Manipulability and tie-breaking in constrained school choice

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Abstract

In school choice problems, we provide an in-depth analysis of the manipulability of the constrained deferred acceptance (DA) and Boston (BOS) mechanisms. We characterize dominant strategies in both mechanisms and show that constrained DA is less manipulable than constrained BOS in the sense of Arribillaga and Massó (2015). We argue that, from a manipulability perspective, tie-breakers should be revealed before preferences are reported. When this is the case, we are able to compare the manipulability of DA for different tie-breaking rules. We show that single tie-breaking (STB) outperforms multiple tie-breaking in terms of manipulability. We also show that other tie-breaking rules share the desirable manipulability properties of STB while improving on STB’s ex-post fairness, an important concern for practitioners.

JEL Classification: C78, D47, D82, I20.  
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1 Introduction

In 2005, the Boston School Committee replaced its school choice mechanism known as the Boston mechanism (BOS) by a deferred acceptance mechanism (DA). An important motivation for this reform was that, when students can rank all the schools they could potentially attend, DA is non-manipulable whereas BOS is not (Abdulkadiroğlu et al., 2006; Pathak and Sönmez, 2008).

However, it is rare for mechanisms used in practice to let students rank all the schools in a district. Instead, school choice mechanisms are

1 Schools districts in Chicago and England abandoned mechanisms similar to BOS in 2007 and 2009 out of analogous manipulability concerns (Pathak and Sönmez, 2013).
typically constrained (Haeringer and Klijn, 2009), with students allowed to report preferences on a limited number of schools only. Under such constraints, DA is manipulable and it is unclear whether replacing BOS by DA actually reduces manipulability.

Constraints on the number of schools students can report are pervasive in practice (Pathak and Sönmez, 2013). Such constraints may be viewed as a way to keep mechanisms simple to interact with and operate. Recent results also suggest that constraints may have a positive effect on efficiency in the case of DA, either by alleviating stability requirements (Dur and Morrill, 2016), or by pushing students to reveal information about their cardinal preferences (Van der Linden, 2017).

In spite of their prevalence, little is known about the manipulability of the constrained versions of DA and BOS, henceforth denoted by $DA^k$ and $BOS^k$, where $k$ is the maximum number of schools students can rank. This paper contributes to filling this gap in the literature. We provide an in-depth study of the manipulability of $DA^k$ and $BOS^k$, focusing on dominant strategies. For $DA^k$ and $BOS^k$, we characterize preferences and priorities for which a student has a dominant strategy. Although both mechanisms sometimes fail to be strategy-proof, we show that the mechanisms can be ordered according to the “amount” of dominant strategies they provide in the sense of Arribillaga and Massó (2015). According to the same criterion, manipulability also decreases as $k$ increases in $DA^k$, but not in $BOS^k$. These results confirm parallel results from Pathak and Sönmez (2013) and provide further justifications for recent reforms where a number of districts were observed switching from $BOS^k$ to $DA^k$, or increasing the number of schools student can report in $DA^k$ (Pathak and Sönmez, 2013, Table 1).

Our manipulability comparisons are not the mere consequence of some students being able to report all their acceptable schools. Instead, our comparisons relate to correlations in students’ priorities at different schools. In $DA^k$, correlations in priorities provide students with what we call safe sets of school: sets of school which, if they are reported, protect a student from ending up unassigned. We show that students who have safe sets can have dominant strategies even when they are unable to report all their acceptable schools.

The above results assumed that priorities are strict and known to students before they report their preferences. When priorities are coarse, school districts must rely on tie-breakers before $DA^k$ or $BOS^k$ can be applied. When a tie-breaking rule is used, the dominant practice seems to consist in breaking ties after preferences are reported (Abdulkadiroglu et al., 2009; Calsamiglia et al., 2014), forcing students to report their preference with only partial information on priorities. We show that unless the profile of cardinal utilities is extreme, this practice can decrease the number of students with dominant strategies in constrained mechanisms.

Even when post tie-breaking priorities are revealed before students report their preferences, the selection of the tie-breaking rule remains an important decision for district officials. The two most common tie-breaking rules are the single tie-breaking rule (STB).
and the *multiple* tie-breaking rule (MTB), with STB breaking ties in the same way at all schools whereas MTB draws a different tie-breaker for each school. The literature suggests that $DA$ is more *efficient* when used with STB than with MTB. In contrast, we provide the first analysis of the *manipulability* of tie-breaking rules when used with $DA^k$. In a special case (one seat per school and $k \leq 2$), we show that priority profiles in the support of STB give dominant strategies to more students than profiles in the support of MTB. We also provide results from simulations suggesting that this result generalizes beyond the special case for which we have analytical results.

Our simulations reveal that the incentive advantage of STB over MTB can be sizable for some value of the parameters. However, this advantage is limited for other values of the parameters. This is troubling because ex-post, STB selects priority profiles in which the same tie-breaker is given to every student at every school, which officials and parents view as unfair (Pathak, 2011). A natural question is therefore whether the incentive properties of STB can be harnessed while avoiding its fairness costs.

To answer this question, we develop a new metric of the fairness of priority profiles. In a strict priority profile, students are given a priority rank at each school. The sum of a student’s priority ranks reflects how high a priority the student is given across the different schools (e.g., a student who occupies the first rank at each of the $m$ schools has a sum of ranks of $m$). We compare priority profiles based on the distribution across students of these sums of rank. We say that a profile $F$ is more fair than another profile $F'$ if the distribution induced by $F$ can be obtained from the distribution induced by $F'$ through a series of progressive transfers. A tie-breaking rule $g$ is more ex-post fair than another tie-breaking rule $g'$ if any profile in the support of $g$ is more fair than any profile in the support of $g'$. We show that it is possible to construct tie-breaking rules that significantly improve upon STB in terms of fairness while preserving STB’s incentive properties.

**Related Literature.** The closest paper to ours is Pathak and Sönmez (2013) who compare the manipulability of constrained school choice mechanism from the perspective of truthful Nash equilibria. In contrast, we focus on dominant strategies and rely on a comparison criterion introduced by Arribillaga and Massó (2015). As we explain on pg. 10, most comparison criteria have limitations and our analysis should be viewed as complementary to that of Pathak and Sönmez (2013), who use a criterion independent from that of Arribillaga and Massó (2015). Our paper also provides a number of newer insights that have no counterpart in Pathak and Sönmez (2013), including our study of the manipulability of tie-breaking rules in $DA^k$.6

Our paper contributes to the relatively thin literature on constrained school choice mechanisms. The constrained school choice problem was introduced by Haeringer and Klijn (2009), who studied the efficiency and stability properties of the Nash equilibria of constrained school choice mechanisms. Calsamiglia et al. (2010) studied constrained

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4 See Abdulkadiroglu et al. (2009) and De Haan et al. (2015) for simulations based on field data, and Ashlagi et al. (2015), Ashlagi and Nikzad (2015) and Arnosti (2015) for theoretical results in the large.

5 The parameters are the number of schools, number of students, number of seats per schools, and number of schools students can report.

6 We also show that, according to Arribillaga and Massó’s criterion, $DA^k$ is less manipulable than $BOS^\ell$ even when $\ell > k$, whereas Pathak and Sönmez (2013) only prove a parallel result using their criterion for $\ell = k$.  

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mechanisms experimentally replicating the design of Chen and Sönmez (2004) while adding a treatment where mechanisms are constrained. More recently, Dur and Morrill (2016) and Van der Linden (2017) have showed how constraints on the number of schools students can report can make $DA^k$ more efficient than the unconstrained $DA$.

In this paper, our focus is on the manipulability of constrained school choice mechanisms from the perspective of dominant strategies. In a companion paper (Decerf and Van der Linden, 2017), we study the strategies of students who do not have dominant strategies. We show that $DA^k$ outperforms $BOS^k$ in terms of the ability of students to eliminate a large set of strategies using weak dominance, and in terms of students’ maximin assignments.

This paper also contributes to the growing literature comparing the manipulability of pairs of mechanisms that both fail to be strategy proof. Recent work in this area using manipulability criteria from Pathak and Sönmez (2013) and Arribillaga and Massó (2015) include Chen et al. (2016), Van der Linden (2016), Harless (2017), Chen and Kesten (2017) and Turhan (2017).\footnote{Other manipulability comparison criteria have been suggested by Aleskerov and Kurbanov (1999), Parkes et al. (2002), Maus et al. (2007), Fujinaka and Wakayama (2015), Mennle and Seuken (2014), Barberà and Gerber (2017) and Andersson et al. (2014), among others.}

Whereas we argue that revealing information on priorities may have a positive impact on manipulability, Li (2017) finds that when priorities depend on exam scores, asking students to report their preferences before scores are known may have a positive effect on ex-ante utility.

To our knowledge, our paper is the first to formally analyze the ex-post fairness of tie-breaking rules, and to propose a criterion for ex-post fairness comparisons.

2 The school choice model and constrained school choice mechanisms

The model is similar to Haeringer and Klijn (2009). There is a finite set of schools $S := \{s_1, \ldots, s_m\}$ with $m \geq 2$, and a finite set of students $T := \{t_1, \ldots, t_n\}$. As this assumption is always satisfied in practice and simplifies some of our results, we impose that $n \geq m$.

A typical school is denoted by $s_j$, or sometimes $s$. Every school $s_j \in S$ has a capacity $q_j$ and a priority profile $F_j$. Capacity $q_j$ represents the number of seats available at school $s_j$. A set of schools $\bar{S} \subseteq S$ is in oversupply if together, the schools in $\bar{S}$ can accept all the students, i.e., $\sum_{s_j \in \bar{S}} q_j \geq n$. A set of schools is in short-supply otherwise. Throughout this paper, we assume that no pair of schools is in oversupply.\footnote{Chen (2014) shows that if every pair of schools is in oversupply, $BOS^m$ is strategy-proof. In this case, $DA^k$ is also strategy-proof for any $k \in \{2, \ldots, m\}$ (see Proposition 2). Even if every pair of schools is in oversupply, it is not hard to show that $BOS^k$ may still be manipulable when $k < m$.} Priorities $F_j$ are linear orderings of the students in $T$. A profile of priorities $F := (F_1, \ldots, F_m)$ is a list containing the priorities of every $s_j \in S$ and the domain of all priority profiles is $\mathcal{F}$.

A typical student is denoted by $t_i$, or sometimes $t$. Every student $t_i \in T$ has a preference $R_i$. Preference $R_i$ is a linear ordering on $S \cup \{t_i\}$. The domain of all preferences for $t_i$ is $\mathcal{R}_i$. A preference profile $R := (R_1, \ldots, R_n)$ is a list containing the preference of every
$t_i \in T$. For a given preference profile $R$, the list containing the preferences of everyone but $t_i$ is $R_{-i}$.

A strict preference of $t_i$ for school $s$ over school $s'$ is denoted by $s \ P_i s'$, while $s \ R_i t_i$ denotes a weak preference, allowing for $s = s'$. A school $s \in S$ is acceptable for $t_i$ if $s \ R_i t_i$. To avoid trivialities, we assume that every student has at least one acceptable school. For simplicity, we abuse the notation and write $s \in R_i$ when $s$ is acceptable given $R_i$, and $\#R_i$ for the number of acceptable schools in $R_i$. By the same token, $S' \subseteq R_i$ indicates that all schools in $S'$ are acceptable for $t_i$ given $R_i$.

An assignment is a function $\mu : T \rightarrow S \cup T$ that matches every student with a school or with herself ($\mu(t) \in S \cup \{t\}$ for any $t \in T$). If $\mu(t) = t$, we say that $t$ is unassigned in $\mu$. An assignment is feasible if no school exceeds its capacity, i.e., for any $s_j \in S$, we have $\#\{t \in T \mid \mu(t) = s_j\} \leq q_j$, where for any set $A$, $\#A$ denotes the cardinality of $A$.

A (school choice) mechanism $M$ associates every profile of reported preferences $Q := (Q_1, \ldots, Q_n)$ in some domain $Q := \times_{i \in T} Q_i$ with a feasible assignment $\mu$.\footnote{As is common in school choice, we assume that schools are non-strategic players and that the priority profile is known to the social decision maker.} The notation and terminology for preferences extend to reported preferences: (a) $s \ Q_i s'$ means that $t_i$ reports $s$ weakly before $s'$ in $Q_i$ (where possibly $s = s'$), (b) school $s \in S$ is reported by $t_i$ in $Q_i$ if $s \ Q_i t_i$, (c) $s \in Q_i$ indicates that $s$ is reported in $Q_i$, (d) $\#Q_i$ is the number of reported schools in $Q_i$, (e) $S \subseteq Q_i$ indicates that all schools in $S$ are reported in $Q_i$, (f) a typical profile of reported preferences is $Q := (Q_1, \ldots, Q_n)$, and (g) given reported profile $Q$, the list of reported preferences of every student but $t_i$ is $Q_{-i}$.

For a domain of reported profiles $Q$ and a student $t_i \in T$, the set of possible subprofiles for all $t_j \in T \setminus \{t_i\}$ is $Q_{-i}$ (i.e., $Q_{-i} := \{Q_{-i} \in \times_{t_j \in T \setminus \{t_i\}} Q_j \mid (Q_i, Q_{-i}) \in Q$ for some $Q_i \in Q_i\}$). In a constrained mechanism $M^k$, the domain is $Q^k := \times_{t \in T} Q^k_t$, where for every $t_i \in T$, $Q^k_i$ is the set of all reported preferences in which $t_i$ reports no more than $k \leq m$ schools.

For any reported profile $Q$ and any student $t_i$, the school $t_i$ is assigned to in $M(Q)$ is $M_i(Q)$. Student $t_i$ is assigned in $M$ given $Q$ if $M_i(Q) \neq t_i$ and unassigned if $M_i(Q) = t_i$.

A pair $(M, R)$ defines a strategic form game known as a game of school choice (Ergin and Sönmez, 2006). As a consequence, we sometimes refer to a reported preference $Q_i$ as a strategy. Given mechanism $M$, $Q_i$ is a dominant strategy if

$$M_i(Q_i, Q_{-i}) \ R_i M_i(Q'_i, Q_{-i}),$$

for any $Q_{-i} \in Q_{-i}$ and any $Q'_i \in Q_i$.

The two classes of mechanisms we focus on correspond to constrained versions of BOS and DA identified by Haeringer and Klijn (2009). We first describe the well-known unconstrained BOS.

**Round 1:** Students apply to the school they reported as their most-preferred acceptable school (if any). Every school that receives more applications than its capacity starts rejecting the lowest applicants in its priority ranking, up to the point where it meets its capacity. All other applicants are definitively accepted at the schools they applied to and capacities are adjusted accordingly.
Round \( \ell \) : Students who are not yet assigned apply to the school they reported as their \( \ell \)th acceptable school (if any). Every school that receives more new applications in round \( \ell \) than its remaining capacity starts rejecting the lowest new applicants in its priority ranking, up to the point where it meets its capacity. All other applicants are definitively accepted at the schools they applied to and capacities are adjusted accordingly.

The algorithm terminates when all reported schools have been considered, or when every student is assigned to a school. The constrained versions of BOS which we denote by \( BOS^k \) are identical to BOS except that no student is allowed to report more than \( k \) schools.

We now turn to DA. Again, we first describe the famous unconstrained version of DA.

Round 1: Students apply to the school they reported as their most-preferred acceptable school (if any). Every school that receives more applications than its capacity definitively rejects the lowest applicants in its priority ranking, up to the point where it meets its capacity. All other applicants are temporarily accepted at the schools they applied to (this means they could still be rejected in a later round).

Round \( \ell \) : Students who were rejected in round \( \ell - 1 \) apply to their next acceptable school (if any). Every school considers the new applicants of round \( \ell \) together with the students it temporarily accepted. If needed, each school definitely rejects the lowest students in its priority ranking, up to the point where it meets its capacity. All other applicants are temporarily accepted at the schools they applied to (this means they could still be rejected in a later round).

The algorithm terminates when all reported schools have been considered, or when every student is assigned to a school. The constrained versions of DA which we denote by \( DA^k \) are identical to DA except that no student is allowed to report more than \( k \) schools.

3 Safe sets and safe strategies

When students cannot report all available schools, they face the risk of “running out” of reported schools and being unassigned. Experimental evidence shows that students understand this risk. When students cannot report all available schools, students are more likely to report a school they dislike but at which they have a high priority in order to protect themselves from being unassigned (Calsamiglia et al., 2010).

In \( DA^k \), if student \( t \) reports a school \( s_j \) where \( t \) is among the \( q_j \) students with highest priority, \( t \) cannot be assigned to a school that she reported lower than \( s_j \). In particular, \( t \) cannot be unassigned. We call such a school a top-priority school for \( t \). Formally, \( s_j \) is a top-priority school for \( t \) if no more than \( q_j - 1 \) students have a higher priority at \( s_j \) than \( t \).
Interestingly, reporting a top-priority school is not the only way for a student to protect herself from being unassigned in $DA^k$. Students with no top-priority school can often guarantee they will be assigned by reporting an appropriate set of schools. Consider the following profile, where the left panel represents students’ preferences and the right panel represents schools’ priorities. Each school has one seat and “...” indicates that the rest of the ordering is arbitrary.

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(1)

Student $t_4$ does not have a top-priority school. However, only students $t_1$ and $t_2$ have a higher priority than $t_4$ at schools $s_1$, $s_2$ and $s_3$. As a consequence, if $t_4$ reports $s_1$, $s_2$ and $s_3$, she is guaranteed to be assigned to one of these schools. Given (1), any strategy $Q_i^t \in Q_i^k$ with $\{s_1, s_2, s_3\} \subseteq Q_i^t$ is what we call a safe strategy for $t_i$ in $DA^k$. In general, given mechanism $M$, $Q_i$ is a safe strategy for $t_i$ if playing $Q_i$ protects $t_i$ from being unassigned, i.e., $M_i(Q_i, Q_{-i}) \neq t_i$ for any $Q_{-i} \in Q_{-i}$. Given mechanism $M$ and priority profile $F$, a set of schools $S^* \subseteq S$ is a safe set for $t_i$ if any of $t_i$’s strategies in which $S^*$ is reported is safe, i.e., for any $Q_i \in Q_i$, $S^* \subseteq Q_i$ implies that $Q_i$ is safe. For example, $\{s_1, s_2, s_3\}$ is a safe set for $t_4$ in $DA^k$ when the priority profile is (1). In general, any set of schools containing a top-priority school is also a safe set in $DA^k$.

As we show below, safe sets are tightly related to dominant strategies in $DA^k$. In particular, if a student has a safe set that covers her most-preferred schools, she has a dominant strategy in $DA^k$ consisting in reporting the schools in this safe set in the order of her preference. The same is not true in $BOS^k$: even in the presence of a safe set $S^*$, there is no optimal way for a student to report the schools in $S^*$ when the mechanism is $BOS^k$ (different ordering of the schools in $S^*$ are best-responses to different reports from other students).

Safe sets are also more rare in $BOS^k$ than in $DA^k$. Even if $s$ is a top-priority school for $t$, student $t$ must report $s$ first in $BOS^k$ to guarantee herself an assignment at $s$. Thus, unlike in $DA^k$, sets of schools containing a top-priority school are not necessarily safe in $BOS^k$. In general, the existence of safe sets in $BOS^k$ requires that a group of no more than $k$ schools be in oversupply, which is relatively rare in practice. The next proposition characterizes safe sets in $DA^k$ and $BOS^k$ (all the proofs can be found in the Appendix).

