma is a corollary of Lemma 1 in Shorrocks (1983), a well-known result in the literature on income inequality measurement. Shorrocks (1983) compares $n-$dimensional income vectors y and y' that are sorted in such a way that $y_1 \leq \cdots \leq y_n$ and $y'_1 \leq \cdots \leq y'_n$. Letting μ denote the mean value in y, the Lorenz curve associated to vector y is defined as

$$L\left(y, \frac{\ell}{n}\right) = \sum_{i=1}^{\ell} \frac{y_i}{n\mu} \qquad \text{with } \ell = 1, \dots, n.$$

In a nutshell, Lemma 1 in Shorrocks (1983) shows that if $\mu = \mu'$, then y can be obtained from y' by a sequence of mean-preserving progressive transfers if and only if

$$L(y,p) \geq L(y',p) \qquad \text{for any } p \in [0,1].$$

In our setting, the precondition for Lemma 1 is fulfilled given that, by definition, vectors $\hat{r}^F$ and $\hat{r}^{F'}$ have the same mean

$$\frac{1}{n}\sum_i \hat{r}_i^F = \frac{m(n+1)}{2} = \frac{1}{n}\sum_i \hat{r}_i^{F'}.$$

It then follows that

$$L\left(\hat{r}^F, p\right) \geq L\left(\hat{r}^{F'}, p\right) \text{ for any } p \in [0,1] \qquad \Leftrightarrow \qquad \sum_{i=1}^{b} \hat{r}_i^F \geq \sum_{i=1}^{b} \hat{r}_i^{F'} \text{ for any } b \in \{1,\ldots,n\},$$

the desired result. ■

Take any $k \in \{1,\ldots,m\}$ such that $\lfloor kq/m \rfloor \geq 1$ and any profile F^{STB} in the support of g^{STB}.

- **Step 1:** Pattern F^{FTB} constructed from F^{STB} is more ex-post fair than F^{STB}.
  By the construction of F^{STB}, we have for any $b \in \{1,\ldots,n\}$ that

  $$\sum_{i=1}^{b} \hat{r}_i^{F^{STB}} = m + 2m + \cdots + bm = m\frac{b(b+1)}{2}.$$

  We divide the proof of Step 1 into two sub-steps

  - **Step 1.1**: For any $F \in \mathcal{F}$ we have $\sum_{i=1}^{b} \hat{r}_i^F \geq m\frac{b(b+1)}{2}$ for any $b \in \{1,\ldots,n\}$.
    Take any $b \in \{1,\ldots,n\}$. Clearly, the value of $\sum_{i=1}^{b} \hat{r}_i^F$ is minimized when a set of b students receive at all schools the priority ranks 1 to b. Take any $F' \in \mathcal{F}$ meeting this condition. Then, the students in the set are the b students with the smallest total priority ranks in F'. In that case, we have

    $$\sum_{i=1}^{b} \hat{r}_i^{F'} = \sum_{j=1}^{m}\sum_{\ell=1}^{b} \ell = \sum_{j=1}^{m} \frac{b(b+1)}{2} = m\frac{b(b+1)}{2}.$$

  - **Step 1.2**: We have $\sum_{i=1}^{b} \hat{r}_i^{F^{FTB}} > m\frac{b(b+1)}{2}$ for some $b \in \{1,\ldots,n\}$.
    Take $b^* = 1$. As $\lfloor kq/m \rfloor \geq 1$ and given that $m \geq 2$, no student has priority rank 1 at all schools in F^{FTB}. This implies that $\hat{r}_1^{F^{FTB}} > m\frac{(1+1)}{2} = m$.

  Step 1.1 and 1.2 together show that

  $$\sum_{i=1}^{b} \hat{r}_i^{F^{FTB}} \geq \sum_{i=1}^{b} \hat{r}_i^{F^{STB}} \qquad \text{for any } b \in \{1,\ldots,n\}.$$

  and that the inequality is strict for some $b^* = 1$. By Lemma 10, F^{FTB} is more ex-post fair than F^{STB}.

- **Step 2:** All priority profiles in the support of g^{FTB} are equally ex-post fair.

  Any two profiles F^{FTB} and F'^{FTB} in the support of g^{FTB} are permutations of each other. Therefore, their associated vectors of total priority rank $r^{F^{FTB}}$ and $r^{F'^{FTB}}$ are also permutations of each other. As a result we have that $\hat{r}^{F^{FTB}} = \hat{r}^{F'^{FTB}}$. By Lemma 10, any two profiles sharing the same sorted vector of total priority ranks are equally ex-post fair.

- **Step 3:** All priority profiles in the support of g^{STB} are equally ex-post fair.

  The argument is the same as in Step 2.

By steps 1 to 3, we have that any profile in the support of g^{FTB} is more ex-post fair than any profile in the support of g^{STB}, the desired result.

Proof of Proposition 11/ (i). Take any $k \in \{1, \ldots, m\}$. By Corollary 4, we have that $\mathbb{E}^{dom}_{DA^k}(g^{STB}) = \min\{kq, n\}$. There remains to show that $\mathbb{E}^{dom}_{DA^k}(\bar{g}^{F^{SPR}}) = \min\{kq, n\}$. There are two cases to consider: either sets of k schools are oversupplied, or not.

- **Case 1:** Oversupply: $kq \geq n$.

  This case happens when $k = m$ and $mq = n$, which is not ruled out by our assumptions ($mq \leq n$). By the argument used in Case 1 of the proof of Corollary 4, we have $\mathbb{E}^{dom}_{DA^k}(g^{SPR}) = n$.

- **Case 2:** Short-supply: $kq < n$.

  Take any F^{SPR} in the support of g^{SPR}. For each student t with a top-priority at some school s under F^{SPR}, t has a dominant strategy if and only if s is one of t's k most-preferred schools. For each such student, this occurs with probability k/m. If students have at most one top-priority in F^{SPR}, then mq students have a top-priority. In that case, the expected number of students with a dominant strategy is $mq * k/m = kq$, the desired result.

  There remains to show that all students have at most one top-priority in F^{SPR}. Consider any student t who has top-priority at a school, say s_1. We show that the rank of t is between $q+1$ and n for all other schools (s_2 to s_m). Student t has at s_1 a rank between 1 and q. Let $x = \lfloor n/m \rfloor$. By the construction of F^{SPR}, t has at s_2 a rank at least equal to $1 + x$. As we assume $mq \leq n$ we have that $x \geq q$ and therefore $1 + x$ is larger than q. At school s_m, t has a rank at most equal to $q + (m-1)x$. As $x = \lfloor n/m \rfloor$ we have that $q + (m-1)x \leq n$, the desired result.