**Proposition 1.** (i) For any $k \in \{1, \ldots, m\}$, the set of schools $\hat{S} \subseteq S$ with $\#\hat{S} \leq k$ is safe for student $t$ in $DA^k$ if and only if there exists a subset $\hat{S} \subseteq \hat{S}$ such that no more than $\sum_{s \in \hat{S}} q_s$ students have a higher priority at some $s \in \hat{S}$ than $t$, i.e.,

$$\#\{t' \in T \setminus \{t\} \mid t' \ F_s \ t \text{ for some } s \in \hat{S}\} < \sum_{s \in \hat{S}} q_s.$$  

(2)
(ii) For any \( k \in \{2, \ldots, m\} \), the set of schools \( \tilde{S} \subseteq S \) with \( \# \tilde{S} = k \) is safe for student \( t \) in \( BOS^k \) if and only if \( \tilde{S} \) is in oversupply or \( t \) has top-priority at all schools in \( \tilde{S} \).\(^{10}\)

4 Dominant strategies in \( DA^k \) and \( BOS^k \)

One way a student can have a dominant strategy in \( DA^k \) is if she is able to report all her acceptable school. But there are more subtle ways for a student to have a dominant strategy in \( DA^k \). For example, a student who reports her preference truthfully up to one of her top-priority schools is playing a dominant strategy in \( DA^k \). More generally, student \( t \) has a dominant strategy if her \( k \) most-preferred schools form a safe set. When she truthfully reports the safe set made of her \( k \) most-preferred schools, \( t \) does not face the risk of running out of reported schools and being unassigned. Also, \( t \) does not have to "skip" any of her \( k \) most-preferred schools in her report. In this case, it is as if the constraint was not binding for \( t \), and her dominant strategies in \( DA^k \) are essentially the same as in \( DA^m \). (Recall that \( m \) is the total number of schools and \( DA^m \) is therefore the unconstrained version of \( DA \).

The next proposition shows that having less than \( k \) acceptable schools or having a safe set covering one’s \( k \) most-preferred schools are the only cases in which a student has a dominant strategy in \( DA^k \).\(^{11}\)

**Proposition 2** (Dominant strategies in \( DA^k \)). For any \( k \in \{1, \ldots, m\} \), a student has a dominant strategy if and only if (i) she has no more than \( k \) acceptable schools or (ii) her \( k \) most-preferred schools are all acceptable and form a safe set.

As we show in Section 5.2, priority profiles and preferences for which the \( k \) most-preferred schools of a student form a safe set are not uncommon. As a consequence, many students can have dominant strategy in \( DA^k \) even when no student is able to report all her acceptable schools.\(^{12}\)

Dominant strategies are much more rare in \( BOS^k \). The reason for this scarcity of dominant strategies are the same as in \( BOS^m \): In \( BOS^k \) as in \( BOS^m \), there often exist reports of the other students for which a student would benefit from misreporting her first school, and other reports for which she would benefit from reporting her first school truthfully. This is true even if the student has less than \( k \) acceptable schools (provided she has at least two acceptable schools).

**Proposition 3** (Dominant strategies in \( BOS^k \)). For any \( k \in \{1, \ldots, m\} \), a student has a dominant strategy in \( BOS^k \) if and only if (i) she has only one acceptable school or (ii) she has a top-priority at her most-preferred school.

\(^{10}\) If \( \# \tilde{S} < k \), \( \tilde{S} \) can also be safe in \( BOS^k \) if all schools are in oversupply. For \( k = 1 \), \( BOS^1 \) is strategically equivalent to \( DA^1 \) and the characterization in (i) applies.\(^{11}\)

\(^{11}\) Every other strategy \( Q_i \in Q_k \) is either (a) unsafe and does not report all of \( t_i \)'s acceptable schools, or (b) safe but fails to report at least one school \( s^* \) that \( t_i \) prefers to a school she could be assigned to when reporting \( Q_i \). In case (a), \( t_i \) could end up unassigned whereas she would have been assigned to one of her unreported acceptable schools had she reported that school. In case (b), \( t_i \) could be assigned to a school she likes less than \( s^* \) whereas she would have been assigned to \( s^* \) had reported \( s^* \).

\(^{12}\) In the 2003-2004 NYC match, Abdulkadiroğlu et al. (2005) report that about 22,000 of the almost 100,000 students reported the maximum number of 12 schools, suggesting that these 22,000 students may have had more than 12 acceptable schools. According to Abdulkadiroğlu et al. (2009), the number of students reporting 12 schools remained similar in later years (between 28% and 20%).
5 Comparing the occurrence of dominant strategies across mechanisms

Based on our characterizations of dominant strategies, it is possible to compare the incentive properties of $DA^k$ and $BOS^k$.

5.1 Inclusion comparisons

We first follow Arribillaga and Massó (2015) in saying that mechanism $B$ is at least as manipulable as mechanism $A$ if for any $t_i$, whenever $t_i$ has a truthful dominant strategy given $R_i$ in $B$, $t_i$ also has a truthful dominant strategy given $R_i$ in $A$. Formally, mechanism $B$ is at least as manipulable as mechanism $A$ if for any profile of priorities $F \in \mathcal{F}$, for any profile of capacities $(q_1, \ldots, q_m)$, and for any $t_i \in T$,

$$\{R_i \in R_i \mid t_i \text{ has a truthful dominant strategy in } B\} \subseteq \{R_i \in R_i \mid t_i \text{ has a truthful dominant strategy in } A\}.$$  \hspace{1cm} (3)

In the context of a constrained school choice mechanism $M^k$, we consider that any strategy in which $t_i$ reports her $\min\{k, \#R_i\}$ most-preferred schools truthfully is truthful.

Mechanism $B$ is more manipulable than mechanism $A$ if $B$ is at least as manipulable as $B$, but the converse is not true. In the context of constrained school choice, the latter means that there exists $F^* \in \mathcal{F}$, $(q_1^*, \ldots, q_m^*)$, and $t_i \in T$ such that $\subseteq$ replaces $\subset$ in (3). Mechanism $B$ is equally manipulable as $A$ if $A$ is at least as manipulable as $B$ and $B$ is at least as manipulable as $A$, i.e., for any $F \in \mathcal{F}$, any $(q_1, \ldots, q_m)$, and any $t_i \in T$, $=$ replaces $\subseteq$ in (3).

The three next corollaries follow from our characterizations of dominant strategies in $DA^k$ and $BOS^k$ and are summarized in Figure 1. From our characterizations it is easy to see that, for any $k \in \{2, \ldots, m\}$ and any $\ell \in \{2, \ldots, m\}$, students who have dominant strategies in $BOS^\ell$ also have dominant strategies in $DA^k$, which yields the following result.

**Corollary 1.** For any $k \in \{2, \ldots, m\}$ and any $\ell \in \{2, \ldots, m\}$, $BOS^\ell$ is more manipulable than $DA^k$.\(^{13}\)

In particular, even the heavily constrained $DA^2$ is less manipulable than the unconstrained $BOS^m$.\(^{13}\)

Proposition 1 implies that the collection of a student’s safe sets can only grow with $k$ in $DA^k$, which together with Propositions 2 yields the following corollary.

**Corollary 2.** For any $k \in \{1, \ldots, m - 1\}$, $DA^{k+1}$ is less manipulable than $DA^k$.

Finally, because dominant strategies in $BOS^k$ result from students having a single acceptable school or having a top-priority at their most-preferred school, dominant strategies in $BOS^k$ are insensitive to changes in $k$.

**Corollary 3.** For any $k \in \{1, \ldots, m - 1\}$, $BOS^{k+1}$ is equally manipulable as $BOS^k$.

\(^{13}\)For $k = 1$, $DA^k$ and $BOS^k$ are strategically equivalent and the two mechanisms are therefore equally manipulable.
$DA^m > \cdots > DA^1 = BOS^m = \cdots = BOS^1$

Figure 1: Manipulability comparisons of $BOS^k$ and $DA^k$ in the sense of Arribillaga and Massó (2015), where $A > B$ indicates that $A$ is less manipulable than $B$ and $A = B$ indicates that $A$ and $B$ are equally manipulable.

Corollaries 1 to 3 parallel and confirm Proposition 2 and Corollary 2 in Pathak and Sönmez (2013). As Pathak and Sönmez (2013), we find that the manipulability advantage of $DA^m$ over $BOS^m$ carries over to $DA^k$ and $BOS^k$. Also, like Pathak and Sönmez (2013), we find that increasing the number of schools student can report reduces manipulability in $DA^k$. Our results provide further incentive justifications for reforms identified in Pathak and Sönmez (2013, Table 1) where a number of districts were observed switching from $BOS^k$ to $DA^k$ or increasing the number of schools student can report in $DA^k$.

Pathak and Sönmez (2013, Proposition 2 and Corollary 2) use a manipulability partial order which is not related to the partial order from Arribillaga and Massó (2015) we use here. Corollaries 1 to 3 are therefore independent from results in Pathak and Sönmez (2013). As noted above, we also show that for $k \geq 2$, $DA^k$ is less manipulable than $BOS^\ell$ even if $\ell > k$, a result that has no counterpart in Pathak and Sönmez (2013).

Most manipulability partial orders have limitations and our analysis should be viewed as complementary to that of Pathak and Sönmez (2013). By relying on the partial order developed by Arribillaga and Massó (2015), we focus on dominant strategies. An advantage of focusing on dominant strategies is they provide students with clearcut incentives that are independent of any beliefs about other students’ reported preferences. The cost of focusing on dominant strategies is our disregard for the effect of choosing one mechanism over another on the incentives of students who have dominant strategies in neither mechanisms.\textsuperscript{14} Differently, Pathak and Sönmez (2013) compare mechanisms based on the existence of truthful Nash equilibria, which includes cases where students do not have dominant strategies, but disregards the well-documented difficulty for students to coordinate on equilibria.\textsuperscript{15}

5.2 Quantitative comparisons for random priority profiles

In most school districts, priorities are based on a few criteria (such as living within walking distance of a school) which do not enable a strict ordering of all students. When this is the case, ties in priorities must be broken before standard school choice mechanisms — such as $DA^k$ and $BOS^k$ — can be applied. We call pre-existing priorities the (possibly weak) profile of priorities determined by these criteria. The (strict) priority profile $F$ is then selected according to some distribution $g$ among the set of profiles that respect pre-existing priorities. We call the realization of $g$ the ex-post priority profile.

\textsuperscript{14} In a companion paper (Decerf and Van der Linden, 2017), we show that even for students who do not have a dominant strategy, $DA^k$ outperforms $BOS^k$ in terms of the ability of students to eliminate a large set of strategies using weak dominance.

\textsuperscript{15} Especially in the presence of multiple equilibria, as can be the case in $DA^k$ and $BOS^k$ (Haeringer and Klijn, 2009).
The distribution $g$ typically follows from the application of a tie-breaking rule to the profile of pre-existing priorities. The two most common tie-breaking rules are the *single* tie-breaking rule (STB) and the *multiple* tie-breaking rule (MTB). In STB, a *unique* ordering of the students is drawn uniformly at random. At *every* school, ties in pre-existing priorities are then broken according to this unique ordering, which induces high levels of correlation between ex-post priorities. MTB breaks ties in pre-existing priorities at each school according to a *new* ordering of students, drawn uniformly at random specifically for this school. On average, MTB therefore induces much less correlation between ex-post priorities than STB.

Recall that, by definition, mechanism $B$ is more manipulable than mechanism $A$ if (3) is true for all $F \in \mathcal{F}$ (and for all $(q_1, \ldots, q_m)$ and $t_i \in T$). Therefore, assuming that the realization of $g$ is revealed to students before preferences are submitted — an issue we get back to in the next section, we know that $DA^k$ provides more students with a dominant strategy than $BOS^k$ regardless of $g$ (Corollary 1). We also know that, irrespective of $g$, the number of students with a dominant strategy increases with $k$ in $DA^k$ (Corollary 2). These are strong qualitative comparisons, but they lack quantitative content.

To obtain quantitative estimates, we focus on STB. We also focus on the commonly studied case of no pre-existing priorities (priorities are exclusively determined by the tie-breaker) and homogeneous quotas, i.e., $q_s = q$ for every $s \in S$. Besides making the model analytically tractable (Miralles, 2009; Abdulkadiroğlu et al., 2011, 2015), the absence of pre-existing priorities can be viewed as an approximation of the common real world scenario where pre-existing priorities are rare. Homogeneous quotas are another common assumption that is required to obtain clear analytical results. These two assumptions are maintained throughout the rest of this paper.

As Propositions 2 and 3 make clear, any student with less than $k$ acceptable schools (weakly) adds to the difference between the number of students with a dominant strategy in $DA^k$ and $BOS^k$. Also, the number of students with at most $k$ acceptable schools increases with $k$. This increase adds to the difference between the number of students with a dominant strategy in $DA^{k+1}$ and $DA^k$. Since these effects are well-understood, we focus on quantifying other sources of dominant strategies that do not stem from students having less than $k$ acceptable schools. To do so while keeping the results simple, we assume that all students have more than $k$ acceptable schools. For a student $t_i \in T$, we denote by $\bar{R}_i$ the set of $t_i$’s preferences in which more than $k$ schools are acceptable.

To assess the magnitude of the difference in incentives between mechanisms, we compute the expected number of students with a dominant strategy in $DA^k$ and $BOS^k$. Formally, for any $g$ and any $k \in \{1, \ldots, m\}$, the expected number of students with a dominant strategy under mechanism $M^k$ is

$$E^{dom}_{M^k}(g) := \frac{1}{\# R_1} \sum_{t_i \in T} \sum_{R_i \in \bar{R}_i} \sum_{F \in \mathcal{F}} 1(R_i, F) g(F),$$

---

16STB has been shown to have better efficiency properties than MTB (see footnote 4). In Section 7, we show that STB also has better incentive properties than MTB.

17 Some real world school choice problems do exclude pre-existing priorities (Abdulkadiroğlu et al., 2011).

18 In fact, the stronger assumption $q_s = 1$ for every $s \in S$ is often used, for example in Immorlica and Mahdian (2005) and Abdulkadiroğlu et al. (2017).
where $1(R_i, F)$ is an indicator function which takes value 1 when $t_i$ has a dominant strategy in $M^k$ given preference $R_i$ and priorities $F$, and zero otherwise. In the above definition of $E^dom_{M^k}(g)$, preferences are implicitly drawn uniformly at random. We slightly abuse the notation and denote by $\tilde{F}$ the degenerate distribution $\tilde{g}$ for which $\tilde{g}(\tilde{F}) = 1$. In particular, $E^dom_{M^k}(\tilde{F}) = \frac{1}{\tilde{g}\{R_i \mid t_i \in T \}} \sum_{t_i \in T} \sum_{R_i \in \bar{R}} 1(R_i, \tilde{F})$.

When $DA^k$ is used with STB, students who are given priorities 1 to $q$ have a top-priority at every school and clearly have a dominant strategy. Students who are given priorities $q + 1$ to $2q$ do not have top-priorities. However, they know that if they are rejected from a school, the school must be filled with $q$ students whose priority is higher than theirs. Therefore, whichever school a student with priority $q + 1$ to $2q$ ranks second is a school at which she has a top-priority among the students who are not assigned to her most-preferred school. As a consequence, any pair of schools is a safe set for such a student, and by Proposition 2 this student has a dominant strategy in $DA^2$. More generally, the following result is a corollary of Proposition 2.

**Corollary 4.** For every $k \in \{1, \ldots, m\}$, $E^dom_{DA^k}(g^{STB}) = \min\{kq, n\}$.

Corollary 4 is illustrated in Figure 2 for the cases $n = 100$, $m = 10$, and $q \in \{3, 5, 8, 10\}$. As Corollary 4 shows, the expected number of students with a dominant strategy increases with $k$ by a factor of $q$. In contrast, the expected number of students with a top-priority at their most-preferred school is independent of $k$, and the following is therefore a corollary of Proposition 3.

**Corollary 5.** For every $k \in \{1, \ldots, m\}$ and every distribution $g$, the expected number of students with a dominant strategy in $BOS^k$ is $E^dom_{BOS^k}(g) = q$.

As Corollaries 4 and 5 show, $E^dom_{DA^k}(g^{STB}) - E^dom_{BOS^k}(g)$ is of the order of $q(k - 1)$, regardless of $g$. Besides being robust in the sense of Corollary 1, the incentive advantage of $DA^k$ over $BOS^k$ can therefore be sizable. The same is true of the incentive advantage of $DA^x$ over $DA^y$ when $x > y$. For example, when $n = 100$, $m = 10$ and $q = 8$, $DA^6$ on average provides 48% of the students with a dominant strategy whereas $DA^2$ only gives a dominant strategy to 16% of the students, and $BOS^k$ to 8% of the students (regardless of $k$).

In contrast, when the total number of seats is in drastic short-supply, even $DA^{m-1}$ provides only a limited improvement over $BOS^{m-1}$. For example, when $n = 100$, $m = 10$, and $q = 1$, $DA^9$ on average provides 9% of the students with a dominant strategy, versus 1% for $BOS^9$. The same is true when $k$ is small compared to $m$. Assuming that there are as many seats as students ($mq = n$) and considering the parameters of the NYC match (2003-2004) $m = 500$, $n = 100,000$, and $k = 12$, we find that, on average, only 2.4% of the students have a dominant strategy in $DA^{12}$.

Recall that Corollaries 4 and 5 only consider students with more than $k$ acceptable schools. As explained above, the fact that some students are able to report all their acceptable schools further favors $DA^k$ over $BOS^k$, and $DA^{k+1}$ over $DA^k$. Therefore, in spite of Corollaries 4 and 5, $DA^k$ can have a sizable incentive advantage over $BOS^k$ (and $DA^k$ a sizable incentive advantage over $DA^{k+1}$) even when $q$ is small or $k$ is small compared to $m$. However, in these cases, incentive improvements predominantly come
Figure 2: Expected number of students with a dominant strategy in $DA^k$ from Corollary 4 ($n = 100$, $m = 10$ and $q \in \{3, 5, 8, 10\}$).

from students being able to report all their acceptable schools, and not from safe sets induced by correlations in priorities.\textsuperscript{19}

6 Timing of tie-breaking and information on priorities

All previous results assumed that priorities are known to the students when they report their preferences. In $DA^k$, knowledge of the priorities allows students to identify their safe sets and determine whether they have a dominant strategy. Even when a student does not have a dominant strategy, knowing her safe sets can help the student rule out dominated strategies and identify maximin strategies (Decerf and Van der Linden, 2017). In $BOS^k$, knowledge of the priorities can help the chosen few who have a top-priority at their most-preferred school.

Revealing the priorities before preferences are reported is compatible with priorities being drawn at random. Officials simply have to announce the realization of the draw before students report their preferences. In most school districts, however, the dominant practice seems to consist in breaking ties in priorities after preferences are reported.

\textsuperscript{19} As explained in footnote 12, about 80\% of the students who participated in the NYC match (2003-2004) reported less than 12 schools. These students are likely to have had less than 12 acceptable schools, and therefore had a dominant strategy in $DA^{12}$ regardless of the priorities. Unless some had a single acceptable school, only a few of these students would have had a dominant strategy in $BOS^{12}$. 

13
Students must then choose their reported preference under partial information on priorities. Under partial information (on priorities), students only know the distribution of priority profiles before reporting their preferences. Let $M(Q; F)$ be the outcome of $M$ when the priority profile is $F$ and the profile of (ordinal) reported preferences is $Q$. The expected utility of student $t_i$ in mechanism $M$ when reported preferences are $Q$ and priorities are drawn according to $g$ is

$$\mathbb{E}_i(Q; M) := \sum_{F \in \mathcal{F}} u_i(M(Q; F)) g(F).$$

Given $M$ and $g$, $Q_i$ is an expected dominant strategy for $t_i$ if

$$\mathbb{E}_i(Q_i, Q_{-i}; M) \geq \mathbb{E}_i(Q'_i, Q_{-i}; M) \quad \text{for any } Q'_i \in Q_i \text{ and any } Q_{-i} \in Q_{-i}. \quad (5)$$

Observe that unlike dominant strategies, expected dominant strategies need not be safe strategies in $DA^k$ (see Example 2).

We are interested in comparing the occurrence of dominant strategies when ex-post priorities are concealed and when they are revealed before preferences are reported. In $DA^k$, students who have less than $k$ acceptable schools have an expected dominant strategy regardless of whether priorities are revealed or concealed, and these students do not impact the comparison. Similarly, students who have a single acceptable school have an expected dominant strategy in $BOS^k$ regardless of information on priorities, and the number of acceptable schools does not otherwise impact expected dominant strategies in $BOS^k$. Therefore, as in the previous section, we focus on students with more than $k$ acceptable schools. To keep the results simple, we again assume that all students have more than $k$ acceptable schools.

In many cases, a lack of information on ex-post priorities leads to a significant decrease in the number of students with an expected dominant strategy in $DA^k$.

**Example 1** (Revealing priorities and dominant strategies). Consider $DA^1$ with two schools, each having a single seat. Suppose that there are three students with identical preferences, where $u_i(s_1) = 2$, $u_i(s_2) = 1$ and $u_i(t_i) = 0$ for $i \in \{1, 2, 3\}$. Further suppose that STB is used, i.e., the priority profile is drawn uniformly at random among the six following profiles

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If students only know that STB is used but do not know which of the six profiles will be drawn, every student would prefer to report $s_1$ if no other student reported $s_1$. But every student would prefer to report $s_2$ if both other students reported $s_1$. Hence, no student has an expected dominant strategy.

In contrast, if the realization of STB is revealed before students report their preference, then one of the three students systematically has a top-priority at her most-preferred school, and therefore has a dominant strategy (Proposition 2).
Given Proposition 2, it may seem that revealing more information on priorities can only increase the number of students with an expected dominant strategy. Although we show below that this is true in most cases, there are cardinal utility profiles for which maintaining uncertainty can provide more students with an expected dominant strategy than if the ex-post priority profile is revealed.

**Example 2.** This example is identical to Example 1 but with \( u_i(s_1) = 10 \) instead of 1 for \( i \in \{1, 2, 3\} \). Under this more extreme priority profile, reporting \( s_1 \) is always a best response. Reporting \( s_1 \) yields an expected utility of at least \( 10/3 \) (when both other students report \( s_1 \)), which is higher than 1, the maximum expected utility any of student can secure when reporting \( s_2 \) instead of \( s_1 \). Hence, any student has an expected dominant strategy under partial information, whereas only one of the three students has a dominant strategy if the ex-post priorities are revealed.

As Example 1 suggests, in many cases, no student has an expected dominant strategy under partial information, whereas a number of students would have a dominant strategy if ex-post priorities were revealed (see the simulations in Section 7 for quantitative estimates of the latter). However, as Example 2 illustrates, when the difference between schools’ cardinal utilities is particularly large, partial information can yield a larger number of dominant strategies. The next proposition makes these observations more general and more precise.

Recall that we focus on the case of no pre-existing priorities and homogeneous quotas. In this context, distribution \( g \) is **ex-ante fair** if students are given any priority rank at any school with the same probability. Formally, for \( p \in \{1, \ldots, n\} \), let \( F_s(t) = p \) indicate that student \( t \) has priority rank \( p \) at school \( s \). Then, \( g \) is ex-ante fair if

\[
\sum_{\{F \in \mathcal{F} \mid F_s(t) = p\}} g(F) = \frac{1}{n} \quad \text{for any } p \in \{1, \ldots, n\}, \text{ any } s \in S, \text{ and any } t \in T.
\]

Observe that ex-ante fair distributions allow for correlation between priorities ex-post. For example, STB is ex-ante fair although it features high levels of correlations between priorities ex-post. MTB is also ex-ante fair but on average features lower levels of correlations between priorities than STB. For any utility function over schools \( u_i \), let \( u_i(x) \) be the utility of the \( x \)-th highest ranked school (i.e., \( u_i(1) \) is the utility of the most-preferred school, \( u_i(2) \) is the utility of the second most-preferred school, and so on). To simplify the statement of our results, we normalize \( u_i(t_i) = 0 \), where \( u_i(t_i) \) is the utility of being unassigned.\(^{20}\)

**Proposition 4** (Expected dom. strat. require extreme preference in \( DA^k \)). For any \( k \in \{1, \ldots, m - 1\} \) and any ex-ante fair tie-breaking rule \( g \), if the \( k \) most-preferred schools of \( t_i \) are in short-supply and if

\[
u_i(k+1) \left( \frac{n}{q} - (k-1) \right) > u_i(k),
\]

then student \( t_i \) does not have an expected dominant strategy in \( DA^k \).

\(^{20}\) Without normalization, inequality (6) would become \( [u_i(k+1) - u_i(t_i)] \left( \frac{n}{q} - (k-1) \right) > u_i(k) - u_i(t_i) \).
When there are as many students as seats, which implies \( m = \frac{n}{q} \), condition (6) simplifies to
\[
 u_i(k + 1)(m - (k - 1)) > u_i(k) \tag{7}
\]
Consider parameters \( m = 500 \) and \( k = 12 \) from the NYC match (2003-2004). Condition (7) then says that, under partial information, the (normalized) utility a student attaches to her 12th most-preferred school must be over 489 times higher than the utility she attaches to her 13th most-preferred school for her to have an expected dominant strategy in \( DA^k \), which seems unrealistic. In this case, under partial information, most students likely fail to have an expected dominant strategy in \( DA^k \). Although when \( m = 500 \) and \( k = 12 \) the number of students with a dominant strategies under full information is also small (Corollary 4), the advantage of revealing ex-post priorities can be much larger for other larger values of the parameters.

Full information can be preferred to partial information even when there are less (expected) dominant strategies under full information than under partial information. Dominant strategies are cognitively less demanding than expected dominant strategies. First, identifying expected dominant strategies requires students to form cardinal preferences over schools, which takes more effort than forming ordinal preferences (a student who forms cardinal preference necessarily has ordinal preference, whereas the converse is not true). Significant effort is also required to perform expected utility computations, and district officials may want to spare parents the costs associated with these efforts.

In light of Proposition 4 and the lower cognitive load required by dominant strategies, there appears to be an incentive justification for revealing ex-post priorities before students report their preferences, which is not the current dominant practice. Districts may be reluctant to revealing the full ex-post priority profile out of practical and privacy concerns. Revealing ex-post priorities may also exacerbate feelings of unfairness. Although tie-breaking rules are typically fair \textit{ex-ante}, a student may \textit{ex-post} feel that she has been treated unfairly if she learns that she has received a low tie-breaker at all schools (see Section 7). Finally, the full priority profile may be too much information for a student to process. As condition (2) shows, computing a student’s safe sets is relatively straightforward computationally, but it can be tedious.

For these reasons, instead of revealing the raw priorities and quotas, we recommend that districts set up decision-support platforms through which students can learn about their safe sets and dominant strategies. A proof of concept of such a platform can be found at https://martinvanderlinden.shinyapps.io/Decision_support_DAk. Based on a student’s preference input, the platform uses its knowledge of the ex-post priority profile to compute the student’s safe sets. The platform then provides the student with a recommendation on the list of schools to report. If a student has a dominant strategy, she is advised to report her \( k \) most-preferred schools truthfully. If she does not have a dominant strategy and has no safe set, the student is advised to include a school she knows is in low-demand as a protection against being unassigned. Even if a student does

\[21\] Suppose that \( k \) increases to 100. Then, 20% of the students have a dominant strategy in \( DA^{100} \) when ex-post priorities are revealed. However, for a student to have an \textit{expected} dominant strategy in \( DA^{100} \) under partial information, condition (6) requires that the utility the student attaches to her 12th most-preferred school be over 400 times larger than the utility she attaches to her 13th most-preferred school, which remains unlikely.
not have a dominant strategy, the platform informs her of some of her safe sets, which can help her rule out dominated strategies and identify maximin strategies (Decerf and Van der Linden, 2017). Complete examples of advice and screen-shots of the platform can be found in Appendix B.

Somewhat extreme cardinal preferences are also required for students to have expected dominant strategies in $BOS^k$.

**Proposition 5** (Expected dom. strat. require extreme preference in $BOS^k$). For any $k \in \{2, \ldots, m\}$ and any ex-ante fair tie-breaking rule $g$, if

$$u_i(1) \frac{q}{n-q} > u_i(2) - u_i(3),$$

then student $t_i$ does not have an expected dominant strategy in $BOS^k$.

When there are as many students as seats, which implies $m = \frac{n}{q}$, condition (8) simplifies to

$$u_i(1) \frac{1}{m-1} > u_i(2) - u_i(3).$$

Consider again parameters $m = 500$ and $k = 12$ from the NYC match (2003-2004). Condition (9) then says that, under partial information, a student must have an extreme preference for her first school over her second school (or be almost indifferent between her second and third schools) to have an expected dominant strategy in $BOS^k$ ($u_i(1) \frac{1}{500-1} > u_i(2) - u_i(3)$). In this case, expected dominant strategies should therefore be rare. However, unlike in $DA^k$, dominant strategies are also rare in $BOS^k$ when priorities are revealed before preferences are reported (Corollary 5). Gains or losses associated with revealing or concealing priorities in $BOS^k$ are therefore likely to be marginal.

7 Comparing tie-breaking rules

Although STB is usually considered more efficient than MTB (see footnote 4), the issue of tie-breaking selection remains vividly debated. Of particular concern for school board officials is the parents’ perception of STB’s unfairness. Pathak (2011) reports that during the reform of the NYC assignment mechanism, an official remarked:

“I believe that the equitable approach is for a child to have a new chance with each [...] program. [i.e, use MTB] [...] If we use only one random number [i.e, use STB], and I had the bad luck to be the last student in the line this would be repeated 12 times and I would never get a chance. I do not know how we could explain this to a parent.” (Pathak, 2011)

As the quote suggests, although STB and MTB are both ex-ante fair, STB is perceived as unfair ex-post because it gives the same tie-breaking order to every student at every school.

To determine whether STB is worth its apparent ex-post fairness cost, it is useful to get a more complete picture of STB’s advantages over MTB. Previous research has focused
on efficiency. In this section, we provide new insights into STB’s incentive performances. We show that, in a special case, STB dominates any tie-breaking rule (including MTB) in terms of the expected number of students it provides with a dominant strategy. For more general cases, we provide quantitative estimates of STB’s incentive advantage over MTB through simulations. Our simulations suggest that STB’s incentive advantage over MTB is not specific to the special case for which we have analytical results. Finally, we show that the incentive performances of STB can be preserved while improving on STB’s ex-post fairness.

Because dominant strategies are not affected by tie-breaking rules in BOS\(^k\) (Corollary 5), we only study DA\(^k\), the mechanism for which the use of STB and MTB is usually debated. Motivated by the results of Section 6, we assume that the realization of tie-breaking rules is disclosed before preferences are reported. Also, we again assume that all students have more than \(k\) acceptable schools (as students with less than \(k\) acceptable schools have dominant strategies in DA\(^k\) regardless of priorities, these students do not impact comparisons between tie-breaking rules).

7.1 STB v. MTB: analytical results for \(k \leq 2\) and \(q = 1\)

For the degenerate distribution \(F\), the probability that student \(t_i\) has a dominant strategy depends on the probability that \(t_i\) has a safe set covering her \(k\) most-preferred schools (Proposition 2). If \(R_i\) is drawn uniformly at random, this probability is a linear function of the number of \(t_i\)'s safe sets of size \(k\) given \(F\), and is independent across students. Thus, the expected number of students with a dominant strategy \(E_{\text{dom}}^{DA}(F)\) is directly proportional to the number of safe sets of size \(k\) generated by \(F\) (see Lemma 5 in the Appendix for a formal proof). Tie-breaking rules that, on average, select priorities generating more safe sets therefore provide more students with dominant strategies.

The safe sets generated by a priority profile \(F\) can be decomposed into two categories, depending on the properties of \(F\) from which these safe sets originate:

(a) Students having top-priorities: When a student has a top-priority at a school \(s^*\), any set of schools containing \(s^*\) is a safe set which we call a first-order safe set.

(b) Correlations in priorities: A student can also have higher-order safe sets (i.e., safe sets that are not first-order) if she is among the \(kq\) highest-priority students at a number of schools and students who have higher priorities than her are correlated across schools, as illustrated in (1).

Because STB produces maximal correlations between priorities, STB generates numerous higher-order safe sets. However, STB performs poorly when it comes to first-order safe sets because of a form of “decreasing return” to correlations in top-priorities.

Example 3 (Decreasing return to correlations in top-priorities). Suppose that there are four schools, each with one seat, and four students \(\{t_1, \ldots, t_4\}\). Consider DA\(^2\) and the first-order safe sets of size \(k = 2\) (2-FSS) induced by different distribution of the top-priorities.

We focus on student \(t_1\). Suppose that \(t_1\) is awarded the top-priority at school \(s_1\). Then \(\{s_1, s_2\}, \{s_1, s_3\},\text{ and }\{s_1, s_4\}\) are 2-FSS for student \(t_1\).
Now, consider the increase in the number of $t_1$’s 2-FSS if she is also awarded the top-priority at school $s_2$. Alone, a top-priority at $s_2$ gives $t_1$ three 2-FSS, namely $\{s_2,s_1\}, \{s_2,s_3\},$ and $\{s_2,s_4\}$. However, $\{s_2,s_1\}$ is already a 2-FSS for $t_1$ because of her top-priority at $s_1$. Hence, awarding $t_1$ the top-priority at school $s_2$ only gives her two additional 2-FSS, namely $\{s_2,s_3\}$ and $\{s_2,s_4\}$.

In terms of the total number of 2-FSS, it is better to award the top-priority at $s_2$ to another student, say $t_2$, than to give it to $t_1$. Assuming $t_2$ does not yet have any 2-FSS, awarding the top-priority at $s_2$ to $t_2$ gives her three additional 2-FSS ($\{s_2,s_1\}, \{s_2,s_3\}$, and $\{s_2,s_4\}$), instead of the two additional 2-FSS given to $t_1$ if she is awarded the top-priority at $s_2$.

Based on Example 3, it is not hard to see that STB is among the least effective tie-breaking rules at generating first-order safe sets. Surprisingly, for $k \in \{1,2\}$ and $q = 1$, STB is nevertheless optimal in the following sense.

**Proposition 6.** For $k \in \{1,2\}$ and $q = 1$, for any tie-breaking rule $B$, any $F^B \in \text{supp}(B)$, and any $F^{STB} \in \text{supp}(STB)$, we have $E^\text{dom}_{DA}(F^{STB}) \geq E^\text{dom}_{DA}(F^B)$.

As Proposition 6 shows, when $k \in \{1,2\}$ and $q = 1$, any gains in terms of first-order safe sets that may follow from spreading top-priorities in some $F^B \notin \text{supp}(STB)$ is at least compensated by the larger number of higher-order safe sets generated by any $F^{STB} \in \text{supp}(STB)$. For a number of profiles outside of $\text{supp}(STB)$ (including some profiles in $\text{supp}(MTB)$), the gains from spreading top-priorities are, in fact, more than compensated by the larger number of higher-order safe sets generated by profiles in $\text{supp}(STB)$.

**Example 4** (First-order gains do not compensate higher-order losses). Consider $DA^2$ and suppose that there are four schools, each with one seat. Let $\tilde{F}$ be any priority profile in $\text{supp}(MTB)$ with the following two top-priorities\(^{22}\)

$$
\begin{array}{llll}
\tilde{F}_1 & \tilde{F}_2 & \tilde{F}_3 & \tilde{F}_4 \\
& t_1 & t_1 & t_1 & t_2 \\
t_3 & t_4 & t_5 & t_6 \\
& & & & \\
\end{array}
$$

(10)

Observe that because the second-highest priority is never given to the same student twice in $\tilde{F}$, the profile does not take full advantage of opportunities to generate higher-order safe sets of size $k = 2$. As a consequence, all safe sets of size 2 in $\tilde{F}$ are first-order safe sets, with $t_1$ having safe sets $\{s_1,s_2\}, \{s_1,s_3\},$ and $\{s_1,s_4\}$, and $t_2$ having safe sets $\{s_1,s_2\}, \{s_1,s_3\},$ and $\{s_1,s_4\}$. Given this collection of safe sets, $t_1$ always has a dominant strategy whereas $t_2$ has a dominant strategy with probability one-half (specifically, when her two most-preferred schools are $\{s_1,s_2\}, \{s_1,s_3\},$ or $\{s_1,s_4\}$). Hence, given $\tilde{F}$, the expected number of students with a dominant strategy is 1.5, which is lower than 2, the expected number of students with a dominant strategy given any $F^{STB} \in \text{supp}(STB)$.

Profiles like (10) can be found for any $k \geq 2$, which yields the following proposition

**Proposition 7.** For every $k \in \{2,\ldots,m\}$ and every $q \in \mathbb{N}$, either

\(^{22}\)Because $\text{supp}(MTB) = \mathcal{F}$, $\text{supp}(MTB)$ contains many profiles satisfying (10).
(a) every set of \( k \) schools is in oversupply, and hence, \( \mathbb{E}^{\text{dom}}_{DA^k}(F) = n \) for any \( F \in \mathcal{F} \), or
(b) there exists \( \tilde{F}^{MTB} \in \text{supp}(MTB) \) such that \( \mathbb{E}^{\text{dom}}_{DA^k}(F^{STB}) > \mathbb{E}^{\text{dom}}_{DA^k}(\tilde{F}^{MTB}) \) for any \( F^{STB} \in \text{supp}(STB) \).

In practice, most districts allow students to report more than two schools, with \( DA^3 \) being an especially common mechanism (see Pathak and Sönmez, 2013, Table 1, for data on England). Even for \( DA^2 \), Propositions 6 and 7 show that STB clearly outperforms MTB for the case \( q = 1 \) only, which is again uncommon.

As Corollary 4 shows, when STB is used, it is possible to compute the expected number of students with a dominant strategy for a variety of parameters. In general, the same expected number can be hard to compute for MTB, and direct analytical comparisons remain out of reach.\(^{23}\) To compare STB with MTB when \( k > 2 \) or \( q > 1 \), we therefore resort to simulations which we present in the next section. Although a formal proof is left as an open question, our simulations suggest that Proposition 7 generalizes beyond \( k \leq 2 \) and \( q = 1 \).