(ii). The proof that $\bar{g}^{F^{SPR}}$ is more ex-post fair than g^{STB} when $\lfloor n/m \rfloor \geq 1$ is similar to the proof of part (ii) of Proposition 10 and is therefore omitted.

Using Lemma 10, we use three steps to prove the second part of Proposition 11(ii) (i.e., that $\bar{g}^{F^{SPR}}$ is more ex-post fair than $\bar{g}^{F^{FTB}}$ when $\lfloor n/m \rfloor \geq 1$ and $kq < n$). In Step 1, we show that inequality (27) is true for all $b \in \{1, \ldots, kq\}$. Based on Step 1, we show in Step 2 that (27) is also true for all $b \in \{kq+1, \ldots, n\}$. Together, Steps 1 and 2 show that F^{SPR} is at least as ex-post fair as F^{FTB}. Step 3 shows that if $kq < n$, the inequality in (27) is strict for $b^* = kq$. Therefore, F^{SPR} is more ex-post fair than F^{FTB}.

<u>**Step 1:**</u> for all $b \in \{1, \ldots, kq\}$, $\sum_{i=1}^{b} \hat{r}_i^{FTB} \leq \sum_{i=1}^{b} \hat{r}_i^{SPR}$.

The proof of Step 1 relies on the fact that the construction of F^{SPR} is identical to the construction of the first kq priorities in F^{FTB}. If $kq = n$, then F^{SPR} is identical to F^{FTB}. Thus, it is sufficient to prove

$$\sum_{i=1}^{b} \hat{r}_i^{FTB_{kq}} \leq \sum_{i=1}^{b} \hat{r}_i^{FTB_n}, \qquad \text{for all } b \in \{1, \ldots, kq\}, \tag{28}$$

where $\hat{r}_i^{FTB_{kq}}$ is the ordered vector of total ranks under $F^{FTB_{kq}} = F^{FTB}$, and $\hat{r}_i^{FTB_n}$ is the ordered vector of total ranks under F^{FTB_n}, the version of F^{FTB} where all n students have a high priority.

To do so, we show that for all kq,

$$\sum_{i=1}^{b} \hat{r}_i^{FTB_{kq}} \leq \sum_{i=1}^{b} \hat{r}_i^{FTB_{kq+1}}, \qquad \text{for all } b \in \{1, \ldots, kq\}, \tag{29}$$

where $\hat{r}_i^{FTB_{kq+1}}$ is the ordered vector of total ranks under $F^{FTB_{kq+1}}$, the version of F^{FTB} where $kq + 1$ students have a high priority (and the rest of the students have a low priority). The inequalities in (28) then follow from the inequalities in (29) by induction.

The inequalities in (29) are established by considering three cases. We first characterize the total rank of students in $F^{FTB_{kq}}$ as a function of their position in the priority profile. We then use this characterization to show that (29) is true (Case 1) when neither kq/m nor $(kq + 1)/m$ are integers, (Case 2) when kq/m is an integer but $(kq + 1)/m$ is not, and (Case 3) when kq/m is not integer but $(kq + 1)/m$ is. This completes the proof of Step 1, as these are all possible cases of (29).

Characterization: the total rank of a student in $F^{FTB_{kq}}$ as a function of her position in the priority profile.

Consider the case $m = 5$ and $kq = 14$. In this case, FTB^{kq} takes the following form

$$\begin{array}{ccccc} t_1 & t_{13} & t_{11} & t_9 & t_7 \\ t_2 & t_{14} & t_{12} & t_{10} & t_8 \\ t_3 & \vdots & \vdots & \vdots & \vdots \\ t_4 & & & & \\ t_5 & & & & \\ t_6 & & & & \\ \vdots & & & & \end{array} \tag{30}$$

Observe that all students appear once and only once either (a) among the 2 first ranks at one of the schools, or (b) among ranks 3 to 6 at school s_1.

For any k, q, and m, let $\tau^{kq} := \lfloor kq/m \rfloor$ and $\rho^{kq} := kq - m\tau$. Informally, τ^{kq} is the "translation factor" used in the construction of FTB^{kq}, whereas ρ^{kq} is the remainder of kq/m. In general, in FTB^{kq}, all students appear once and only once either (a') in the first τ^{kq} priority rank of some school, or (b') in the $\tau^{kq} + 1$ to ρ^{kq} first rank of the first school. The equivalent of (30) for a general FTB^{kq} pattern can be represented as follows.

$$\begin{array}{cccc} D_1 \subset T \text{ with } \#D_1 = \tau^{kq} & D_2 \subset T \text{ with } \#D_2 = \tau^{kq} & \dots & D_m \subset T \text{ with } \#D_m = \tau^{kq} \\ R \subset T \text{ with } \#R = \rho^{kq} & \vdots & \vdots & \vdots \\ \vdots & & & \end{array} \tag{31}$$

where for any two sets of students $\tilde{T}, \hat{T} \in \{D_1, \dots, D_m, R\}$, we have $\tilde{T} \cap \hat{T} = \emptyset$.

Observe that $kq = m(\tau^{kq} + 1) - x^{kq}$, where x^{kq} is the smallest integer such that $(kq + x^{kq})/m$ is an integer. For example, when $m = 5$ and $kq = 14$, we have $kq = 5 * (2 + 1) - 1$, and $x^{kq} = 1$. Using this formula for kq and replacing in the definition of ρ^{kq}, we have $\rho^{kq} = m(\tau^{kq} + 1) - x^{kq} - m\tau^{kq}$ which implies $\rho^{kq} = m - x^{kq}$. In particular, when $m = 5$ and $kq = 14$, we have $\rho^{kq} = (m - 1)$.

Next, we compute the total rank associated with the position of a student in (31). Consider again $m = 5$ and $kq = 14$, which implies $x^{kq} = 1$ and $\rho^{kq} = (m - 1)$. In this case,the total rank associated with the position of a student in (31) is

$$\begin{array}{ccccc} r & r+m-1 & r+2(m-1) & r+3(m-1) & r+4(m-1) \\ r+m & r+m-1+m & r+2(m-1)+m & r+3(m-1)+m & r+4(m-1)+m \\ r+2m & - & - & - & \\ r+3m & - & - & & \\ r+4m & - & & & \\ r+5m & & & & \end{array} \tag{32}$$

where $r = 25$.

Now, consider the case $m = 5$ and $kq = 13$, which implies $x = 2$ and $\rho^{kq} = (m - 2)$. In this case, the total rank associated with the position of a student in (31) becomes

$$\begin{array}{ccccc} r & r+(m-2) & r+2(m-2) & r+3(m-2) & r+4(m-2) \\ r+m & r+(m-2)+m & r+2(m-2)+m & r+3(m-2)+m & r+4(m-2)+m \\ r+2m & - & - & - & \\ r+3m & - & - & & \\ r+4m & - & & & \\ - & & & & \end{array} \tag{33}$$

In general, the total rank associated with the position of a student in (31) is

$$\begin{array}{ccccc} r & r+\rho^{kq} & r+2\rho^{kq} & \dots & r+(m-1)\rho^{kq} \\ r+m & r+\rho^{kq}+m & r+2\rho^{kq}+m & \dots & r+(m-1)\rho^{kq}+m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r+(\tau^{kq}-1)m & r+\rho^{kq}+(\tau^{kq}-1)m & r+2\rho^{kq}+(\tau^{kq}-1)m & \dots & r+(m-1)\rho^{kq}+(\tau^{kq}-1)m \\ r+\tau^{kq}m & & & & \\ \vdots & & & & \\ r+(\tau^{kq}+\rho^{kq}-1)m & & & & \end{array} \tag{34}$$

where $r = \hat{r}_1^{kq}$ where

$$\begin{aligned}\hat{r}_1^{kq} &= 1 + \sum_{i=1}^{m-1} \left(1 + \tau^{kq} i\right), \\ &= 1 + (m-1) + \tau^{kq} \sum_{i=1}^{m-1} i, \\ &= m + \tau^{kq} \frac{(m-1)m}{2}.\end{aligned}$$

<u>Case 1:</u> Neither kq/m nor $(kq+1)/m$ are integers.

The proof for this case and the next rely on the following lemma, which shows that it is sufficient to prove the existence of a particular increasing injection between the vector of total ranks under kq and the vector of total ranks under $(kq+1)$.

Lemma 11. *Consider any two lists of (not necessarily ordered) total ranks r^{kq} and r^{kq+1} containing respectively kq and $kq+1$ total ranks. If there exist an injection $\alpha : \{1, \dots, kq\} \to \{1, \dots, kq+1\}$ such that for all $i \in \{1, \dots, kq\}$,*

(a) $r_i^{kq} \leq r_{\alpha(i)}^{kq+1}$, and

(b) $r_i^{kq} < r_h^{kq+1}$, for $h \in \{1, \dots, kq+1\}$ such that $\alpha(i) \neq h$ for all $i \in \{1, \dots, kq\}$,

then for all $b \in \{1, \dots, kq\}$, $\sum_{i=1}^{b} \hat{r}_i^{kq} \leq \sum_{i=1}^{b} \hat{r}_i^{kq+1}$.

Proof. In order to derive a contradiction, suppose that for some $c \in \{1, \dots, kq\}$, we have $\hat{r}_c^{kq} > \hat{r}_c^{kq+1}$. This implies $\hat{r}_c^{kq} > \hat{r}_j^{kq+1}$ for all $j \in \{1, \dots, c\}$. That is, there exists at most $(kq+1) - c$ indices $I \subset \{1, \dots, kq+1\}$ such that $\hat{r}_i^{kq+1} > \hat{r}_c^{kq}$ for all $i \in I$ (i.e., $\#I \leq (kq+1) - c$). But by (a) and (b), there must exist at least $(kq+1) - c + 1$ such indices, a contradiction.[28] ■

We now show how such an injection can be constructed when neither kq/m nor $(kq+1)/m$ are integers. Consider the case $kq = 4$ and $m = 3$. In this case, the priorities of the kq and $kq+1$ first students in FTB^{kq} and FTB^{kq+1} are as follows

F_1^{kq}	F_2^{kq}	F_3^{kq}
$\boldsymbol{t_1}$	$\underline{t_4}$	$\underline{t_3}$
$\boldsymbol{t_2}$	$\boldsymbol{t_1}$	$\underline{t_4}$
$\underline{t_3}$	$\boldsymbol{t_2}$	$\boldsymbol{t_1}$
$\underline{t_4}$	$\underline{t_3}$	$\boldsymbol{t_2}$

F_1^{kq+1}	F_2^{kq+1}	F_3^{kq+1}
$\boldsymbol{t_1}$	$\underline{t_5}$	$\underline{t_4}$
$\boldsymbol{t_2}$	$\boldsymbol{t_1}$	$\underline{t_5}$
t_3	$\boldsymbol{t_2}$	$\boldsymbol{t_1}$
$\underline{t_4}$	t_3	$\boldsymbol{t_2}$
$\underline{t_5}$	$\underline{t_4}$	t_3

(35)

In this case, injection $\tilde{\alpha}$ with $\tilde{\alpha}(1) = 1$, $\tilde{\alpha}(2) = 2$, $\tilde{\alpha}(3) = 4$ and $\tilde{\alpha}(4) = 5$ satisfies conditions (a) and (b) in Lemma 11 (with $h = 3$).

As (35) illustrates, students receiving the first kq priorities in FTB^{kq} can in general be divided into two groups:

[28] $(kq+1) - c$ such indices are required to satisfy (a). Furthermore, there must exist one additional index $h \in \{1, \dots, kq+1\}$ such that $r_h^{kq+1} > r_c^{kq}$ and $\alpha(i) \neq h$ for all $i \in \{c, \dots, kq\}$ in order to satisfy (b).

(A) The "diagonal" students : students t_1 to student $t_{\tau^{kq}+\rho^{kq}}$ (students t_1 and t_2 in bold in (35)), and

(B) The "corner" students: students $t_{\tau^{kq}+\rho^{kq}+1}$ to t_{kq} (students t_3 and t_4 underlined in (35)).

Students receiving the first $kq+1$ priorities in FTB^{kq+1} can in general be divided into three groups:

(A') The "first kq diagonal" students : students t_1 to student $t_{\tau^{kq+1}+\rho^{kq+1}-1}$ (students t_1 and t_2 in bold in (35)),

(B') The "last diagonal" student: student $t_{\tau^{kq+1}+\rho^{kq+1}}$ (student t_3 in (35)),

(C') The "corner" students: students $t_{\tau^{kq+1}+\rho^{kq+1}+1}$ to t_{kq+1} (students t_4 and t_5 underlined in (35)).

When neither kq/m nor $(kq+1)/m$ are integers, $\tau^{kq} = \tau^{kq+1}$ and $\rho^{kq+1} = \rho^{kq} + 1$. Thus, groups (A) and (A') and groups (B) and (C') contain the same number of students. Therefore, it is possible to define an injection α^* that matches diagonal students in kq with the first kq diagonal students in $kq+1$ and corner students in kq with corner students in $kq+1$. Formally,

- $\alpha^*(i) = i$ for all $i \in \{1, \ldots, \tau^{kq} + \rho^{kq}\}$, and
- $\alpha^*(i) = i + 1$ for all $i \in \{\tau^{kq} + \rho^{kq} + 1, \ldots, kq\}$.