Figure 3: Sample average (1000 observations) of the number of students with a dominant strategy in \( DA^k \) with MTB as a function of the number of schools that students can report (\( n = 100, m = 10, \) and \( q \in \{3, 5, 8, 10\} \)).

7.2 STB v. MTB: simulations

In order to compare STB with MTB when \( k > 2 \) or \( q > 1 \) (to quantify the differences in dominant strategies between STB and MT), we compute the average number of students

\(^{23}\) Unlike the expected number under STB which is linear in \( k \), the same expected number can be a potentially complex \textit{concave} function of \( k \) under MTB (see the simulations in Figure 3).
with a dominant strategy under each tie-breaking rule for a sample of priority and preference profiles. We report the results of our computational experiment in Figures 3 to 4. In our experiment, there are 10 schools \((m = 10)\) and 100 students \((n = 100)\). Averages are over 1000 random profiles of preferences and priorities. We report results for each \(k \in \{1, \ldots, 10\}\) and each \(q \in \{3, 5, 8, 10\}\).

Our experiment satisfies the conditions of Corollary 4. In particular, the profiles of preferences and priorities are drawn uniformly at random with no pre-existing priorities, and each school has the same number of seats \(q\). Therefore, the expected number of dominant strategies under STB is given by Corollary 4 and illustrated in Figure 2, and we only report the sample average for MTB in Figure 3. Differences between the sample average of MTB and the theoretical average of STB are reported in Figure 5. The results in Figure 5 suggest that Proposition 6 generalizes to the parameters used in our experiment. Under these values of the parameters, the expected number of students with a dominant strategy under STB is higher than the sample average under MTB even when \(k > 2\) (and although \(q > 1\)).

Observe also that, although MTB does not equate the incentive properties of STB, \(DA^k\) with MTB remains preferable to \(BOS^k\), regardless of the tie-breaking rule used in \(BOS^k\). This is illustrated in Figure 4 for \(q = 10\). This turns out to be a general result. Based on Corollary 5, it is not hard to see that \(DA^k\) outperforms \(BOS^k\) regardless of the tie-breaking rules used in either of the mechanisms.
Figure 5: Difference in $D_A^k$ between the expected number of students with a dominant strategy with STB (Corollary 4) and the sample average (1000 observations) of the number of students who have a dominant strategy with MTB ($n = 100$, $m = 10$, and $q \in \{3, 5, 8, 10\}$).

**Proposition 8.** For every $k \in \{1, \ldots, m\}$ and every tie-breaking rules $g^{DA}$ and $g^{BOS}$, $E^{dom}_{DA^k}(g^{DA}) \geq E^{dom}_{BOS^{\ell}}(g^{BOS})$ for any $\ell \in \{1, \ldots, m\}$.

The fact that STB provides more students with dominant strategies than MTB in our experiment is robust to other combinations of values for $n$, $m$, and $q$ that we have investigated. This result is also robust to allowing for heterogeneous quotas and pre-existing priorities.

**7.3 Equating the performances of STB while improving on ex-post fairness**

As the simulations reveal, STB can provide sizable incentive improvements over MTB, but these improvements are limited for some values of the parameters. When improvements are limited, the incentive advantage of STB over MTB may not outweigh the perceived ex-post fairness cost of using STB instead of MTB (see the quote from Pathak, 2011, on page 17). A natural question is therefore whether the incentive advantage of STB can be harnessed while avoiding its ex-post fairness costs.

To answer this question, we develop what we believe to be the first metric of the fairness of (strict) priority profiles. In a priority profile, students are given a priority rank at each school. The sum of a student’s priority ranks reflects how high a priority the student is given across the different schools (e.g., a student who occupies the first
rank at each of the $m$ schools has a sum of ranks of $m$). We propose to compare priority profiles based on the distribution across students of these sums of rank. We then show how STB can be improved upon in terms of ex-post fairness without jeopardizing STB’s incentive properties (and while maintaining ex-ante fairness).

Consider the following priority profile in the support of STB when $n = 8$ and $m = 3$.

\[
\begin{array}{ccc}
F_{STB}^1 & F_{STB}^2 & F_{STB}^3 \\
 t_1 & t_1 & t_1 \\
 t_2 & t_2 & t_2 \\
 t_3 & t_3 & t_3 \\
 t_4 & t_4 & t_4 \\
 t_5 & t_5 & t_5 \\
 t_6 & t_6 & t_6 \\
 t_7 & t_7 & t_7 \\
 t_8 & t_8 & t_8 \\
\end{array}
\] (11)

The sums of students’ priority ranks in $F$ are $(3, 6, 9, \ldots, 24)$, where 3 is the sum of $t_1$’s priority ranks, 6 the sum of $t_2$’s priority ranks, and so on. The remark from the NYC official on pg. 17 suggests that perceptions of unfairness are linked to the spread in this distribution. In the words of the official, given $F_{STB}$ in (11), $t_8$ has a low chance to match with a school she likes because she has the “bad luck” of being the “last student in the line” at each and every school. Being the “last student in the line” at every school gives $t_8$ a sum of priority ranks of 24. Student $t_1$ on the other hand is guaranteed a seat at her most-preferred school as she has been given the top-priority at each of the three schools. This gives $t_1$ a sum of priority ranks of 3.

Among other changes, one would expect that reversing the order of priorities at one of the schools, as in $F'$ below, should alleviate feelings of ex-post unfairness:

\[
\begin{array}{ccc}
F_{STB}^{'} & F_{STB}^{'} & F_{STB}^{'} \\
 t_1 & t_8 & t_1 \\
 t_2 & t_7 & t_2 \\
 t_3 & t_6 & t_3 \\
 t_4 & t_5 & t_4 \\
 t_5 & t_4 & t_5 \\
 t_6 & t_3 & t_6 \\
 t_7 & t_2 & t_7 \\
 t_8 & t_1 & t_8 \\
\end{array}
\] (12)

The sums of students’ priority ranks given $F'$ are $(10, 11, 12, \ldots, 17)$. This distribution features a tighter spread than $(3, 6, 9, \ldots, 24)$. Specifically,

\[(10, 11, \ldots, 16, 17) = (3, 6, \ldots, 21, 24) + (7, 5, 3, 1, -1, -3, -5, -7),\]

where $(7, 5, 3, 1, -1, -3, -5, -7)$ corresponds to a series of “progressive transfers” from students with higher sums of ranks to students with lower sums of ranks.

Formally, for any priority profile $F$, let $r_i^F$ be the sum over all schools of $t_i$’s priority ranks. We call $r_i^F$ the total priority rank of $t_i$ given $F$. For example, for $F_{STB}$ in (11), we have $r_3^{F_{STB}} = 9$ and $r_6^{F_{STB}} = 18$. The vector of total priority ranks given $F$ is

23
Vector $r^F := (r^F_1, \ldots, r^F_n)$. Vector $r^F$ can be obtained from $r^{F'}$ by a progressive transfer if for some positive integer $d$ and for two different students $t_i, t_j \in T$,

$$r^F_i = r^{F'}_i + d \leq r^{F'}_j - d = r^F_j, \quad \text{while } r^F_k = r^{F'}_k \text{ for any } t_k \in T \setminus \{t_i, t_j\}.$$  

We say that $F$ is more ex-post fair than $F'$ if $r^F$ can be obtained from $r^{F'}$ by a sequence of progressive transfers. Similarly, tie-breaking rule $g$ is more ex-post fair than tie-breaking rule $g'$ if any profile in $\text{supp}(g)$ is more ex-post fair than any profile in $\text{supp}(g')$.

It is not hard to find tie-breaking rules that improve upon STB in terms of ex-post fairness. Our goal here is (a) to improve upon STB in terms of ex-post fairness while (b) preserving STB’s incentive properties, and (c) maintaining ex-ante fairness.

Ex-ante fairness turns out to be easy to satisfy. Suppose that we find a priority profile $F^*$ that improves upon all profiles in $\text{supp}(STB)$ in terms of ex-post fairness and incentive. Then, drawing a priority profile uniformly at random from the set of all permutations of $F^*$ guarantees ex-ante fairness (by a permutation of $F^*$, we mean a profile that is obtained from $F^*$ by permuting the identities of a number of students). By symmetry, such a tie-breaking rule also preserves the incentive properties of $F^*$.

Hence, to satisfy (a), (b), and (c), it is sufficient to identify a single priority profile that satisfies (a) and (b). We call (tie-breaking) pattern a profile $F^*$ from which an ex-ante fair tie-breaking rule is generated by permutations. The associated ex-ante fair tie-breaking rule is denoted by $\bar{g}^{F^*}$.

**Proposition 9.** For any $F \in \mathcal{F}$, any tie-breaking rule $\bar{g}^F$ obtained by drawing uniformly at random among the permutations of $F$ is ex-ante fair.

Recall that under STB, any student $t$ who is among the $kq$ highest-priority students has a dominant strategy because it is impossible for more than $kq - 1$ students to have a higher priority than $t$ at all of her $k$ most-preferred schools (hence, $t$’s $k$ preferred schools form a safe set). The same is true if priorities are redistributed among the $kq$ highest-priority students. For example, suppose that $n = 8$, $m = 3$, $q = 2$ and $k = 2$, and consider the following profile

<table>
<thead>
<tr>
<th>$F^\text{FTB}_1$</th>
<th>$F^\text{FTB}_2$</th>
<th>$F^\text{FTB}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>$t_4$</td>
<td>$t_3$</td>
</tr>
<tr>
<td>$t_2$</td>
<td>$t_1$</td>
<td>$t_4$</td>
</tr>
<tr>
<td>$t_3$</td>
<td>$t_2$</td>
<td>$t_1$</td>
</tr>
<tr>
<td>$t_4$</td>
<td>$t_3$</td>
<td>$t_2$</td>
</tr>
<tr>
<td>$t_5$</td>
<td>$t_8$</td>
<td>$t_7$</td>
</tr>
<tr>
<td>$t_6$</td>
<td>$t_5$</td>
<td>$t_8$</td>
</tr>
<tr>
<td>$t_7$</td>
<td>$t_6$</td>
<td>$t_5$</td>
</tr>
<tr>
<td>$t_8$</td>
<td>$t_7$</td>
<td>$t_6$</td>
</tr>
</tbody>
</table>

Clearly, students $t_1$ to $t_4$ have a dominant strategy, just as they would in $F^\text{STB}$. At the same time, priority profile $F^\text{FTB}$ improves upon $F^\text{STB}$ in terms of ex-post fairness, with $r^{F^\text{FTB}} = (6, 9, 8, 7, 18, 21, 20, 19)$ as opposed to $r^{F^\text{STB}} = (3, 6, 9, 12, 15, 18, 21, 24)$.

Sorting the two vectors in increasing order, one finds that $(6, 7, 8, 9, 18, 19, 20, 21) = (3, 6, 9, 12, 15, 18, 21, 24) + (3, 1, -1, -3, 3, 1, -1, -3)$.  

---

\[24\] Sorting the two vectors in increasing order, one finds that $(6, 7, 8, 9, 18, 19, 20, 21) = (3, 6, 9, 12, 15, 18, 21, 24) + (3, 1, -1, -3, 3, 1, -1, -3)$.  

24
The improvement in ex-post fairness brought about by $F^{FTB}$ is more than cosmetic. In particular, four different students are given a top-priority at at least one school in $F^{FTB}$, as opposed to two under $F^{STB}$. Also, unlike in $F^{STB}$, no student is last at every school in $F^{FTB}$.

In general, the pattern in (13) can be constructed by the following series of translations. Fix an arbitrary priority order $F^1_{STB}$ and focus first on priority ranks 1 to $kq$. For any $a \in \{2, \ldots, m\}$, the highest-$kq$ priorities in $F_a$ are obtained by translating the highest-$kq$ priorities in $F_{a-1}$ by $\lfloor kq/m \rfloor$. That is, the rank of a generic “high-priority” student in $F_a$ is equal to her rank in $F_{a-1}$ plus $\lfloor kq/m \rfloor$ (modulo $kq$). Priority ranks $kq+1$ to $n$ are constructed in a similar fashion: For any $a \in \{2, \ldots, m\}$, the $kq+1$ to $n$ priorities in $F_a$ are obtained by translating the same priorities in $F_{a-1}$ by $\lfloor (n-kq)/m \rfloor$.

The next proposition shows that $\bar{g}^{FTB}$ improves upon $g^{STB}$ in terms of ex-post fairness while maintaining the incentive properties of $g^{STB}$.

**Proposition 10.** For any $k \in \{1, \ldots, m\}$, (i) $E_{DA^k}(\bar{g}^{FTB}) = E_{DA^k}(g^{STB}) = \min\{kq, n\}$. (ii) If $\lfloor kq/m \rfloor \geq 1$, then $\bar{g}^{FTB}$ is more ex-post fair than $g^{STB}$.

An obvious pitfall of $F^{FTB}$ is that it still divides students between lucky (top-$kq$ priorities) and unlucky students (priorities $kq+1$ to $n$). Although $F^{FTB}$ does a better job than $F^{STB}$ at distributing priorities fairly among lucky and among unlucky students, unlucky students have a much lower chance of accessing the school of their choice than lucky ones.

In general, further improving upon the ex-post fairness of $F^{FTB}$ while maintaining the incentive properties of STB can be complicated. As Proposition 7 shows, when $k \leq 2$ and $q = 1$, STB is optimal in terms of incentives. At least under such parameters, any alternative profile $F'$ must therefore be itself optimal if it is to equate the incentive properties of STB. In particular, all opportunities for generating higher-order safe sets must be taken advantage of in the construction of $F'$, which poses optimization and combinatorial challenges.

In one important case, however, it is possible to improve upon $F^{FTB}$ while equating the incentive properties of STB. When seats are in short-supply ($mq \leq n$), it is possible to distribute top-priorities between students without awarding any student more than one top-priority. In this case, the decreasing return to correlations in top-priorities described in Example 3 turns out to exactly equate the benefits of correlations in STB, as illustrated in Example 5. This, in turn, allows any priority profile that evenly spreads top-priorities to match STB’s incentives properties while improving upon $F^{FTB}$ in terms of ex-post fairness.

**Example 5** (Decreasing returns to correlations equate benefits of correlations when $mq \leq n$). Suppose that $m = 3$, $q = 3$, and $n = 9$. For any $k$, STB provides a dominant strategy to $3k$ students. Consider the following “spread” priority profile that evenly

---

25 For every real number $x \in \mathbb{R}$, the floor function $\lfloor . \rfloor$ returns the largest integer $\lfloor x \rfloor$ that is not larger than $x$. 

25
The algorithm distributes the 9 top-priorities among the 9 students:

\[
\begin{array}{ccc}
F^{SPR}_1 & F^{SPR}_2 & F^{SPR}_3 \\
t_1 & t_4 & t_7 \\
t_2 & t_5 & t_8 \\
t_3 & t_6 & t_9 \\
\vdots & \vdots & \vdots \\
\end{array}
\]

Under \( F^{SPR} \), students have a dominant strategy if and only if their top-priority is their highest-preferred school. Thus, the expected number of students with a dominant strategy under \( F^{SPR} \) is \( n k/m \), which is equal to \( kq \), the expected number of students with a dominant strategy under \( F^{STB} \).

Again, pattern \( F^{SPR} \) can in general be obtained by a series of translations. Fix an arbitrary \( F^{SPR}_1 \). For any \( a \in \{2, \ldots, m\} \), the priority order \( F^a \) is obtained by translating priority order \( F^{a-1} \) by \( \lfloor n/m \rfloor \). That is, the rank of a generic student in \( F^a \) is equal to her rank in \( F^{a-1} \) plus \( \lfloor n/m \rfloor \) (modulo \( n \)). Observe that \( \lfloor n/m \rfloor \geq q \) when \( mq \leq n \), which implies that, as in Example 5, all students have at most one top-priority in \( F^{SPR} \).

A complete example of \( F^{SPR} \) is illustrated below for the case \( n = 8 \) and \( m = 3 \) (with \( \lfloor 8/3 \rfloor = 2 \)).

\[
\begin{array}{ccc}
F^{SPR}_1 & F^{SPR}_2 & F^{SPR}_3 \\
t_1 & t_7 & t_5 \\
t_2 & t_8 & t_6 \\
t_3 & t_1 & t_7 \\
t_4 & t_2 & t_8 \\
t_5 & t_3 & t_1 \\
t_6 & t_4 & t_2 \\
t_7 & t_5 & t_3 \\
t_8 & t_6 & t_4 \\
\end{array}
\]

One can verify that \( r^{F^{SPR}} = (9, 12, 15, 18, 13, 16, 11, 14) \) which represents an ex-post fairness improvement over \( r^{F^{STB}} = (3, 6, 9, 12, 15, 18, 21, 24) \).\(^{26}\) When \( kq < n \) (which is, e.g., implied by the short-supply assumption together with \( k < m \)), pattern \( F^{SPR} \) also improves upon \( F^{FTB} \) in terms of ex-post fairness.\(^{27}\) The next proposition generalizes these examples.

**Proposition 11.** For any \( k \in \{1, \ldots, m\} \), (i) if \( mq \leq n \), \( E_{DA_k}(\tilde{g}^{F^{SPR}}) = E_{DA_k}(\tilde{g}^{F^{STB}}) = \min\{kq, n\} \). (ii) if \( \lfloor n/m \rfloor \geq 1 \), \( \tilde{g}^{F^{SPR}} \) is more ex-post fair than \( g^{F^{STB}} \), and if in addition \( kq < n \) then \( \tilde{g}^{F^{SPR}} \) is more ex-post fair than \( \tilde{g}^{F^{FTB}} \).

In the short-supply case, it is sometimes possible to find ex-post fairness optimal \(^{26}\)Sorting the two vectors in increasing order, one finds that \( (9, 11, 12, 13, 14, 15, 16, 18) = (3, 6, 9, 12, 15, 18, 21, 24) + (6, 5, 3, 1, -1, -3, -5, -6) \).

\(^{27}\) In particular, in example (13) with \( q = 2 \) and \( k = 2 \), we have \( \tilde{r}^{F^{SPR}} = r^{F^{FTB}} + (3, 4, 4, -4, -4, -4, -3) \), where \( \tilde{r}^{F^{SPR}} \) is the increasing reordering of \( r^{F^{SPR}} \).
patterns that perform as well as STB in terms of incentives. Consider for example

\[
\begin{array}{ccc}
F_{1}^{OPT} & F_{2}^{OPT} & F_{3}^{OPT} \\
t_1 & t_4 & t_8 \\
t_2 & t_6 & t_7 \\
t_3 & t_5 & t_2 \\
t_4 & t_7 & t_3 \\
t_5 & t_8 & t_6 \\
t_6 & t_1 & t_5 \\
t_7 & t_3 & t_1 \\
t_8 & t_2 & t_4
\end{array}
\]

One can check that \( r^{OPT} = (14, 13, 14, 13, 14, 13, 14) \), which is optimal in terms of ex-post fairness (when \( n = 8 \) and \( m = 3 \), the mean total priority rank is always equal to 13.5). Although such patterns can be found (at least computationally), they are harder to generate and describe. The tie-breaking rule often plays an important role in school choice assignment and crystallizes a lot of the parents’ concerns. It is therefore important that the tie-breaking rule be transparent and easy to navigate, and official may prefer the suboptimal \( F^{SPR} \) out of simplicity concerns.