Clearly, α^* satisfies condition (a) in Lemma 11.[29] There remains to prove that α^* also satisfies condition (b), i.e., that

$$r^{kq+1}_{\tau^{kq+1}+\rho^{kq+1}} > r^{kq}_i \text{ for all } i \in \{1, \ldots, kq\}.$$

Because α^* satisfied (a) in Lemma 11, this is equivalent to proving

$$r^{kq+1}_{\tau^{kq+1}+\rho^{kq+1}} > r^{kq+1}_{\alpha^*(i)} \text{ for all } i \in \{1, \ldots, kq\},$$

which by definition of α^* is the same as

$$r^{kq+1}_{\tau^{kq+1}+\rho^{kq+1}} > r^{kq+1}_i \text{ for all } i \in \{1, \ldots, kq\} \backslash \{\tau^{kq+1} + \rho^{kq+1}\}. \tag{36}$$

Based on the first column of (34), we have

$$r^{kq+1}_{\tau^{kq+1}+\rho^{kq+1}} > r^{kq+1}_i \text{ for all } i \in \{1, \ldots, \tau^{kq+1} + \rho^{kq+1} - 1\}.$$

Also, if student s_{j^*} is the student with priority rank τ^{kq+1} at school m, we have

$$r^{kq+1}_{j^*} > r^{kq+1}_i \text{ for all } i \in \{\tau^{kq+1} + \rho^{kq+1} + 1, \ldots, kq + 1\}.$$

[29] A diagonal student in kq has the same total rank as her match in $kq+1$ under α^*. A corner student in kq has a lower total rank than her match in $kq+1$ under α^* (her match receives the same priorities on the "upper-right" corner, and receives a higher priority on the "lower-left" corner due to the addition of student $t_{\tau^{kq+1}+\rho^{kq+1}}$ to the diagonal in $kq+1$).

Hence, in order to prove (36), is it sufficient to prove that

$$r^{kq+1}_{\tau^{kq+1}+\rho^{kq+1}} > r^{kq+1}_{j^*}. \tag{37}$$

By (34), inequality (37) is equivalent to

$$\begin{aligned} r + (\tau^{kq+1} + \rho^{kq+1} - 1)m &> r + (m-1)\rho^{kq+1} + (\tau^{kq} - 1)m \\ \tau^{kq+1}m + \rho^{kq+1}m - m &> \rho^{kq+1}m - \rho^{kq+1} + \tau^{kq+1}m - m \\ 0 &> -\rho^{kq+1}, \end{aligned}$$

which is true because when neither kq/m nor $(kq+1)/m$ are integers, $\rho^{kq+1} > 0$.

Case 2: kq/m is an integer but $(kq+1)/m$ is not

The proof is similar to the proof of Case 1. Consider the case $kq = 6$ and $m = 3$. In this case, the priorities of the kq and $kq+1$ first students in FTB^{kq} and FTB^{kq+1} are as follows

F_1^{kq}	F_2^{kq}	F_3^{kq}
$\boldsymbol{t_1}$	$\underline{t_5}$	$\underline{t_3}$
$\boldsymbol{t_2}$	$\underline{t_6}$	$\underline{t_4}$
$\underline{t_3}$	$\boldsymbol{t_1}$	$\underline{t_5}$
$\underline{t_4}$	$\boldsymbol{t_2}$	$\underline{t_6}$
$\underline{t_5}$	$\underline{t_3}$	$\boldsymbol{t_1}$
$\underline{t_6}$	$\underline{t_4}$	$\boldsymbol{t_2}$

F_1^{kq+1}	F_2^{kq+1}	F_3^{kq+1}
$\boldsymbol{t_1}$	$\underline{t_6}$	$\underline{t_4}$
$\boldsymbol{t_2}$	$\underline{t_7}$	$\underline{t_5}$
t_3	$\boldsymbol{t_1}$	$\underline{t_6}$
$\underline{t_4}$	$\boldsymbol{t_2}$	$\underline{t_7}$
$\underline{t_5}$	t_3	$\boldsymbol{t_1}$
$\underline{t_6}$	$\underline{t_4}$	$\boldsymbol{t_2}$
$\underline{t_7}$	$\underline{t_5}$	t_3

(38)

In this case, injection $\tilde{\alpha}$ with $\tilde{\alpha}(1) = 1$, $\tilde{\alpha}(2) = 2$, $\tilde{\alpha}(3) = 4$, $\tilde{\alpha}(4) = 5$, $\alpha(5) = 6$ and $\alpha(6) = 7$ satisfies conditions (a) and (b) in Lemma 11 (with $h = 3$). In general, injection α^* defined in Case 1 also satisfies (a) and (b) in Lemma 11 when kq/m is an integer but $(kq+1)/m$ is not, the proof being identical to the proof in Case 1.

Case 3: kq/m is not integer but $(kq+1)/m$ is.

Let $\hat{r}_i^{kq+1}$ be the total rank of the i^{th}-student with lowest total rank under pattern FTB^{kq+1}. We need to show that,

$$\sum_{i=1}^{b} \hat{r}_i^{kq} \leq \sum_{i=1}^{b} \hat{r}_i^{kq+1}, \qquad \text{for all } b \in \{1, \ldots, kq\}. \tag{39}$$

Now, consider the $kq+1$-student pattern "$\overline{F}^{kq}$" constructed from the FTB (kq)-students pattern by adding student $kq+1$ at rank $kq+1$ at all schools. For example, if $kq = 5$ and $m = 3$, pattern $\overline{F}^{kq}$ is

$\overline{F}_1^{kq}$	$\overline{F}_2^{kq}$	$\overline{F}_3^{kq}$
t_1	t_5	t_3
t_2	t_1	t_4
t_3	t_2	t_5
t_4	t_3	t_1
t_5	t_4	t_2
$\boldsymbol{t_6}$	$\boldsymbol{t_6}$	$\boldsymbol{t_6}$

(40)

The proof of this case relies on a series of observations about FTB^{kq}, $\bar{F}^{kq}$ and FTB^{kq+1}.

Relationships between $\bar{F}^{kq}$, and FTB^{kq} and FTB^{kq+1}

Observation 1:

$$\sum_{i=1}^{b} \hat{r}_i^{kq} = \sum_{i=1}^{b} \hat{r}_i^{\overline{F}^{kq}}, \qquad \text{for all } b \in \{1, \dots, kq\}. \tag{41}$$

Proof. This follows directly from the construction of $\bar{F}^{kq}$ from FTB^{kq}. ■

Observation 2:

$$\sum_{i=1}^{kq+1} \hat{r}_i^{\overline{F}^{kq}} = \sum_{i=1}^{kq+1} \hat{r}_i^{kq+1}. \tag{42}$$

Proof. This follows from the fact that the $kq+1$ first students receive all $kq+1$ first ranks at all schools in both $\overline{F}^{kq}$ and in FTB^{kq+1}. ■

Properties of $\bar{F}^{kq}$

In the current case, $\rho^{kq} = m - 1$. Hence, the equivalent of (34) for $\bar{F}^{kq}$ is

$$\begin{array}{ccccc}
r & r+m-1 & r+2m-2 & \dots & r+(m-1)m-(m-1) \\
r+m & r+2m-1 & r+3m-2 & \dots & r+mm-(m-1) \\
\vdots & & & & \\
r+(\tau^{kq}-1)m & r+\tau^{kq}m-1 & r+(\tau^{kq}+1)m-2 & \dots & r+(\tau^{kq}+m-2)m-(m-1) \\
r+\tau^{kq}m & - & - & \dots & - \\
\vdots & & & & \\
r+(\tau^{kq}+m-3)m & - & & & \\
r+(\tau^{kq}+m-2)m & & & & \\
- & & & & \\
\vdots & & & & \\
- & & & & \\
\boldsymbol{(kq+1)m} & & & &
\end{array} \tag{43}$$

where, $(kq+1)m$, the rank of student $kq+1$, was simply added in position $(kq+1)$ at school s_1. In (43), $r := \hat{r}_1^{\overline{F}^{kq}} = m + \tau^{kq} \frac{(m-1)m}{2}$.

Observation 3:

$$\hat{r}_i^{\bar{F}^{kq}} < \hat{r}_{i+1}^{\bar{F}^{kq}}, \qquad \text{for all } i \in \{1, \dots, kq+1\}. \tag{44}$$

Proof. Observe that, once r is removed from every total rank, the list of total ranks in (43) can be re-written as

$$
\begin{aligned}
&0,\\
&m-1, m,\\
&2m-2, 2m-1, 2m,\\
&\vdots\\
&(m-1)m-(m-1), (m-1)m-(m-2), \ldots, (m-1)m\\
&mm-(m-1), mm-(m-2), \ldots, mm\\
&\vdots\\
&(\tau^{kq}+m-4)m-(m-1), (\tau^{kq}+m-4)m-(m-2), (\tau^{kq}+m-4)m-(m-3), (\tau^{kq}+m-4)m,\\
&(\tau^{kq}+m-3)m-(m-1), (\tau^{kq}+m-3)m-(m-2), (\tau^{kq}+m-3)m,\\
&(\tau^{kq}+m-2)m-(m-1), (\tau^{kq}+m-2)m,\\
&(kq+1)m
\end{aligned} \tag{45}
$$

In (45), the total ranks in each row are arranged in increasing order. Thus, the smallest and largest elements of each rank are

$$
\begin{aligned}
&0,\\
&m-1, m,\\
&2m-2, 2m,\\
&\vdots\\
&(m-1)m-(m-1), (m-1)m\\
&mm-(m-1), mm\\
&\vdots\\
&(\tau^{kq}+m-4)m-(m-1), (\tau^{kq}+m-4)m,\\
&(\tau^{kq}+m-3)m-(m-1), (\tau^{kq}+m-3)m,\\
&(\tau^{kq}+m-2)m-(m-1), (\tau^{kq}+m-2)m,\\
&(kq+1)m
\end{aligned} \tag{46}
$$

Observe that the difference between the smallest total rank in one row and the largest total rank in the preceding row is always at least $m-(m-1)=1$. Together with the fact that the total ranks in each rows are arranged in increasing order, this proves (44). ■

Observation 4:

$$\hat{r}_{i+m}^{\bar{F}^{kq}} \geq \hat{r}_i^{\bar{F}^{kq}} + m, \qquad \text{for all } i \in \{1, \ldots, kq+1-m\}. \tag{47}$$

Proof. By the argument in the proof of Observation 3, the total ranks in (45) are arranged in increasing order. Then (51) follows from the fact that no row in (45) contains more than m total ranks.[30] ■

[30] When $\tau^{kq} \leq m-2$, all rows in (45) contain at most $\tau^{kq}+1$ total ranks with $\tau^{kq}+1 < m$.

Properties of FTB^{kq+1}

Applying (34) to the case $\tau^{kq}+1=\frac{kq+1}{m}$, we obtain

$$\begin{array}{ccccc} r' & r' & r' & \dots & r' \\ r'+m & r'+m & r'+m & \dots & r'+m \\ \dots & & & & \\ r'+\tau^{kq}m & r'+\tau^{kq}m & r'+\tau^{kq}m & \dots & r'+\tau^{kq}m \end{array} \tag{48}$$

where above, $r' := \hat{r}_1^{kq+1}$, and

$$\begin{aligned} \hat{r}_1^{kq+1} &= 1+\sum_{i=1}^{m-1}\left(1+(1+\tau^{kq})i\right), \\ &= 1+(m-1)+(1+\tau^{kq})\sum_{i=1}^{m-1} i, \\ &= m+(1+\tau^{kq})\frac{(m-1)m}{2}. \end{aligned}$$

From (48), it can bee seen that

$$\hat{r}_b^{kq+1} = r' + ym, \tag{49}$$

where

$$r' = m + (\tau^{kq}+1)\frac{(m-1)m}{2}$$

and

$$y = \begin{cases} 0 & \text{if } b \leq m, \\ 1 & \text{if } b \in \{m+1,\dots,2m\}, \\ 2 & \text{if } b \in \{2m+1,\dots,3m\}, \\ \vdots & \\ \tau^{kq} & \text{if } b \in \{\tau^{kq}m+1,\dots,kq+1\}. \end{cases}$$

Observation 5:

$$\hat{r}_{xm+1}^{kq+1} = \dots = \hat{r}_{xm+a}^{kq+1} = \dots = \hat{r}_{xm+m}^{kq+1}, \qquad \text{for all } x \in \{1,\dots,\tau^{kq}\} \text{ and all } a \in \{1,\dots,m\}. \tag{50}$$

Proof. This follows directly from (49). ■

Observation 6:

$$\hat{r}_{i+m}^{kq+1} = \hat{r}_i^{kq+1} + m, \qquad \text{for all } i \in \{1,\dots,kq+1-m\}. \tag{51}$$

Proof. This follows directly from (49). ■

The proof now follows in two substeps. In Substep 1, we show that for all $x \in \{1, \ldots, \tau^{kq}\}$,

$$\sum_{i=1}^{xm} \hat{r}_i^{\overline{F}^{kq}} \leq \sum_{i=1}^{xm} \hat{r}_i^{kq+1}. \tag{52}$$

Then, in Substep 2, we show that for all $b \neq xm$,

$$\sum_{i=1}^{b} \hat{r}_i^{\overline{F}^{kq}} \leq \sum_{i=1}^{b} \hat{r}_i^{kq+1}. \tag{53}$$

Substep 1: proof of (52).