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**Appendix**

**A Proofs**

**Additional notation.** For any \( x \in \{1, \ldots, m + 1\} \), school \( R_i(x) \) is the school ranked in position \( x \) in \( R_i \). Similarly, for any \( x \in \{1, \ldots, m\} \), \( Q_i(x) \) is the school reported in position \( x \) in \( Q_i \).
Proof of Proposition 1. We first prove part (i). Consider any \(k \in \{1, \ldots, m\}\), any \(t_i \in T\) and any \(\tilde{S} \subseteq S\) with \(#\tilde{S} \leq k\).

(i) **Sufficiency:** If there exists a set of schools \(\hat{S} \subseteq \tilde{S}\) such that
\[
\#\{t' \in T \setminus \{t\} \mid t' F_s t \text{ for some } s \in \hat{S}\} < \sum_{s \in \hat{S}} q_s,
\]
then set \(\tilde{S}\) is safe for student \(t_i\) in \(DA^k\).

We prove the contrapositive: if the set \(\tilde{S}\) is not safe for \(t_i\) in \(DA^k\), then there exists no subset \(\hat{S} \subseteq \tilde{S}\) such that
\[
\#\{t' \in T \setminus \{t\} \mid t' F_s t \text{ for some } s \in \hat{S}\} < \sum_{s \in \hat{S}} q_s.
\]

By definition, if the set \(\tilde{S}\) is not safe for \(t_i\) in \(DA^k\), then there exists a strategy \(Q'_i \in Q^k_i\) reporting all schools in \(\tilde{S}\) and some \(Q'_{-i} \in Q^k_{-i}\) such that \(t_i\) is unassigned in \(DA^k(Q'_i, Q'_{-i})\). This implies that, over the course of mechanism \(DA^k\) applied to profile \(Q'\), student \(t_i\) has been rejected from all schools in \(\tilde{S}\).

In \(DA^k\), in order for \(t_i\) to be rejected from any school \(s \in S\), at least \(q_s\) students with higher priority at \(s\) than \(t_i\) must be temporarily assigned to \(s\). In turn, these students can only be rejected from \(s\) by other students with higher priority at \(s\) than themselves. Altogether, \(t_i\) having been rejected from all schools in \(\tilde{S}\) implies that, in the list of assignments \(DA^k(Q'_i, Q'_{-i})\), each school \(s \in \tilde{S}\) is filled with students having a higher priority at \(s\) than \(t_i\). Therefore, there are at least \(\sum_{s \in \tilde{S}} q_s\) students with higher priority than \(t_i\) at some school in \(\tilde{S}\). Formally, this means that
\[
\#\{t' \in T \setminus \{t\} \mid t' F_s t \text{ for some } s \in \tilde{S}\} \geq \sum_{s \in \tilde{S}} q_s.
\]

Given that student \(t_i\) has been rejected from all schools in \(\tilde{S}\), the same reasoning leads to the same conclusion for any subset \(\hat{S} \subseteq \tilde{S}\). This shows sufficiency.

(i) **Necessity:** If set \(\tilde{S}\) is safe for student \(t_i\) in \(DA^k\), then there exists a set of schools \(\hat{S} \subseteq \tilde{S}\) such
\[
\#\{t' \in T \setminus \{t\} \mid t' F_s t \text{ for some } s \in \hat{S}\} < \sum_{s \in \hat{S}} q_s.
\]

We prove the contrapositive: if for any \(\hat{S} \subseteq \tilde{S}\)
\[
\#\{t' \in T \setminus \{t\} \mid t' F_s t \text{ for some } s \in \hat{S}\} \geq \sum_{s \in \hat{S}} q_s,
\]
then the set \(\tilde{S}\) is not safe, i.e., there exists a strategy \(Q'_i \in Q^k_i\) reporting all schools in \(\tilde{S}\) and some \(Q'_{-i} \in Q^k_{-i}\) such that \(t_i\) is unassigned in \(DA^k(Q'_i, Q'_{-i})\).
Let $Q'_i$ be any strategy for which $t_i$ reports all the schools in $\tilde{S}$ and no other school. We construct $Q'_{-i}$ such that $DA^k_i(Q'_i, Q'_{-i}) = t_i$.

If we have

$$\#\{t' \in T \setminus \{t\} \mid t' \text{ F}_s t \text{ for some } s \in \tilde{S}\} \geq \sum_{s \in \tilde{S}} q_s,$$

for any $\hat{S} \subseteq \tilde{S}$, then it is possible to partition the set of students

$$\{t_j \in T \setminus \{t_i\} \mid t_j \text{ F}_s t_i \text{ for some } s \in \tilde{S}\}$$

into $\#\hat{S}$ subgroups — one subgroup for each $s \in \tilde{S}$ — such that all students in the subgroup associated to $s$ have higher priority at $s$ than $t_i$. Construct $Q'_{-i}$ such that, for each $s \in \hat{S}$, each student $t_k$ in the subgroup associated to $s$ in the above partition reports $Q'_k : s t_k$. The strategies of other students can be picked arbitrarily.

By construction, $DA^k_i(Q'_i, Q'_{-i}) = t_i$ given that every school $s$ reported in $Q'_i$ has all its seats assigned in the first round of $DA^k$ to students having higher priority at $s$ than $t_i$. This shows that the condition is necessary for the set $\tilde{S}$ to be safe.

We now prove part (ii). Consider any $k \in \{2, \ldots, m\}$, any $t_i \in T$ and any $\hat{S} \subseteq S$ with $\#\hat{S} = k$.

(ii) **Sufficiency** ($\tilde{S}$ is in oversupply): If the set of schools $\tilde{S}$ is in oversupply, then $\tilde{S}$ is safe for student $t_i$ in $BOS^k$.

We prove sufficiency by contraposition: if $\tilde{S}$ is not safe for student $t_i$ in $BOS^k$, then $\tilde{S} \subseteq S$ is not in oversupply.

If $\tilde{S}$ is not safe for student $t_i$ in $BOS^k$, then there exists a strategy $Q'_i \in Q^k_i$ reporting all schools in $\tilde{S}$ and some $Q'_{-i} \in Q^k_{-i}$ such that $t_i$ is unassigned in $BOS^k(Q'_i, Q'_{-i})$. As $BOS^k(Q'_i, Q'_{-i}) = t_i$, over the course of $BOS^k$ applied to profile $Q'$, student $t_i$ has been rejected from all schools in $\tilde{S}$. Thus, for each school $s \in \tilde{S}$, $q_s$ students other than $t_i$ are assigned to $s$ in $BOS^k(Q'_i, Q'_{-i})$. Given that each student is assigned to at most to one school, the last statement implies that $n > \sum_{s \in \tilde{S}} q_s$, which, by definition, means that $\tilde{S}$ is in short-supply.

(ii) **Sufficiency** ($\tilde{S}$ is in short-supply): If $t_i$ has top-priority at all schools in $\tilde{S}$, then $\tilde{S}$ is safe for student $t_i$ in $BOS^k$.

Given that $\#\tilde{S} = k$, all strategies $Q_i \in Q^k_i$ reporting all schools in $\tilde{S}$ only report schools in $\tilde{S}$. As $t_i$ has top-priority at all schools in $\tilde{S}$, student $t_i$ has top-priority at $Q_i(1)$. Therefore $BOS^k(Q_i, Q_{-i}) = Q_i(1)$ for any $Q_{-i} \in Q^k_{-i}$. This shows that $\tilde{S}$ is safe for student $t_i$ in $BOS^k$.

(ii) **Necessity**: If the set of schools $\tilde{S}$ is safe for student $t_i$ in $BOS^k$, then $\tilde{S}$ is in oversupply or $t_i$ has top-priority at all schools in $\tilde{S}$.

We prove sufficiency by contraposition: if $\tilde{S}$ is not in oversupply and there is $s^* \in \tilde{S}$ at which $t_i$ does not have top-priority, then $\tilde{S}$ is not safe for student $t_i$ in $BOS^k$, i.e. there exists a strategy $Q'_i \in Q^k_i$ reporting all schools in $\tilde{S}$ and some $Q'_{-i} \in Q^k_{-i}$ such that $t_i$ is unassigned in $BOS^k(Q'_i, Q'_{-i})$. 29
Let $Q'_i$ be any strategy for which $t_i$ declares all the schools in $\tilde{S}$ and $Q'_i(1) = s^*$. We construct $Q'_{-i}$ such that the $q_{s^*}$ students $t_j$ with top-priority at $s^*$ declare $Q'_j : s^* t_j$. The remaining students $t_k \neq t_i$ are partitioned into $k - 1$ subgroups, one for each $s' \in \tilde{S} \setminus \{ s^* \}$. For each $s' \in \tilde{S} \setminus \{ s^* \}$, the subgroup of students associated with $s'$ has size at least $q_{s'}$. Such a partition is possible given that $\tilde{S}$ is not in oversupply. Any student $t_k$ in the subgroup associated with $s'$ reports $Q'_j : s' t_j$

By construction of $Q'_i$ and $Q'_{-i}$, $t_i$ is rejected from $s^*$ in the first round of $BOS^k$ and all seats at all schools in $\tilde{S}$ are occupied at the end of the first round. Therefore, we have $BOS_i^k(Q'_i, Q'_{-i}) = t_i$, the desired result.

**Proof of Proposition 2.** Consider any $k \in \{1, \ldots, m \}$ and any $t_i \in T$

**Sufficiency:** Sufficiency is directly implied by Lemma 1.

**Lemma 1.** For any $k \in \{1, \ldots, m \}$ and any student $t_i \in T$, (i) if $t_i$ has no more than $k$ acceptable schools, then the truthful strategy $Q'_i : R_i(1) \ldots R_i(\#R_i) t_i$ belongs to $Q^k_i$ and is dominant in $DA^k_i$, (ii) if her $k$ most-preferred schools are all acceptable and form a safe set, then the truthful truncated strategy $Q'_{i}^{*} : R_i(1) \ldots R_i(k) t_i$ belongs to $Q^k_i$ and is dominant in $DA^k_i$

Proof. We prove both claims in turn.

(i) Let $R^{Q'_i}$ be the preference relation over $S \cup \{ t_i \}$ defined as

$$R^{Q'_i} : Q'_i t_i R^{S \setminus Q'_i},$$

where $R^{S \setminus Q'_i}$ is the sub-ordering of the schools in $S \setminus Q'_i$ corresponding to that of preferences $R_i$

Because $DA^m$ is non-manipulable (Dubins and Freedman, 1981), we have

$$DA_i(Q'_i, Q'^m_{-i}) R^{Q'_i} DA_i(Q_i, Q'^m_{-i}),$$

for any $Q'^m_{-i} \in Q_i$ and any $Q_i \in Q_i$

In particular,

$$DA_i(Q'_i, Q^k_{-i}) R^{Q'_i} DA_i(Q_i, Q^k_{-i}),$$

for any $Q^k_{-i} \in Q^k_{-i}$ and any $Q_i \in Q_i$

But because $DA^k_i$ is obtained from $DA^m$ by considering only the profiles $Q^k \in Q^k$, the last displayed relation implies

$$DA^k_i(Q'_i, Q^k_{-i}) R^{Q'_i} DA^k_i(Q_i, Q^k_{-i}),$$

for any $Q^k_{-i} \in Q^k_{-i}$ and any $Q_i \in Q_i$.

By construction, $Q'_i$ is without swap and reports all acceptable schools, and therefore, the last displayed relation implies

$$DA^k_i(Q'_i, Q^k_{-i}) R_i DA^k_i(Q_i, Q^k_{-i}),$$

for any $Q^k_{-i} \in Q^k_{-i}$ and any $Q_i \in Q^k_i$ such that $DA^k_i(Q'_i, Q^k_{-i}), DA^k_i(Q_i, Q^k_{-i}) \in Q^*_i \cup \{ t_i \}$.

(14)
Given that $DA_i^k(Q_i^*, Q_{i-1}^*) \in Q_i^* \cup \{t_i\}$ for any $Q_{i-1}^k \in Q_{i-1}^k$, (14) simplifies to

$$DA_i^k(Q_i^*, Q_{i-1}^k) \ R_i \ DA_i^k(Q_i, Q_{i-1}^k),$$

for any $Q_{i-1}^k \in Q_{i-1}^k$ and any $Q_i \in Q_i^k$ such that $DA_i^k(Q_i, Q_{i-1}^k) \in Q_i^* \cup \{t_i\}$. (15)

By construction of $Q_i^*$, if $DA_i^k(Q_i, Q_{i-1}^k) \notin Q_i^* \cup \{t_i\}$ then $t_i \ P_i \ DA_i^k(Q_i, Q_{i-1}^k)$. Therefore, (14) further simplifies to

$$DA_i^k(Q_i^*, Q_{i-1}^*) \ R_i \ DA_i^k(Q_i, Q_{i-1}^k),$$

for any $Q_{i-1}^k \in Q_{i-1}^k$ and any $Q_i \in Q_i^k$ (16) the desired result.

(ii) Let $R^{Q_i^{**}}$ be any preference relation over $S \cup \{t_i\}$ of the form

$$R^{Q_i^{**}} : Q_i^{**} \ t_i \ Q_i^{S \setminus Q_i^{**}},$$

where $Q_i^{S \setminus Q_i^{**}}$ is any sub-orderings of the schools in $S \setminus Q_i^{**}$. Because $DA$ is non-manipulable (Dubins and Freedman, 1981), we have

$$DA_i(Q_i^{**}, Q_{i-1}^m) \ R^{Q_i^{**}} \ DA_i(Q_i, Q_{i-1}^m),$$

for any $Q_{i-1}^m \in Q_{i-1}^m$ and any $Q_i \in Q_i^i.$

In particular,

$$DA_i(Q_i^{**}, Q_{i-1}^k) \ R^{Q_i^{**}} \ DA_i(Q_i, Q_{i-1}^k),$$

for any $Q_{i-1}^k \in Q_{i-1}^k$ and any $Q_i \in Q_i^k.$

But because $DA_i^k$ is obtained from $DA$ by considering only the profiles $Q^k \in Q_i^k$, the last displayed relation implies

$$DA_i^k(Q_i^{**}, Q_{i-1}^k) \ R^{Q_i^{**}} \ DA_i^k(Q_i, Q_{i-1}^k),$$

for any $Q_{i-1}^k \in Q_{i-1}^k$ and any $Q_i \in Q_i^k.$

By construction, $Q_i^{**}$ is without swap, and therefore, the last displayed relation implies

$$DA_i^k(Q_i^{**}, Q_{i-1}^k) \ R_i \ DA_i^k(Q_i, Q_{i-1}^k),$$

for any $Q_{i-1}^k \in Q_{i-1}^k$ and any $Q_i \in Q_i^k$ such that $DA_i^k(Q_i^{**}, Q_{i-1}^k), DA_i^k(Q_i, Q_{i-1}^k) \in Q_i^{**}. (17)$

Given that the $k$ most-preferred schools of $t_i$ form a safe set, we have that

$$DA_i^k(Q_i^{**}, Q_{i-1}^k) \in Q_i^{**},$$

for any $Q_{i-1}^k \in Q_{i-1}^k,$ and (17) simplifies to

$$DA_i^k(Q_i^{**}, Q_{i-1}^k) \ R_i \ DA_i^k(Q_i, Q_{i-1}^k),$$

for any $Q_{i-1}^k \in Q_{i-1}^k$ and any $Q_i \in Q_i^k$ such that $DA_i^k(Q_i, Q_{i-1}^k) \in Q_i^{**}. (18)$

Finally, because only acceptable schools are reported in $Q_i^{**}$ and because $DA_i^k$ is individually rational, (18) is also true for $Q_{i-1}^k$ such that $DA_i^k(Q_i, Q_{i-1}^k) \notin Q_i^{**}$, and we have

$$DA_i^k(Q_i^{**}, Q_{i-1}^k) \ R_i \ DA_i^k(Q_i, Q_{i-1}^k),$$

for any $Q_{i-1}^k \in Q_{i-1}^k$ and any $Q_i \in Q_i^k. (19)$

the desired result.

■
**Necessity:** If $t_i$ has at least $k+1$ acceptable schools and the $k$ most-preferred schools in $R_i$ do not form a safe set, then $t_i$ does not have a dominant strategy in $DA^k$.

The proof is by contradiction: assume that strategy $Q'_i \in Q^k_i$ is dominant for $t_i$.

- **Case 1:** $Q'_i$ reports the $k$ most-preferred schools in $R_i$.

  As the $k$ most-preferred schools in $R_i$ do not form a safe set, there exists $Q'_{i-1} \in Q^k_{i-1}$ such that

  $$DA^k_i(Q'_i, Q'_{i-1}) = t_i.$$

  Let $s^*$ denote the acceptable school $s^* = R_i(k+1)$ that is not reported in $Q'_i$. By assumption, $t_i$ finds at least $k+1$ schools acceptable and hence $s^* P_i t_i$.

  We construct a profile $Q''_{i-1} \in Q^k_{i-1}$ for which $t_i$ ends up unassigned when playing $Q'_i$, but for which she is assigned to $s^*$ when playing an alternative strategy $Q''_i \in Q^k_i$.

  Formally, $Q''_{i-1}$ such that

  $$DA^k_i(Q'_i, Q''_{i-1}) = t_i \quad \text{and} \quad DA^k_i(Q''_i, Q''_{i-1}) = s^*$$

  for $Q''_i : s^* t_i$. If such profile $Q''_{i-1}$ can be constructed, then $Q'_i$ is not a dominant strategy.

  Profile $Q''_{i-1}$ is constructed as follows. In the course of $DA^k$ applied to $(Q'_i, Q'_{i-1})$, $t_i$ is rejected from all the schools reported in $Q'_i$. Therefore, it must be that, in assignment $DA^k(Q'_i, Q'_{i-1})$, there is another student assigned to each of the available seats in each of the schools reported in $Q'_i$. Let $A \subseteq T$ be the set of all these students who are assigned to a school $t_i$ applied to but was rejected from in $DA^k_i(Q'_i, Q'_{i-1})$.

  Now construct $Q''_{i-1}$ as follows:

  - For any $t_j \in A$, let $Q''_j$ be the strategy in which $t_j$ reports only $DA^k_j(Q'_i, Q'_{i-1})$.
  - For any $t_h \in T \setminus (A \cup \{t_i\})$, let $Q''_h$ be any strategy in which $t_h$ does not report $s^*$.

  By construction, for every school $s_j \in Q'_i$, there are at least $q_j$-students with higher priority at $s_j$ than $t_i$ who rank $s_j$ first in $Q''_{i-1}$. Thus, $t_i$ will be rejected from any of these schools over the course of $DA^k$ given that the reported profile is $(Q'_i, Q''_{i-1})$. Therefore, we have

  $$DA^k_i(Q'_i, Q''_{i-1}) = t_i.$$

  By construction again, no student reports $s^*$ in $Q''_{i-1}$. Therefore,

  $$DA^k_i(Q''_i, Q''_{i-1}) = s^*.$$

- **Case 2:** $Q'_i$ does not report all of the $k$ most-preferred schools in $R_i$.