In order to derive a contradiction, assume that there is a $\tilde{x} \in \{1, \ldots, \tau^{kq}\}$ for which

$$\sum_{i=1}^{\tilde{x}m} \hat{r}_i^{\overline{F}^{kq}} > \sum_{i=1}^{\tilde{x}m} \hat{r}_i^{kq+1}. \tag{54}$$

For any $x \in \{1, \ldots, \tau^{kq}\}$, let $N_x^F := \{xm - (m-1), \ldots, xm\}$ be the set of the m students whose total ranks lies between the $xm - (m-1)$-th and the xm-th position in the ordered vector of total rank $\hat{r}^F$ for pattern F. Inequality (54) implies that for some $x' \leq \tilde{x}$ we have

$$\sum_{i \in N_{x'}^{\overline{F}^{kq}}} \hat{r}_i^{\overline{F}^{kq}} > \sum_{i \in N_{x'}^{kq+1}} \hat{r}_i^{kq+1}. \tag{55}$$

Notice that, by (51) (Observation 6) we have that

$$\sum_{i \in N_{x'+1}^{kq+1}} \hat{r}_i^{kq+1} = m^2 + \sum_{i \in N_{x'}^{kq+1}} \hat{r}_i^{kq+1}.$$

Also, by (47) (Observation 4), we have that

$$\sum_{i \in N_{x'+1}^{\overline{F}^{kq}}} \hat{r}_i^{\overline{F}^{kq}} \geq m^2 + \sum_{i \in N_{x'}^{\overline{F}^{kq}}} \hat{r}_i^{\overline{F}^{kq}}.$$

More generally, together with (47) and (51), (55) implies that

$$\sum_{i \in N_{x''}^{\overline{F}^{kq}}} \hat{r}_i^{\overline{F}^{kq}} > \sum_{i \in N_{x''}^{kq+1}} \hat{r}_i^{kq+1} \qquad \text{for all } x'' \geq x'. \tag{56}$$

Therefore, together with (56), (54) implies that

$$\sum_{i=1}^{kq+1} \hat{r}_i^{\overline{F}^{kq}} > \sum_{i=1}^{kq+1} \hat{r}_i^{kq+1},$$

which contradicts (42) (Observation 2).

Substep 2: proof of (53).

Take any $b' = xm + a$ with $a \in \{1, 2, \dots, m\}$ and with $x \in \{0, 1, \dots, \tau^{kq}\}$. We must show that

$$\sum_{i=1}^{b'} \hat{r}_i^{\overline{F}^{kq}} \leq \sum_{i=1}^{b'} \hat{r}_i^{kq+1},$$
$$\sum_{i=1}^{xm} \hat{r}_i^{\overline{F}^{kq}} + \left(\hat{r}_{xm+1}^{\overline{F}^{kq}} + \dots + \hat{r}_{xm+a}^{\overline{F}^{kq}}\right) \leq \sum_{i=1}^{xm} \hat{r}_i^{kq+1} + \left(\hat{r}_{xm+1}^{kq+1} + \dots + \hat{r}_{xm+a}^{kq+1}\right).$$

If it was the case that

$$\sum_{i=1}^{b'} \hat{r}_i^{\overline{F}^{kq}} > \sum_{i=1}^{b'} \hat{r}_i^{kq+1},$$

then by (52) we have that

$$\hat{r}_{xm+1}^{\overline{F}^{kq}} + \dots + \hat{r}_{xm+a}^{\overline{F}^{kq}} > \hat{r}_{xm+1}^{kq+1} + \dots + \hat{r}_{xm+a}^{kq+1}$$

By (50) (Observation 5) and (44) (Observation 3), last inequality implies that $\hat{r}_{xm+a}^{\overline{F}^{kq}} > \hat{r}_{xm+a}^{kq+1}$ and also

$$\hat{r}_{xm+a+1}^{\overline{F}^{kq}} + \dots + \hat{r}_{xm+m}^{\overline{F}^{kq}} > \hat{r}_{xm+a+1}^{kq+1} + \dots + \hat{r}_{xm+m}^{kq+1}$$

and therefore

$$\sum_{i=1}^{xm+m} \hat{r}_i^{\overline{F}^{kq}} > \sum_{i=1}^{xm+m} \hat{r}_i^{kq+1},$$

contradicting (52). Notice that this argument also applies to the particular case $b' = a$. For the particular case $b' = \tau^{kq} + a$, we have a contradiction with (42). The desired result follows then from (41).

Step 2: for all $b \in \{kq + 1, \dots, n\}$, $\sum_{i=1}^{b} \hat{r}_i^{FTB} \leq \sum_{i=1}^{b} \hat{r}_i^{SPR}$.

In order to derive a contradiction, assume that, for some $b^* \in \{kq + 1, \dots, n\}$, $\sum_{i=1}^{b^*} \hat{r}_i^{FTB} > \sum_{i=1}^{b^*} \hat{r}_i^{SPR}$. We know that

$$\sum_{i=1}^{n} \hat{r}_i^{FTB} = \sum_{i=1}^{n} \hat{r}_i^{SPR}. \tag{57}$$

Hence, proving

$$\sum_{i=b^*+1}^{n} \hat{r}_i^{FTB} \geq \sum_{i=b^*+1}^{n} \hat{r}_i^{SPR}. \tag{58}$$

would result in the desired contradiction.

Consider the patterns $F^{F\bar{T}B}$ and $F^{S\bar{T}B}$ constructed from F^{FTB} and F^{STB} by orthogonal symmetry around rank $n/2$. That is, the rank of t at school s in $F^{\bar{X}}$ equals $n+1$ minus the rank of t at school s in F^X. These constructions are illustrated below for the case $kq = 3$, where the orthogonal symmetry is taken with respect to the dashed line, and, the double plain lines in FTB and $F\bar{T}B$ represents the separation between the students with high and low priorities.

F_1^{FTB}	F_2^{FTB}	F_3^{FTB}
t_1	t_3	t_2
t_2	t_1	t_3
t_3	t_2	t_1
$\uparrow kq$		$\downarrow (n-kq)$
t_4	t_8	t_7
t_5	t_4	t_8
t_6	t_5	t_4
t_7	t_6	t_5
t_8	t_7	t_6

$F_1^{F\bar{T}B}$	$F_2^{F\bar{T}B}$	$F_3^{F\bar{T}B}$
t_8	t_7	t_6
t_7	t_6	t_5
t_6	t_5	t_4
t_5	t_4	t_8
t_4	t_8	t_7
$\uparrow (n-kq)$		$\downarrow kq$
t_3	t_2	t_1
t_2	t_1	t_3
t_1	t_3	t_2

(59)

F_1^{SPR}	F_2^{SPR}	F_3^{SPR}
t_1	t_7	t_5
t_2	t_8	t_6
t_3	t_1	t_7
t_4	t_2	t_8
t_5	t_3	t_1
t_6	t_4	t_2
t_7	t_5	t_3
t_8	t_6	t_4

$F_1^{S\bar{P}R}$	$F_2^{S\bar{P}R}$	$F_3^{S\bar{P}R}$
t_8	t_6	t_4
t_7	t_5	t_3
t_6	t_4	t_2
t_5	t_3	t_1
t_4	t_2	t_8
t_3	t_1	t_7
t_2	t_8	t_6
t_1	t_7	t_5

(60)

By definition of $\bar{X}$, we have $r_i^{\bar{X}} = m(n+1) - r_i^X$, which implies $\hat{r}_i^{\bar{X}} = m(n+1) - \hat{r}_{(n+1)-i}^X$. Thus, we have for any $z \in \{1, \ldots, n-1\}$ that

$$\sum_{i=1}^{z} \hat{r}_i^{F\bar{T}B} = zm(n+1) - \sum_{i=n-z}^{n} \hat{r}_i^{FTB} \quad \text{and} \quad \sum_{i=1}^{z} \hat{r}_i^{S\bar{P}R} = zm(n+1) - \sum_{i=n-z}^{n} \hat{r}_i^{SPR}$$

which implies that (58) is equivalent to

$$(n+1-b^*)m(n+1) - \sum_{i=1}^{(n+1)-b^*} \hat{r}_i^{F\bar{T}B} \geq (n+1-b^*)m(n+1) - \sum_{i=1}^{(n+1)-b^*} \hat{r}_i^{S\bar{P}R}$$
$$\sum_{i=1}^{(n+1)-b^*} \hat{r}_i^{F\bar{T}B} \leq \sum_{i=1}^{(n+1)-b^*} \hat{r}_i^{S\bar{P}R}. \tag{61}$$

But observe that because $b^* > kq$, we have $(n+1) - b^* \leq n - kq$. That is, in $F\bar{T}B$, the sum in (61) is over students with high priorities only. But then, (61) follows from Step 1, and we have the desired contradiction.

To see why, observe that, from the point of view of ex-post fairness, $F\bar{T}B$ is equivalent to the FTB pattern where exactly $(n-kq)$ students have high priority, and kq students have low priority. This can be seen by observing that the pattern $F\bar{\bar{T}}B$ obtained from $F\bar{T}B$ by yet another orthogonal symmetry, this time around the $m/2$'s column yields the same distribution of total ranks as $F\bar{T}B$, and is indeed of the form of an FTB pattern with $(n-kq)$ high priority students. This is illustrated below, where the orthogonal symmetry is with respect to the second column.

$F_1^{F\bar{T}B}$	$F_2^{F\bar{T}B}$	$F_3^{F\bar{T}B}$
t_8	t_7	t_6
t_7	t_6	t_5
t_6	t_5	t_4
t_5	t_4	t_8
t_4	t_8	t_7
$\uparrow (n-kq)$		$\downarrow kq$
t_3	t_2	t_1
t_2	t_1	t_3
t_1	t_3	t_2

$F_1^{F\bar{\bar{T}}B}$	$F_2^{F\bar{\bar{T}}B}$	$F_3^{F\bar{\bar{T}}B}$
t_6	t_7	t_8
t_5	t_6	t_7
t_4	t_5	t_6
t_8	t_4	t_5
t_7	t_8	t_4
$\uparrow (n-kq)$		$\downarrow kq$
t_1	t_2	t_3
t_3	t_1	t_2
t_2	t_3	t_1

(62)

<u>**Step 3:**</u> If $kq < n$ then $\sum_{i=1}^{kq} \hat{r}_i^{FTB} < \sum_{i=1}^{kq} \hat{r}_i^{SPR}$.

Given that in F^{FTB} the kq smallest priority ranks are attributed at all schools to the same kq students, we have for the case $b^* = kq$ that

$$\sum_{i=1}^{kq} \hat{r}_i^{FTB} = \sum_{i=1}^{kq} \hat{r}_i^{STB}.$$

There remains to show for F^{SPR} that, when $kq < n$, there is a student t that has a priority rank smaller than kq at some school and a priority rank larger than kq at another school. If this is the case, we have

$$\sum_{i=1}^{kq} \hat{r}_i^{SPR} > \sum_{i=1}^{kq} \hat{r}_i^{STB}.$$

There are two cases

- **Case 1**: $\lfloor n/m \rfloor < kq$.

  Consider student t_n who has priority rank n at school s_1. Given that $kq < n$, t_n has a priority rank larger than kq at s_1. The priority rank of t_n at school s_2 is equal to $n + \lfloor n/m \rfloor$ modulo n, which is equal to $\lfloor n/m \rfloor$. This case being such that $\lfloor n/m \rfloor < kq$, t_n has a priority rank smaller than kq at s_2, the desired result.

- **Case 2**: $\lfloor n/m \rfloor \geq kq$.

  Consider student t_1 who has the priority rank 1 at school s_1. Given that $q \geq 1$, t_1 has a priority rank smaller than kq at s_1. The priority rank of t_1 at school s_2 is equal to $1 + \lfloor n/m \rfloor$ modulo n. This case being such that $\lfloor n/m \rfloor \geq kq$, t_1 has a priority rank larger than kq at s_2 if $1 + \lfloor n/m \rfloor \leq n$. By assumption $m \geq 2$ and therefore $\lfloor n/m \rfloor < n$, implying that $1 + \lfloor n/m \rfloor \leq n$, the desired result.