  This case is such that there is an acceptable school $s^* = R_i(r)$ for some $r \leq k$ such that $s^* \not\in Q'_i$. As the $k$ most-preferred schools in $R_i$ do not form a safe set, there exists $Q'_{i-1} \in Q^k_{i-1}$ such that

  $$s^* P_i DA^k_i(Q'_i, Q'_{i-1}).$$
We show that there also exists $Q''_{i} \in Q_{-i}^{k}$ such that

$$s^* \; P_i \; DA^k_i(Q'_i, Q''_{-i}) \quad \text{and} \quad DA^k_i(Q''_i, Q''_{-i}) = s^*$$

for $Q''_i : s^* \; t_i$. If it is the case, then $Q'_i$ is not a dominant strategy.

Let $S' = \{s \in Q'_i | s \in DA^k_i(Q'_i, Q'_{-i})\}$ be the set of schools that $t_i$ ranks in $Q'_i$ above her assignement $DA^k_i(Q'_i, Q'_{-i})$. Let $A$ be the subset of students who are assigned to a school in $S'$ in the list of assignments $DA^k(Q'_i, Q'_{-i})$. Now, construct $Q''_{i}$ as follows:

- For any $t_j \in A$, let $Q''_{j}$ be the strategy in which $t_j$ reports only $DA^k_j(Q'_i, Q'_{-i})$.
- For any $t_h \in T \setminus \{A \cup \{t_i\}\}$, let $Q''_{h}$ be any strategy in which $t_h$ does not report $s^*$.

By construction, for every school $s_j \in S'$, there are at least $q_j$-students with higher priority at $s_j$ than $t_i$ who rank $s_j$ first in $Q''_{-i}$. Therefore,

$$s^* \; P_i \; DA^k_i(Q'_i, Q''_{-i}).$$

Also, no student reports $s^*$ in $Q''_{-i}$, which implies

$$DA^k_i(Q''_i, Q''_{-i}) = s^*.$$ 

**Proof of Proposition 3.** Consider any $k \in \{1, \ldots, m\}$ and any $t_i \in T$.

**Sufficiency** is directly implied by Lemma 2.

**Lemma 2.** For any $k \in \{1, \ldots, m\}$ and any student $t_i \in T$, (i) if $t_i$ has only one acceptable school, then the truthful strategy $Q^*_i : R_i(1) \; t_i \; \text{belongs to} \; Q^k_i$ and is dominant in $BOS^k_i$, (ii) if she has top-priority at her most-preferred school, then the truthful truncated strategy $Q^{**}_i : R_i(1) \; t_i \; \text{belongs to} \; Q^k_i$ and is dominant in $BOS^k_i$.

Proof. We prove both claims in turn.

(i) We show that

$$BOS^k_i(Q^*_i, Q_{-i}) \; R_i \; BOS^k_i(Q_i, Q_{-i}), \quad \text{for any} \; Q_{-i} \in Q^k_{-i} \; \text{and any} \; Q_i \in Q^k_i. \quad (20)$$

Take any $Q_i \in Q^k_i$ and any $Q_{-i} \in Q^k_{-i}$. Two cases may arise:

- **Case 1:** $BOS^k_i(Q_i, Q_{-i}) = R_i(1)$. Given that school $R_i(1)$ is ranked first in $Q'_i$, we have that

$$BOS^k_i(Q, Q_{-i}) = R_i(1) \quad \text{implies that} \quad BOS^k_i(Q^*_i, Q_{-i}) = R_i(1),$$

and therefore (20) holds.

- **Case 2:** $BOS^k_i(Q_i, Q_{-i}) \neq R_i(1)$.

As $t_i$ only has one acceptable school, this case is such that $t_i \; R_i \; BOS^k_i(Q_i, Q_{-i})$. By definition of $BOS^k_i$, we have that $BOS^k_i(Q^*_i, Q_{-i}) \in \{R_i(1), t_i\}$ and therefore (20) holds.
(ii) Given that \( t_i \) has top-priority at \( R_i(1) \), strategy \( Q_i^{**} \) is such that
\[
BOS_i^k(Q_i^{**}, Q_{-i}) = R_i(1) \quad \text{for any} \quad Q_{-i} \in Q_i^{-}
\]
and is therefore dominant.

\[\blacksquare\]

**Necessity:** If \( t_i \) has at least 2 acceptable schools and does not have top-priority at her most-preferred school, then \( t_i \) does not have a dominant strategy in \( BOS_i^k \). Take any strategy \( Q_i' \in Q_i^k \). We show that \( Q_i' \) cannot be a dominant strategy for \( t_i \) in \( BOS_i^k \).

- **Case 1:** \( Q_i' \) does not rank \( R_i(1) \) first.
  
  This case is such that \( Q_i': s^* \ldots \) for some \( s^* \neq R_i(1) \). Consider any \( Q_{-i} \in Q_i^{-} \) such that all students \( t_j \neq t_i \) report neither \( s^* \) nor \( R_i(1) \). By construction, we have
  
  \[
  BOS_i^k(Q_i', Q_{-i}) = s^* \quad \text{and} \quad BOS_i^k(Q_i^*, Q_{-i}) = R_i(1),
  \]
  
  which shows that \( Q_i' \) is not a dominant strategy.

- **Case 2:** \( Q_i' \) ranks \( R_i(1) \) first.
  
  We show that there exists a profile \( Q_{-i}' \in Q_i^{-} \) such that \( t_i \) is assigned to a school different from \( R_i(1) \) and \( R_i(2) \) when playing \( Q_i' \), but is assigned to \( R_i(2) \) when playing \( Q_i'' = R_i(2) \).

  - **Subcase 2.1:** \( t_i \) has top-priority at \( R_i(2) \).

    Profile \( Q_{-i}' \) is constructed as follows
    
    * Take the \( q_{R_i(1)} \) students who have a top-priority at \( R_i(1) \). These students only report \( R_i(1) \). Observe that there are at least \( q_{R_i(1)} \) such students given that \( t_i \) does not have top-priority at this school.
    
    * All remaining students only report \( R_i(2) \).

    The model assumes that no two schools have enough seats to jointly host all students. As a result, all the seats in schools \( R_i(1) \) and \( R_i(2) \) are filled in the first round of \( BOS_i^k \) applied to \( (Q_i', Q_{-i}) \). Therefore, we have \( R_i(2) \) \( P_i \) \( BOS_i^k(Q_i', Q_{-i}) \).

    Yet, given that \( t_i \) has a top-priority at \( R_i(2) \), we have \( BOS_i^k(Q_i'', Q_{-i}) = R_i(2) \).

    Together, we have
    
    \[
    BOS_i^k(Q_i', Q_{-i}) \quad P_i \quad BOS_i^k(Q_i'', Q_{-i}),
    \]
    
    and \( Q_i' \) is not a dominant strategy.

  - **Subcase 2.2:** \( t_i \) does not have top-priority at \( R_i(2) \).

    Profile \( Q_{-i}' \) is constructed as follows
    
    * Take the \( q_{R_i(1)} \) students that have a top-priority at \( R_i(1) \). These students only report \( R_i(1) \).
    
    * If among the remaining students, there are at least \( q_{R_i(2)} \) students that have higher priority than \( t_i \) at \( R_i(2) \), then for any remaining student \( t_j \) let \( Q_j': R_i(1) R_i(2) t_j \), otherwise all remaining students only report \( R_i(2) \).
Because there are not enough seats in $R_i(1)$ and $R_i(2)$ to jointly host all students, we have by construction that

$$R_i(2) \ P_i \ BOS_i^k(Q_i', Q_{-i}') \quad \text{and} \quad BOS_i^k(Q_i'', Q_{-i}') = R_i(2),$$

which shows that $Q_i'$ is not a dominant strategy.

**Proof of Corollary 1.** Consider any $\ell \in \{2, \ldots, m\}$, any $k \in \{2, \ldots, m\}$ and any $t_i \in T$.

**Step 1.** For each $R_i$ at which $t_i$ has a dominant strategy in $BOS^\ell$, $t_i$ also has a dominant strategy in $DA^k$.

By Proposition 3, $t_i$ has a dominant strategy in $BOS^\ell$ only in two cases

- **Case 1:** $t_i$ has only one acceptable school.
  This case is such that $t_i$ has no more than $k$ acceptable schools. Then, by Proposition 2, $t_i$ also has a dominant strategy in $DA^k$.

- **Case 2:** $t_i$ has a top-priority at her most-preferred school.
  - Subcase 2.1: $t_i$ has no more than $k$ acceptable schools.
    By Proposition 2, $t_i$ has a dominant strategy in $DA^k$.
  - Subcase 2.2: $t_i$ has more than $k$ acceptable schools.
    As $t_i$ has top-priority at her most-preferred school, there are at most $q_{R_i(1)} - 1$ students with a higher priority at $R_i(1)$ than $t_i$. Then, by Proposition 1, $t_i$’s $k$ most-preferred schools form a safe set. By Proposition 2, $t_i$ then has a dominant strategy in $DA^k$.

**Step 2.** There exist $R_i$, $F$ and $q$ such that $t_i$ has a dominant strategy in $DA^k$ but not in $BOS^\ell$.

Take any $R_i$, $F$ and $q$ such that $t_i$ (a) finds all schools acceptable, (b) does not have top-priority at $R_i(1)$, but (c) has a top-priority at $R_i(2)$. By Proposition 3, $t_i$ does not have a dominant strategy in $BOS^\ell$. By Proposition 1, her 2 most-preferred schools form a safe set. By Proposition 1 again, her $k$ most-preferred schools form a safe set. Then, by Proposition 2, $t_i$ has a dominant strategy in $DA^k$.

**Proof of Corollary 2.** Consider any $k \in \{1, \ldots, \min(m-1, n-1)\}$ and any $t_i \in T$.

**Step 1.** For each $R_i$ at which $t_i$ has a truthful dominant strategy in $DA^k$, $t_i$ has a truthful dominant strategy in $DA^{k+1}$.

By Proposition 2, $t_i$ has a dominant strategy in $DA^k$ only in two cases

- **Case 1:** $t_i$ has no more than $k$ acceptable schools.
  This case is such that $t_i$ has no more than $k + 1$ acceptable schools. By Proposition 2, $t_i$ has a dominant strategy in $DA^{k+1}$.

35
• **Case 2:** the $k$ most-preferred schools of $t_i$ are all acceptable and form a safe set.

  If $t_i$ has no more than $k + 1$ acceptable schools, then $t_i$ has a dominant strategy in $DA^{k+1}$ by Proposition 2. Otherwise, $t_i$ has more than $k + 1$ acceptable schools. Given that her $k$ most-preferred schools form a safe set, her $k + 1$ most-preferred schools also form a safe set (Proposition 1). Then, by Proposition 2, $t_i$ has a dominant strategy in $DA^{k+1}$.

**Step 2.** There exist $R_i$, $F$ and $q$ such that $t_i$ has a dominant strategy in $DA^{k+1}$ but not in $DA^k$.

  Take any $R_i$ with $k + 1$ acceptable schools. Take $q$ such that there is only one seat in each school. Take any $F$ such that $t_i$ has top-priority at $R_i(k + 1)$ but has the lowest priority of all students at all other schools. As $k < n$, the $k$ most-preferred schools of $t_i$ are not oversupplied. By Proposition 1, the $k$ most-preferred schools of $t_i$ do not form a safe set. Thus, by Proposition 2, $t_i$ does not have a dominant strategy in $DA^k$. By Proposition 1, the $k+1$ most-preferred schools of $t_i$ form a safe set. Thus, by Proposition 2, $t_i$ has a dominant strategy in $DA^{k+1}$.

**Proof of Corollary 3.** Consider any $k \in \{1, \ldots, m - 1\}$ and any $t_i \in T$. For each $R_i$ at which $t_i$ has a dominant strategy in $BOS^k$, $t_i$ has a dominant strategy in $BOS^r$ for any $r \in \{1, \ldots, m\}$. By Proposition 3, $t_i$ has a dominant strategy in $BOS^k$ only in two cases

  • **Case 1:** $t_i$ has only one acceptable school.

  By Proposition 2, $t_i$ has a dominant strategy in $DA^r$.

  • **Case 2:** $t_i$ has a top-priority at her most-preferred school.

  By Proposition 2, $t_i$ has a dominant strategy in $DA^r$.

**Proof of Corollary 4.** There are two cases to consider: either sets of $k$ schools are oversupplied, or not.

  • **Case 1:** Over-supply: $kq \geq n$.

    If $kq \geq n$, then all sets of $k$ schools are oversupplied. By Proposition 1, all sets of $k$ schools are safe sets for any students. Hence, the $k$ most-preferred schools of all students form a safe set. By Proposition 2, all students have a dominant strategy in $DA^k$. Hence we have $\mathbb{E}_{DA^k}(g^\text{STB}) = n$. Given that $kq \geq n$, we have $\mathbb{E}_{DA^k}(g^\text{STB}) = \min\{kq, n\}$, the desired result.

  • **Case 2:** Short-supply: $kq < n$.

    STB creates perfect correlation across priorities at all schools. We show that all students whose priority rank is smaller or equal to $kq$ have a dominant strategy, unlike students with a priority rank larger than $kq$.

    Take any $t_i$ with priority rank smaller or equal to $kq$. Denote by $S^k$ the set of her $k$ most-preferred schools. Given the perfect correlation in priorities created by STB, we have that

    $\#\{t' \in T \setminus \{t\} \mid t' \not\sim s \text{ for some } s \in S^k\} < kq = \sum_{s \in S^k} q_s.$
By Proposition 1, the set $S^k$ is safe for $t_i$. By Proposition 2, $t_i$ has a dominant strategy in $DA^k$.

Take any $t_i$ with priority rank larger than $kq$. Denote by $S^k$ the set of her $k$ most-preferred schools. Given the perfect correlation in priorities created by STB, we have that

$$\#\{t' \in T \setminus \{t\} \mid t' F_s t \text{ for some } s \in S^k\} \geq kq = \sum_{s \in S^k} q_s.$$ 

By Proposition 1, the set $S^k$ is not safe for $t_i$. We assumed that all students have more than $k$ acceptable schools. Therefore, by Proposition 2, only students for whom the $k$ most-preferred schools form a safe set have a dominant strategy in $DA^k$. Therefore, $t_i$ does not have a dominant strategy in $DA^k$.

Given that there are $kq$ students whose priority rank is smaller or equal to $kq$, we have that $E_{dom}^{DA^k}(g^{STB}) = kq$. Given that $kq < n$, we have $E_{dom}^{DA^k}(g^{STB}) = \min\{kq, n\}$, the desired result.

**Proof of Corollary 5.** Take any $F$ in $\text{supp}(g)$. Given $F$, the expected number of students with a dominant strategy in $BOS^k$ is given by (4):

$$E_{dom}^{BOS^k}(F) := \frac{1}{\#R_1} \sum_{t_i \in T} \sum_{R_i \in R_i} 1(R_i, F),$$

where $1(R_i, F)$ is an indicator function which takes value 1 when $t_i$ has a dominant strategy in $BOS^k$ given preferences $R_i$ and priorities $F$, and zero otherwise. We assumed that all students have more than $k$ acceptable schools. Therefore, by Proposition 3, only students who have top-priority at their most-preferred school have a dominant strategy in $BOS^k$.

For any $t_i \in T$, we have that $1(R_i, F) = 1$ if $t_i$ has top-priority at $R_i(1)$ in profile $F$ and $1(R_i, F) = 0$ otherwise. Letting $p_i(F) \in \{0, \ldots, m\}$ denote the number of schools at which $t_i$ has top-priority for the priority profile $F$, we have that

$$\frac{1}{\#R_1} \sum_{R_i \in R_i} 1(R_i, F) = \frac{p_i(F)}{m}.$$ 

Aggregating over all students in $T$, we obtain

$$E_{dom}^{BOS^k}(F) = \frac{1}{m} \sum_{t_i \in T} p_i(F).$$ 

Given that exactly $q$ students receive top-priority at each school, we have that $\sum_{t_i \in T} p_i(F) = mq$. Therefore, the last equation becomes

$$E_{dom}^{BOS^k}(F) = q,$$

where $q < n$ because no two schools have enough capacity to host all students. The last equation shows that $E_{dom}^{BOS^k}(F)$ does not depend on the priority profile $F$. Hence, all distributions $g$ on $F$ are such that $E_{BOS^k}^{dom}(g) = q$, the desired result.
Proof of Proposition 4. The following lemma shows that student $t_i$ has an expected dominant strategy in $DA^k$ only when the strategy obtained by truncating preferences $R_i$ after $R_i(k)$ is expected dominant. Recall that we assume every student $t_i$ to have at least $k + 1$ acceptable schools.

Lemma 3. For any $k \in \{1, \ldots, m\}$, in the absence of pre-existing priorities and given an ex-ante fair tie-breaking rule $g$, a student $t_i$ whose $k$ most-preferred schools are in short-supply has an expected dominant strategy in $DA^k$ only if the strategy $Q_i^*$: $R_i(1) R_i(2) \ldots R_i(k)$ $t_i$ is an expected dominant strategy.

Proof. We show that no strategy $Q_i' \in Q_i^k$ dominates $Q_i^*$ (in expected terms) in $DA^k$. If this is the case, then it is only when $Q_i^*$ is an expected dominant strategy that $t_i$ has an expected dominant strategy. To do so, we identify for any strategy $Q_i' \in Q_i^k$ different from $Q_i^*$ a profile $Q_{-i}' \in Q_{-i}^k$ for which strategy $Q_i$ is such that

$$\mathbb{E}_i(Q_i^*, Q_{-i}; DA^k) > \mathbb{E}_i(Q_i', Q_{-i}; DA^k).$$

Take any strategy $Q_i' \in Q_i^k$ different from $Q_i^*$. Let $a \in \{1, \ldots, k\}$ be the smallest rank for which $Q_i'(a) \neq R_i(a)$. That is, strategies $Q_i^*$ and $Q_i'$ coincide for ranks smaller than $a$: $Q_i'(1) = Q_i^*(1), \ldots, Q_i'(a-1) = Q_i^*(a-1)$, but not at rank $a$ for which $Q_i'(a) \neq Q_i^*(a) = R_i(a)$ (they need not coincide either at ranks larger than $a$).

Let $Q_{-i}'$ be such that all other students report the $a - 1$ most-preferred schools of $t_i$ without swaps, and no other schools. That is, for any $t_j \neq t_i$, we have $Q_j'(1) = R_i(1), \ldots, Q_j'(a-1) = R_i(a-1)$, and $Q_j(a) = t_j$.

By construction, for any priority profiles, we have either that

$$DA^k_i(Q_i^*, Q_{-i}') = DA^k_i(Q_i', Q_{-i}')$$

or that

$$DA^k_i(Q_i^*, Q_{-i}') = R_i(a) \quad \text{and} \quad DA^k_i(Q_i', Q_{-i}') = Q_i'(a).$$

As we assume that (i) the $k$ most-preferred schools are in short-supply, (ii) there are no pre-existing priorities, and (iii) the tie breaking rule is ex-ante fair, there exist priority profiles in the support of $g$ for which (22) holds. By the definition of $a$, we have that $R_i(a) P_i Q_i'(a)$, and hence $u_i(R_i(a)) > u_i(Q_i'(a))$. This shows that

$$\mathbb{E}_i(Q_i^*, Q_{-i}; DA^k) > \mathbb{E}_i(Q_i', Q_{-i}; DA^k),$$

the desired result.