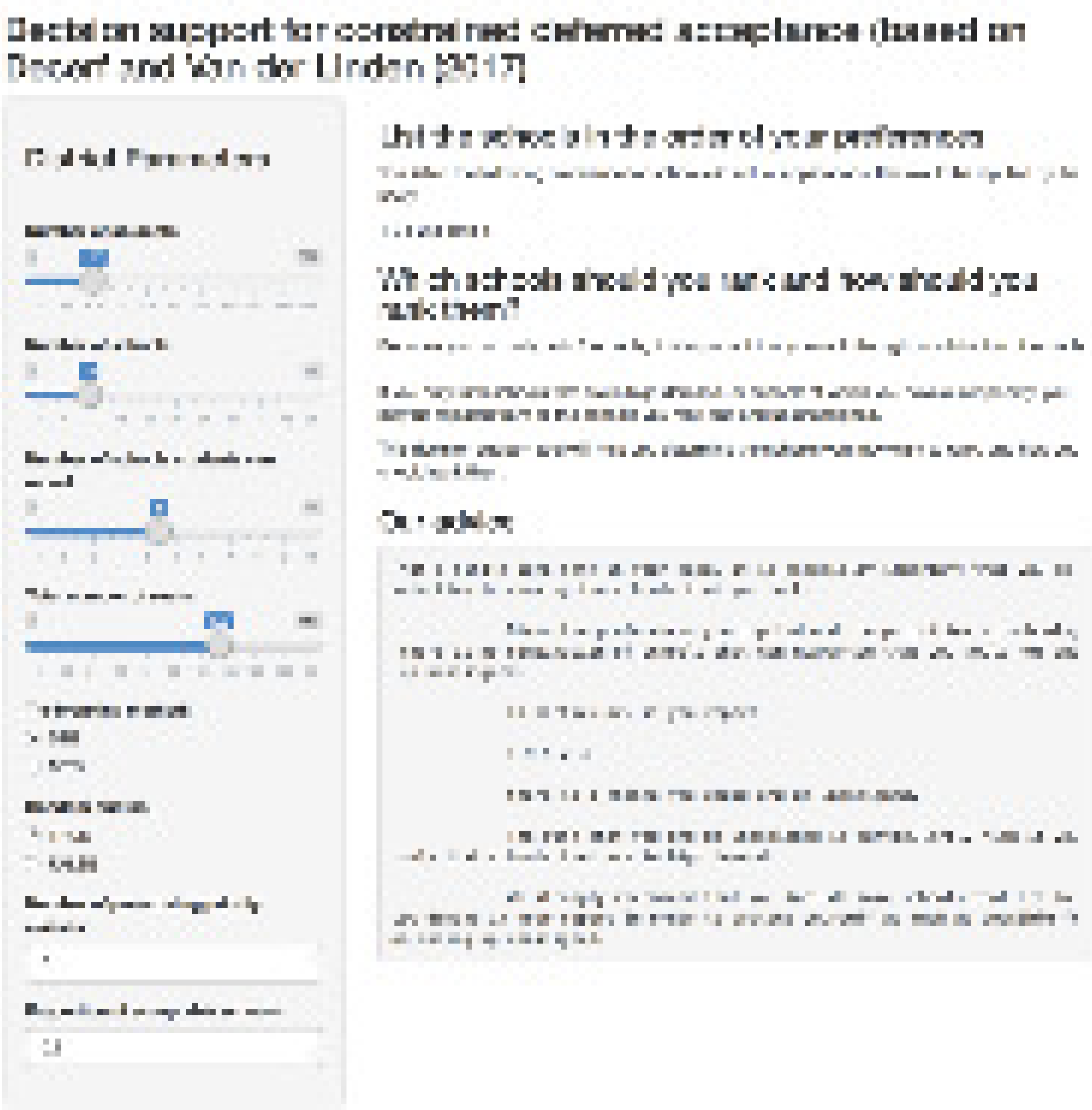

Figure 6: Screen-shot of the proof of concept for a decision support app.

B Decision support app

A screenshot of the app can be found in Figure 6. This is an example of the message received by a student who, given the preference she entered in the app, has a dominant strategy ($k = 5$).

List the schools in the order of your preferences

You listed the following preference for schools (in actual applications, this would be inputed by the user):

2 10 1 4 7 6 5 8

Which schools should you report and how should you rank them?

Because you can only report 5 schools, it is important that you report the right combination of schools.

If you only report schools that are in high demand, or schools at which you have a low priority, you may be rejected from all the schools you report and end up unassigned.

This decision support tool will help you determine the schools you may want to report, and how you should rank them.

Our advice

(Dominant Strategy) Given the preferences you inputed and the priorities at schools, you have a clear best strategy.

Listing your 5 most-preferred schools in the order of your preference will guarantee you a better assignment than with any other ranking you could submit.

We strongly recommend that you report the following preference list:

2 10 1 4 7

This is an example of the message received by a student who, given the preference she entered in the app, does not have a single safe set ($k = 5$).

List the schools in the order of your preferences

You listed the following preference for schools (in actual applications, this would be inputed by the user):

5 1 7 3 9 6 2 4

Which schools should you report and how should you rank them?

Because you can only report 5 schools, it is important that you report the right combination of schools.

If you only report schools that are in high demand, or schools at which you have a low priority, you may be rejected from all the schools you report and end up unassigned.

This decision support tool will help you determine the schools you may want to report, and how you should rank them.

Our advice

(Not a single safe set) In your case, it is especially important that you be selective in chosing the schools that you report.

Given the preferences you inputed and the priorities at schools, there is no combination of schools that can guarantee that you would not end up unassigned.

In particular, if you report

5 1 7 3 9

there is a chance you could end up unassigned.

The risk that you end up unassigned is particularly high if you only list schools that are in high demand.

We strongly recommend that you include some schools that are in low demand in your report in order to protect yourself as much as possible from ending up unassigned.

Finally, this is an example of the message received by a student who, given the preference she entered in the app, does not have a dominant strategy but does have safe sets ($k = 5$)

List the schools in the order of your preferences

You listed the following preference for schools (in actual applications, this would be inputed by the user):

7 8 2 4 3 1 5 9

Which schools should you report and how should you rank them?

Because you can only report 5 schools, it is important that you report the right combination of schools.

If you only report schools that are in high demand, or schools at which you have a low priority, you may be rejected from all the schools you report and end up unassigned.

This decision support tool will help you determine the schools you may want to report, and how you should rank them.

Our advice

(Safe sets, but no dominant strategy) Given the preferences you inputed and the priorities at schools, you do not have a clear best strategy.

In particular, if you report

7 8 2 4 3

there is a chance you could end up unassigned.

However, some of the list of schools you can report can protect your from ending up unassigned.

One of the best strategies for you is to report the following preference list:

7 2 3 1 5

This strategy will guarantee that you are at worst assigned to school 5.

No other strategy can guarantee you a better worst case assignment.

The following are other rankings that guarantee that you will be assigned and that you may be interested in reporting:

c(7, 8, 3, 1, 5), c(8, 2, 3, 1, 5)

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