Take any $k \in \{1, \ldots, m - 1\}$ and any ex-ante fair tie-breaking rule $g$. By Lemma 3, student $t_i$ has a dominant strategy if and only if $Q_i^*$: $R_i(1) R_i(2) \ldots R_i(k)$ $t_i$ is such that

$$\mathbb{E}_i(Q_i^*, Q_{-i}; DA^k) \geq \mathbb{E}_i(Q_i', Q_{-i}; DA^k) \quad \text{for any } Q_i' \in Q_i^k \text{ and any } Q_{-i} \in Q_{-i}^k.$$
Let the strategy $Q''_i$ be defined as

$$Q''_i : R_i(1) \ R_i(2) \ldots \ R_i(k-1) \ R_i(k+1) \ t_i$$

Strategy $Q''_i$ differs from $Q'_i$ only because $Q''_i(k) = R_i(k+1) \neq R_i(k) = Q'_i(k)$.

Let the profile $Q''_{-i}$ be such that $Q''_j = Q'_j$ for any $t_j \neq t_i$. In profile $Q''_{-i}$, all other students report the same preferences as $t_i$ when she reports $Q'_i$.

Under strategy profile $(Q''_i, Q''_{-i})$, the situation is ex-ante symmetric: all students report the same preferences, there are no pre-existing priorities and the tie-breaking rule is ex-ante fair. As a result, all students have the same probability of being assigned to any of the schools they report. Hence, for any school $s \in Q'_i$, the probability that $t_i$ be assigned to $s$ is

$$\text{Proba}(t_i \text{ assigned to } s|(Q'_i, Q''_{-i})) = \frac{q}{n}.$$ 

Under strategy profile $(Q''_i, Q''_{-i})$, the situation is also ex-ante symmetric for the $k-1$ most-preferred schools of $t_i$. Hence, for any school $s \in Q'_i$ such that $s R_i R_i(k-1)$, the probability that $t_i$ be assigned to $s$ is

$$\text{Proba}(t_i \text{ assigned to } s|(Q''_i, Q''_{-i})) = \frac{q}{n}.$$ 

In profile $(Q''_i, Q''_{-i})$, $t_i$ is the only student who reports $R_i(k+1)$. Therefore, if $t_i$ is not assigned to a school she ranks above $R_i(k)$, then she is assigned to $R_i(k+1)$ with probability one. Hence, the probability that $t_i$ is assigned to $R_i(k+1)$ is

$$\text{Proba}(t_i \text{ assigned to } R_i(k+1)|(Q''_i, Q''_{-i})) = 1 - \frac{q}{n} * (k-1),$$

where $\frac{q}{n} * (k-1) < 1$ given that the $k$ most-preferred schools are in short-supply.

When the sub-profile is $Q''_{-i}$, the difference in expected utility $t_i$ associates with playing $Q'_i$ instead of $Q''_i$ is

$$\mathbb{E}_i(Q'_i, Q''_{-i}; DA^k) - \mathbb{E}_i(Q''_i, Q''_{-i}; DA^k) = u_i(k) \frac{q}{n} + \left(1 - \frac{kq}{n}\right) u_i(t_i) - u_i(k+1) \left(1 - \frac{q}{n} * (k-1)\right),$$

(recall that $u_i(b)$ denotes the utility that $t_i$ associates with school $R_i(b)$.

Therefore, the necessary condition is obtained from

$$\mathbb{E}_i(Q'_i, Q''_{-i}; DA^k) \geq \mathbb{E}_i(Q''_i, Q''_{-i}; DA^k)$$

$$u_i(k) \frac{q}{n} + \left(1 - \frac{kq}{n}\right) u_i(t_i) \geq u_i(k+1) \left(1 - \frac{q}{n} * (k-1)\right),$$

$$(u_i(k) - u_i(t_i)) \frac{q}{n} \geq (u_i(k+1) - u_i(t_i)) \left(1 - \frac{q}{n} * (k-1)\right).$$

Because we normalize $u_i(t_i) = 0$, this yields

$$u_i(k) \frac{q}{n} \geq u_i(k+1) \left(1 - \frac{q}{n} * (k-1)\right),$$

$$u_i(k) \geq u_i(k+1) \left(\frac{n}{q} - (k-1)\right),$$

39
the desired result.

Proof of Proposition 5. The following lemma shows that student \( t_i \) has an expected dominant strategy in \( BOS^k \) only when the strategy obtained by truncating preferences \( R_i \) after \( R_i(k) \) is expected dominant. Remember we assume that \( t_i \) has at least \( k+1 \) acceptable schools.

Lemma 4. For any \( k \in \{1, \ldots, m\} \), in the absence of pre-existing priorities and given an ex-ante fair tie-breaking rule \( g \), a student \( t_i \) has an expected dominant strategy in \( BOS^k \) only if the strategy \( Q_i^* : R_i(1) R_i(2) \ldots R_i(k) t_i \) is an expected dominant strategy.

Proof. The proof uses the same argument as the proof of Lemma 3 and is therefore omitted.

Take any \( k \in \{1, \ldots, m-1\} \) and any ex-ante fair tie-breaking rule \( g \). By Lemma 4, student \( t_i \) has an expected dominant strategy in \( BOS^k \) only if strategy \( Q_i^* : R_i(1) R_i(2) \ldots R_i(k) t_i \) is an expected dominant strategy. The necessary condition stated in Proposition 5 is obtained by considering the particular strategy \( Q''_i : R_i(2) t_i \) and a particular profile \( Q''_{-i} \in Q^k_{-i} \) and checking whether

\[
\mathbb{E}_i(Q_i^*, Q''_{-i}; BOS^k) \geq \mathbb{E}_i(Q''_i, Q''_{-i}; BOS^k).
\]

Let the profile \( Q''_{-i} \) be such that \( Q''_j = Q_i^* \) for any \( j \neq i \).

Under strategy profile \((Q''_i, Q''_{-i})\), the situation is ex-ante symmetric: all students report the same preferences, there are no pre-existing priorities and the tie-breaking rule is ex-ante fair. As a result, all students have the same probability of being assigned to any of the school they report. Hence, for any school \( s \in Q_i^* \), the probability that \( t_i \) be assigned to \( s \) is

\[
\text{Prob}(t_i \text{ assigned to } s | (Q_i^*, Q''_{-i})) = \frac{q}{n}.
\]

Under strategy profile \((Q''_i, Q''_{-i})\), student \( t_i \) is the only one to rank \( R_i(2) \) as her favorite school and therefore \( BOS^k_i(Q''_i, Q''_{-i}) = R_i(2) \).

When the profile is \( Q''_{-i} \), a necessary condition for \( Q_i^* \) to be an expected dominant strategy is

\[
\mathbb{E}_i(Q_i^*, Q''_{-i}; BOS^k) \geq \mathbb{E}_i(Q''_i, Q''_{-i}; BOS^k),
\]

\[
\sum_{\ell=1}^k u_i(\ell) \frac{q}{n} + \left(1 - \frac{kq}{n}\right) u_i(t_i) \geq u_i(2),
\]

and because we normalize \( u_i(t_i) = 0 \), this is

\[
\sum_{\ell=1}^k u_i(\ell) \frac{q}{n} \geq u_i(2),
\]

and given that \((k-2)u_i(3) \geq \sum_{\ell=3}^k u_i(\ell)\), the following is also a (weaker) necessary condition

\[
u_i(1) \frac{q}{n} + u_i(2) \frac{q}{n} + \left(1 - \frac{2q}{n}\right) u_i(3) \geq u_i(2),
\]

\[
u_i(1) \frac{q}{n} + \left(1 - \frac{2q}{n}\right) u_i(3) \geq \left(1 - \frac{q}{n}\right) u_i(2),
\]
and given that \((1 - \frac{q}{n}) \geq (1 - \frac{2q}{n})\), the following is also a (weaker) necessary condition
\[
\frac{1}{n} \left(1 - \frac{q}{n}\right) u_i(3) \geq \left(1 - \frac{q}{n}\right) u_i(2),
\]
\[
\frac{1}{n} \left(1 - \frac{q}{n}\right) u_i(3) \geq u_i(2) - u_i(3),
\]
the desired result.

**Proof of Proposition 6.** We first present Lemmas 5 and 6 that provide new expressions for \(E_{DA_{\text{dom}}}^\text{dom}(F)\). Observe that by (4)
\[
E_{DA_{\text{dom}}}^\text{dom}(F) = \frac{1}{\# \mathcal{R}_1} \sum_{R_i \in \mathcal{R}_1} \sum_{t_i \in T} 1(R_i, F).
\]
Recall that \(\# \mathcal{R}_1\) is the total number of preference profiles and \(1(R_i, F)\) is an indicator function equal to 1 when \(t_i\) has a dominant strategy in \(DA^k\) with preferences \(R_i\) and priorities \(F\), and zero otherwise.

For any priority profile \(F\), any \(k \in \{1, \ldots, m\}\) and any \(t_i \in T\), let \(SS_i^k(F)\) be the number of sets of \(k\) schools that are safe for \(t_i\) given \(F\).

**Lemma 5** (\(E_{DA_{\text{dom}}}^\text{dom}\) depends on the number of safe sets). For any priority profile \(F\) and any \(k \in \{1, \ldots, m\}\),
\[
\frac{1}{\# \mathcal{R}_1} \sum_{R_i \in \mathcal{R}_1} \sum_{t_i \in T} 1(R_i, F) = c \sum_{t_i \in T} SS_i^k(F),
\]
for some positive constant \(c\).

Proof. For any priority profile \(F\), any \(k \in \{1, \ldots, m\}\) and any \(t_i \in T\), let \(1^k(R_i, F)\) equal 1 if \(t_i\) has a safe set covering her \(k\) most-preferred schools given \(F\) and \(R_i\). Because we assumed that all students find at least \(k + 1\) schools acceptable, we have by Proposition 2 that \(1(R_i, F) = 1^k(R_i, F)\). Slightly abusing the notation, for any set of \(k\) schools \(S^k\), let \(\#S_i^k\) be the number of preferences in \(\mathcal{R}_i\) for which \(t_i\)'s \(k\) most-preferred schools are the schools in \(S^k\). As \(\mathcal{R}_i\) contains all linear orders on \(S\) with at least \(k + 1\) acceptable schools, \(\#S_i^k\) is identical for every \(S^k\). Therefore, we can let \(\tau := \#S_i^k\) and have \(S_i^k = \tau\) for all \(t_i \in T\), where \(\tau\) is a positive constant. Also, let \(1_i(S^k, F)\) equal 1 if \(S^k\) is a safe set for \(t_i\) given \(F\) and zero otherwise. We then have
\[
\frac{1}{\# \mathcal{R}_1} \sum_{R_i \in \mathcal{R}_1} \sum_{t_i \in T} 1(R_i, F) = \frac{1}{\# \mathcal{R}_1} \sum_{R_i \in \mathcal{R}_1} \sum_{t_i \in T} 1^k(R_i, F)
\]
\[
= \frac{1}{\# \mathcal{R}_1} \sum_{t_i \in T} \sum_{R_i \in \mathcal{R}_1} 1^k(R_i, F)
\]
\[
= \frac{1}{\# \mathcal{R}_1} \sum_{t_i \in T} \sum_{\{S^k \subseteq S \mid \#S^k = k\}} 1_i(S^k, F) \#S_i^k
\]
\[
= \frac{1}{\# \mathcal{R}_1} \sum_{t_i \in T} \tau \sum_{\{S^k \subseteq S \mid \#S^k = k\}} 1_i(S^k, F)
\]
\[
= \frac{\tau}{\# \mathcal{R}_1} \sum_{t_i \in T} SS_i^k(F),
\]
41
where \( c := \frac{r}{\#R_i} \) is a positive constant.

For any \( k \in \{1, \ldots, m\} \), any \( F \), and any \( t_i \in T \), a set of \( k \) schools \( S^k \) is a safe set of order 1 if \( S^k \) contains a subset 1 \( \tilde{S} \subseteq S^k \) that is is safe for \( t_i \) in \( DA^1 \). A set \( S^k \) is a safe set of order 2 if \( S^k \) does not contain a subset \( \tilde{S} \subseteq S^k \) that is is safe for \( t_i \) in \( DA^1 \), but \( S^k \) contains a subset \( \tilde{S} \subseteq S^k \) that is safe for \( t_i \) in \( DA^2 \). Similarly, \( S^k \) is a safe set of order \( \ell \leq k \) if \( S^k \) does not contain a subset \( \tilde{S} \subseteq S^k \) that is is safe for \( t_i \) in \( DA^{\ell-h} \) for any \( h \in \{1, \ldots, \ell - 1\} \), but \( S^k \) contains a subset \( \tilde{S} \subseteq S^k \) that is safe for \( t_i \) in \( DA^\ell \).

We let \(#S^k_i(F, \ell)\) denote the number of sets of \( k \) schools that are safe sets of order \( \ell \) for student \( t_i \) in \( DA^k \) given that the priority profile is \( F \).

**Lemma 6** (Safe sets can be counted order by order).

\[
\sum_{t_i \in T} SS^k_i(F) = \sum_{\ell=1}^k \sum_{t_i \in T} \#S^k_i(F, \ell).
\]

Proof. Simply observe that (i) every safe set of size \( k \) is a safe set of order \( \ell \) for some \( \ell \leq k \) and (ii) by definition, no safe set can be of two different orders.

Next, for \( k = 2 \), \( q = 1 \) and for an arbitrary \( F \), we present Lemma 7 that computes the number of safe sets of order 1 generated by \( F \). Lemma 7 relies the \( n \) dimensional vector \( x(F) \) whose components represent the number of schools at which each student has a top-priority given \( F \). For example, if \( T = \{t_1, \ldots, t_6\} \) and there are 4 schools with \( F_1 : t_1 \ldots, F_2 : t_1 \ldots, F_3 : t_2 \ldots, F_4 : t_6 \ldots \), then \( x(F) = (2, 1, 0, 0, 0, 1) \). The value of \( x(F) \) for student \( t_i \in T \) is denoted by \( x_i(F) \). Observe that when \( q = 1 \) we have for every \( F \) that

\[
\sum_{t_i \in T} x_i(F) = m
\]

(23)

Vector \( x(F) \) turns out to summarize all the information on \( F \) that is necessary for our purpose.

**Lemma 7** (Computing \(#S^k_i(F, 1)\) based on \( x(F) \) when \( k = 2 \)). For \( q = 1 \),

\[
\sum_{t_i \in T} \#S^2_i(F, 1) = \sum_{t_i \in T} \frac{(m - 1) + (m - x_i(F))}{2} x_i(F).
\]

Proof. Without loss of generality, suppose that \( t_i \) has a top-priority at schools 1 to \( x_i(F) \). Then every pair of schools \( \{s_x(s_y) \mid s_x \in \{1, \ldots, x_i(F)\}\} \) is a safe sets of order 1 for \( t_i \) (and not other set is). These pairs are

\[
(s_1, s_2), \ldots, (s_1, s_m) \rightarrow (m - 1) \text{ pairs},
\]
\[
(s_2, s_3), \ldots, (s_2, s_m) \rightarrow (m - 2) \text{ pairs},
\]
\[
\vdots
\]
\[
(s_{x_i(F)}, s_{x_i(F) + 1}), \ldots, (s_{x_i(F)}, s_m) \rightarrow (m - (x_i(F))) \text{ pairs}.
\]

In total, there are \( \frac{(m-1)+(m-x_i(F))}{2} x_i(F) \) such pairs. The desired result.
Next, for $k = 2$, $q = 1$ and for an arbitrary $F$, we present Lemma 8 that provides an upper-bound on the number of safe sets of order 2 generated by $F$.

**Lemma 8** (Bounding $\# S_k^2(F,2)$ based on $x(F)$ when $k = 2$). For $q = 1$,

$$\sum_{t_i \in T} \# S_k^2(F,2) \leq \sum_{t_i \in T} \frac{(x_i(F) - 1)x_i(F)}{2}.$$  

Proof. Given the distribution of top-priorities, the maximum value of $\sum_{t_i \in T} \# S_k^2(F,2)$ when $q = 1$ is obtained when a student $t^*$ who does not have any top-priority at any school has the second highest priority at each and every school. In this case, for any student $t_j \neq t^*$, any pair $(s_x, s_y)$ such that $t_j$ has a top-priority at $s_x$ and $s_y$ is a safe set for $t^*$.

Again, assume without loss of generality that $t_j$ has a top-priority at schools 1 to $x_j(F)$.

These pairs are

$$\begin{align*}
(s_1, s_2), & \quad \ldots, \quad (s_1, s_{x_j(F)}) \quad \rightarrow x_j(F) - 1 \text{ pairs,} \\
(s_2, s_3), & \quad \ldots, \quad (s_2, s_{x_j(F)}) \quad \rightarrow x_j(F) - 2 \text{ pairs,} \\
& \quad \vdots \\
(s_{x_j(F)-2}, s_{x_j(F)-1}), & \quad (s_{x_j(F)-2}, s_{x_j(F)}) \quad \rightarrow 2 \text{ pairs,} \\
(s_{x_j(F)-1}, s_{x_j(F)}) & \quad \rightarrow 1 \text{ pair.}
\end{align*}$$

In total, there are $\frac{(x_j(F)-1)+1}{2} (x_j(F) - 1) = \frac{(x_j(F)-1)x_j(F)}{2}$ such pairs for student $t_j$. Because $t^*$ does not have top-priorities, all these safe sets are of order 2. The total number of safe sets for $t^*$ is the sum over all students $t_j$ of $\frac{(x_j(F)-1)x_j(F)}{2}$. But because the number of safe sets for $t^*$ is the maximum of $\sum_{t_i \in T} \# S_k^2(F,2)$, we have

$$\sum_{t_i \in T} \# S_k^2(F,2) \leq \sum_{t_i \in T} \frac{(x_i(F) - 1)x_i(F)}{2}.$$  

$\blacksquare$

Put together, Lemmas 7 and 8 yield Lemma 9, which computes an upper-bound on the number of safe sets of size 2 created by $F$ when $k = 2$ and $q = 1$.

**Lemma 9** (Bounding $\sum_{\ell=1}^{2} \sum_{t_i \in T} \# S_k^\ell(F,\ell)$ based on $x(F)$ for $k = 2$). For $q = 1$,

$$\sum_{\ell=1}^{2} \sum_{t_i \in T} \# S_k^\ell(F,\ell) \leq m(m - 1).$$
Proof. By Lemmas 7 and 8

\[
\sum_{\ell=1}^{2} \sum_{t_i \in T} \#S_{1}^{2}(F, \ell)
\]
\[
\leq \sum_{t_i \in T} \frac{(m - 1) + (m - x_i(F))}{2} x_i(F) + \sum_{t_i \in T} \frac{(x_i(F) - 1)x_i(F)}{2}
\]
\[
= \sum_{t_i \in T} \frac{x_i(F)(m - 1) + x_i(F)m - x_i(F)^2}{2} + \sum_{t_i \in T} \frac{x_i(F)^2 - x_i(F)}{2}
\]
\[
= \frac{(m - 1)}{2} \sum_{t_i \in T} x_i(F) + \frac{m}{2} \sum_{t_i \in T} x_i(F) - \frac{1}{2} \sum_{t_i \in T} x_i(F)
\]
\[
= \frac{(m - 1)}{2} \sum_{t_i \in T} x_i(F) = (m - 1)m,
\]

where the last equality follows from \(\sum_{t_i \in T} x_i(F) = m\). \(\square\)

Proof of Proposition 6. Take any \(F^B \in F\) and any \(F^{STB} \in supp(STB)\). We prove the proposition by considering in turn the cases \(k = 1\) and \(k = 2\).

- **Case 1:** \(k = 1\).
  When \(k = 1\), the number of safe sets of size 1 for \(t_i\) is simply equal to \(x_i(F^B)\). Hence, the total number of safe sets generated is equal to \(\sum_{t_i \in T} x_i(F^B) = m\). This value does not depend on \(F\). Therefore, we have \(\mathbb{E}_{DA^k}(F^{STB}) = \mathbb{E}_{DA^k}(F^B)\).

- **Case 2:** \(k = 2\).
  The proof follows from the above lemmas. By Lemma 5, \(\mathbb{E}_{DA^k}(F^{STB}) \geq \mathbb{E}_{DA^k}(F^B)\) if and only if

\[
\sum_{t_i \in T} SS_{1}^{2}(F^{STB}) \geq \sum_{t_i \in T} SS_{1}^{2}(F^B).
\]

By Lemma (6), this is equivalent to

\[
\sum_{\ell=1}^{2} \sum_{t_i \in T} \#S_{1}^{2}(F^{STB}, \ell) \geq \sum_{\ell=1}^{2} \sum_{t_i \in T} \#S_{1}^{2}(F^B, \ell).
\] (24)

By Lemma 8, we have that

\[
(m - 1)m \geq \sum_{\ell=1}^{2} \sum_{t_i \in T} \#S_{1}^{2}(F^B, \ell).
\]

When \(F^{STB}\) is used, all sets of two schools are safe for the student who has a top-priority at all schools. These are all safe sets of order 1, and there are \(\binom{m}{2} = \frac{(m - 1)m}{2}\) of them. Similarly, all sets of two schools are safe for the student who has
a top-priority at all schools. These are safe sets of order 2 and there are again \( \binom{m}{2} = \frac{(m-1)m}{2} \) of them. Together, we have

\[
\sum_{\ell=1}^{2} \sum_{t_i \in T} \#S_i^2(F^{STB}, \ell) = (m - 1)m.
\]

Therefore, inequality (24) holds, which completes the proof. □

**Proof of Proposition 7.** Take any \( k \in \{2, \ldots, m\} \). There are two cases to consider: either sets of \( k \) schools are oversupplied, or sets of \( k \) schools are in short-supply.

- **Case 1:** Oversupply: \( kq \geq n \).

  If \( kq \geq n \), then all sets of \( k \) schools are oversupplied. By Proposition 1, all sets of \( k \) schools are therefore safe sets for any students. Hence, the \( k \) most-preferred schools of any student form a safe set. But then, by Proposition 2, all students have a dominant strategy in \( DA^k \). Hence, we have \( \mathbb{E}^{dom}_{DA^k}(F) = n \) for any \( F \in \mathcal{F} \), which corresponds to case (ii.a) in Proposition 7.

- **Case 2:** Short-supply: \( kq < n \).

  For a particular \( F^{STB} \in supp(STB) \), we construct \( F^{MTB} \in supp(MTB) \) such that \( \mathbb{E}^{dom}_{DA^k}(F^{STB}) > \mathbb{E}^{dom}_{DA^k}(F^{MTB}) \). The argument used for Case 2 in the proof of Corollary 4 shows that the expected number of dominant strategies is the same for any \( F^{STB} \in supp(STB) \). As a result, if \( \mathbb{E}^{dom}_{DA^k}(F^{STB}) > \mathbb{E}^{dom}_{DA^k}(F^{MTB}) \), then \( \mathbb{E}^{dom}_{DA^k}(F^{STB}) > \mathbb{E}^{dom}_{DA^k}(F^{MTB}) \) for any \( F^{STB} \in supp(STB) \).

  Let the priority profile \( F^{STB} \in supp(STB) \) be such that the priority rank of \( t_1 \) is 1 at all schools, the priority rank of \( t_2 \) is 2 at all schools, and so on. This priority profile is as follows

<table>
<thead>
<tr>
<th>( F^{STB} )</th>
<th>( F^{STB} )</th>
<th>( \ldots )</th>
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<tbody>
<tr>
<td>( t_1 )</td>
<td>( t_1 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>( t_2 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( t_{kq} )</td>
<td>( t_{kq} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \vdots )</td>
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</tbody>
</table>

Given that \( kq < n \), the priority rank of \( t_{kq} \) in \( F^{STB} \) is lower than the priority rank of \( t_n \) in \( F^{STB} \).

Profile \( F^{MTB} \) is constructed from \( F^{STB} \) by swapping the priority ranks of students \( t_{kq} \) and student \( t_n \) at \( s_1 \). As a result, in \( F^{MTB} \), the priority rank of \( t_n \) is \( kq \) and the priority rank of \( t_{kq} \) is \( n \).
We assumed that all students find at least \( k + 1 \) schools acceptable. By Proposition 2, only students for whom the \( k \) most-preferred schools form a safe set have a dominant strategy in \( DA^k \).

We show that all sets of \( k \) schools containing school \( s_1 \) are safe for \( t_{kq} \) in profile \( \tilde{F}^{STB} \), whereas no set of \( k \) schools containing school \( s_1 \) is safe for \( t_{kq} \) in profile \( \tilde{F}^{MTB} \). Let \( \hat{S}_1^k \) be any set of \( k \) schools containing school \( s_1 \). For profile \( \tilde{F}^{STB} \), we have that

\[
\# \{ t_j \in T \mid t_j \neq t_{kq} \text{ and } t_j \tilde{F}^{STB} t_{kq} \text{ for some } s \in \hat{S}_1^k \} = kq - 1.
\]

Given that \( \sum_{s \in \hat{S}_1^k} q_s = kq \), Proposition 1 implies that set \( \hat{S}_1^k \) is safe for \( t_{kq} \) in profile \( \tilde{F}^{STB} \). For profile \( \tilde{F}^{MTB} \), we have that

\[
\# \{ t_j \in T \mid t_j \neq t_{kq} \text{ and } t_j \tilde{F}^{MTB} t_{kq} \text{ for some } s \in \hat{S}_1^k \} = kq.
\]

By Proposition 1, set \( \hat{S}_1^k \) is not safe for \( t_{kq} \) in profile \( \tilde{F}^{MTB} \). This shows that \( t_{kq} \) has less safe sets of size \( k \) in \( \tilde{F}^{MTB} \) than in \( \tilde{F}^{STB} \). Also, no student has strictly more safe sets of size \( k \) in \( \tilde{F}^{MTB} \) than in \( \tilde{F}^{STB} \). In particular, if no set of \( k \) schools has enough seats to host all students, then \( t_n \) does not have any safe set of size \( k \) in \( \tilde{F}^{MTB} \). Therefore, the number of safe sets of size \( k \) is strictly smaller in \( \tilde{F}^{MTB} \) than in \( \tilde{F}^{STB} \). By Lemma 5, this implies that \( \mathbb{E}^{dom}_{DA^k}(\tilde{F}^{STB}) > \mathbb{E}^{dom}_{DA^k}(\tilde{F}^{MTB}) \), the desired result.

**Proof of Proposition 8.** By Corollary 5, we have \( \mathbb{E}^{dom}_{BOS^\ell}(g^{BOS}) = q \) for any \( \ell \in \{1, \ldots, m\} \) and any distribution \( g^{BOS} \) on \( F \). We must show that we have \( \mathbb{E}^{dom}_{DA^k}(g^{DA}) \geq q \) for any \( k \in \{1, \ldots, m\} \) and any distribution \( g^{DA} \) on \( F \). The proof for the latter claim follows the same argument as the proof of Corollary 5 and is therefore omitted. We only observe that the inequality in \( \mathbb{E}^{dom}_{DA^k}(g^{DA}) \geq q \) is strict for \( k \geq 2 \). The reason is that the necessary condition for having a dominant strategy in \( BOS^\ell \) is sufficient but not necessary in \( DA^k \). We assumed that all students have more than \( k \) acceptable schools. Therefore, by Proposition 3, only students who have top-priority at their most-preferred school have a dominant strategy in \( BOS^\ell \). Yet, by Proposition 2, students who have top-priority at any of their \( k \) most-preferred schools have a dominant strategy in \( DA^k \).

**Proof of Proposition 9.** Take any \( F^* \in F \) and let the tie-breaking rule \( \bar{g}^{F^*} \) be defined as the uniform distribution over the set of all permutations of \( F^* \). Let the support
of the tie-breaking rule $\bar{g}^{F^*}$ be denoted as $\bar{F}^*$. We show that the rule $\bar{g}^{F^*}$ is ex-ante fair, i.e., for any $t_i \in T$, any $s \in S$, and any priority rank $p \in \{1, \ldots, n\}$ we have

$$\text{Proba}(F_s(t_i) = p \mid \bar{g}^{F^*}) = \sum_{\{F \in \bar{F}^* \mid F_s(t_i) = p\}} \bar{g}^{F^*}(F) = \frac{1}{n}.$$  

Take any $s \in S$ and any priority rank $p \in \{1, \ldots, n\}$. Given that rank $p$ at school $s$ is assigned to exactly one student in all $F \in \bar{F}$, we must have

$$\sum_{t_i \in T} \sum_{\{F \in \bar{F}^* \mid F_s(t_i) = p\}} \bar{g}^{F^*}(F) = 1. \quad (25)$$

Given that $\bar{g}^{F^*}$ is a uniform distribution over $\bar{F}^*$, we have for any $F \in \bar{F}^*$ that

$$\bar{g}^{F^*}(F) = \frac{1}{\# \bar{F}^*},$$

which implies that

$$\sum_{\{F \in \bar{F}^* \mid F_s(t_i) = p\}} \bar{g}^{F^*}(F) = \frac{\#\{F \in \bar{F}^* \mid F_s(t_i) = p\}}{\# \bar{F}^*}. \quad (26)$$

By construction of $\bar{F}^*$, we have that for any $t_j \neq t_i$,

$$\#\{F \in \bar{F}^* \mid F_s(t_j) = p\} = \#\{F \in \bar{F}^* \mid F_s(t_i) = p\}.$$

Therefore, we have that

$$\sum_{t_i \in T} \sum_{\{F \in \bar{F}^* \mid F_s(t_i) = p\}} \bar{g}^{F^*}(F) = n \ast \frac{\#\{F \in \bar{F}^* \mid F_s(t_i) = p\}}{\# \bar{F}^*},$$

and (25) implies that

$$\frac{\#\{F \in \bar{F}^* \mid F_s(t_i) = p\}}{\# \bar{F}^*} = \frac{1}{n}.$$

The last equation together with (26), yields the desired result.

**Proof of Proposition 10.** (i) Take any $k \in \{1, \ldots, m\}$. By Corollary 4, we have that $\mathbb{E}^{\text{dom}}_{DA^k}(g^{STB}) = \min\{kq, n\}$. There remains to show that $\mathbb{E}^{\text{dom}}_{DA^k}(\bar{g}^{FFT_B}) = \min\{kq, n\}$. We consider two cases: either sets of $k$ schools are oversupplied, or not.

- **Case 1:** Over-supply: $kq \geq n$.
  By the argument used in Case 1 of the proof of Corollary 4, we have $\mathbb{E}^{\text{dom}}_{DA^k}(g^{FFT_B}) = n$.

- **Case 2:** Short-supply: $kq < n$.
  We show that, unlike students whose priority ranks are never smaller or equal to $kq$, all students whose priority rank is smaller or equal to $kq$ at some school have a dominant strategy in $DA^k$ under $F^{FFT_B}$.

47
Take any $t_i$ with priority rank smaller or equal to $kq$ at some school. Denote by $S^k$ the set of her $k$ most-preferred schools. By construction of the pattern $F^{FTB}$, there are exactly $kq - 1$ students other than $t_i$ who have at some school a priority rank no larger than $kq$. Therefore, we have that

$$\#\{t' \in T \setminus \{t\} \mid t' F_s t \text{ for some } s \in S^k\} < kq = \sum_{s \in S^k} q_s.$$ 

By Proposition 1, the set $S^k$ is safe for $t_i$. Then, by Proposition 2, $t_i$ has a dominant strategy in $DA^k$.

Take any $t_i$ whose priority rank is never smaller or equal to $kq$ at any school. Denote by $S^k$ the set of her $k$ most-preferred schools. This case is such that there are at least $kq$ students other than $t_i$ who receive at all schools a priority rank no larger than $kq$. These $kq$ students have a higher priority than $t_i$ at every school. Therefore, we have that

$$\#\{t' \in T \setminus \{t\} \mid t' F_s t \text{ for some } s \in S^k\} \geq kq = \sum_{s \in S^k} q_s.$$ 

By Proposition 1, the set $S^k$ is not safe for $t_i$. We assumed that all students have more than $k$ acceptable schools. Therefore, by Proposition 2, only students for whom the $k$ most-preferred schools form a safe set have a dominant strategy in $DA^k$. Thus, $t_i$ does not have a dominant strategy in $DA^k$.

Given that, under $F^{FTB}$, there are exactly $kq$ students who receive at some school a priority rank no larger than $kq$, we have that $\mathbb{E}^{dom}_{STB}(g^{FTB}) = kq$. Given that $kq < n$, we have $\mathbb{E}^{dom}_{DA^k}(g^{STB}) = \min\{kq, n\}$, the desired result.

(ii) The proof is based on Lemma 10, which characterizes the ex-post fairness partial ordering of priority profiles.

**Lemma 10** (Second-order dominance). Take any two $F$ and $F' \in F$. Let $\hat{r}^F$ and $\hat{r}^{F'}$ denote the vectors obtained when sorting in increasing order the vectors of total priority ranks $r^F$ and $r^{F'}$ associated to $F$ and $F'$. $F$ is at least as ex-post fair as $F'$ if and only if

$$\sum_{i=1}^b \hat{r}^F_i \geq \sum_{i=1}^b \hat{r}^{F'}_i \quad \text{for any } b \in \{1, \ldots, n\}. \quad (27)$$

Moreover, $F$ is more ex-post fair than $F'$ if the inequality is strict for some $b^*$.

Proof. This lemma is a corollary of Lemma 1 in Shorrocks (1983), a well-known result in the literature on income inequality measurement. Shorrocks (1983) compares $n$-dimensional income vectors $y$ and $y'$ that are sorted in such a way that $y_1 \leq \cdots \leq y_n$ and $y'_1 \leq \cdots \leq y'_n$. Letting $\mu$ denote the mean value in $y$, the Lorenz curve associated to vector $y$ is defined as

$$L\left(y, \frac{\ell}{n}\right) = \sum_{i=1}^\ell \frac{y_i}{n\mu} \quad \text{with } \ell = 1, \ldots, n.$$
In a nutshell, Lemma 1 in Shorrocks (1983) shows that if \( \mu = \mu' \), then \( y \) can be obtained from \( y' \) by a sequence of mean-preserving progressive transfers if and only if

\[
L(y, p) \geq L(y', p) \quad \text{for any } p \in [0, 1].
\]

In our setting, the precondition for Lemma 1 is fulfilled given that, by definition, vectors \( \hat{r}_F \) and \( \hat{r}_{F'} \) have the same mean

\[
\frac{1}{n} \sum_i \hat{r}_F = \frac{m(n + 1)}{2} = \frac{1}{n} \sum_i \hat{r}_{F'}.
\]

It then follows that

\[
L(\hat{r}_F, p) \geq L(\hat{r}_{F'}, p) \quad \text{for any } p \in [0, 1] \quad \iff \quad \sum_{i=1}^{b} \hat{r}_i^F \geq \sum_{i=1}^{b} \hat{r}_i^{F'} \quad \text{for any } b \in \{1, \ldots, n\},
\]

the desired result. ■

Take any \( k \in \{1, \ldots, m\} \) such that \( \lfloor kq/m \rfloor \geq 1 \) and any profile \( F^{STB} \) in the support of \( g^{STB} \).

- **Step 1:** Pattern \( F^{FTB} \) constructed from \( F^{STB} \) is more ex-post fair than \( F^{STB} \).

  By the construction of \( F^{STB} \), we have for any \( b \in \{1, \ldots, n\} \) that

  \[
  \sum_{i=1}^{b} \hat{r}_{F^{STB}} = m + 2m + \cdots + bm = \frac{b(b + 1)}{2}.
  \]

  We divide the proof of Step 1 into two sub-steps

  - **Step 1.1:** For any \( F \in \mathcal{F} \) we have \( \sum_{i=1}^{b} \hat{r}_i^F \geq \frac{mb(b+1)}{2} \) for any \( b \in \{1, \ldots, n\} \).

    Take any \( b \in \{1, \ldots, n\} \). Clearly, the value of \( \sum_{i=1}^{b} \hat{r}_i^F \) is minimized when a set of \( b \) students receive at all schools the priority ranks 1 to \( b \). Take any \( F' \in \mathcal{F} \) meeting this condition. Then, the students in the set are the \( b \) students with the smallest total priority ranks in \( F' \). In that case, we have

    \[
    \sum_{i=1}^{b} \hat{r}_i^{F'} = \sum_{j=1}^{m} \sum_{\ell=1}^{b} \ell = \sum_{j=1}^{m} \frac{b(b+1)}{2} = b(b+1).
    \]

  - **Step 1.2:** We have \( \sum_{i=1}^{b} \hat{r}_i^{F^{FTB}} > \frac{mb(b+1)}{2} \) for some \( b \in \{1, \ldots, n\} \).

    Take \( b^* = 1 \). As \( \lfloor kq/m \rfloor \geq 1 \) and given that \( m \geq 2 \), no student has priority rank 1 at all schools in \( F^{FTB} \). This implies that \( \hat{r}_i^{F^{FTB}} > \frac{m(1+1)}{2} = m \).

Step 1.1 and 1.2 together show that

\[
\sum_{i=1}^{b} \hat{r}_i^{F^{FTB}} \geq \sum_{i=1}^{b} \hat{r}_i^{F^{STB}} \quad \text{for any } b \in \{1, \ldots, n\}.
\]

and that the inequality is strict for some \( b^* = 1 \). By Lemma 10, \( F^{FTB} \) is more ex-post fair than \( F^{STB} \).
• **Step 2:** All priority profiles in the support of \( g^{FTB} \) are equally ex-post fair.

Any two profiles \( F^{FTB} \) and \( F'^{FTB} \) in the support of \( g^{FTB} \) are permutations of each other. Therefore, their associated vectors of total priority rank \( r^{FTB} \) and \( r'^{FTB} \) are also permutations of each other. As a result we have that \( \hat{r}^{FTB} = \hat{r}'^{FTB} \). By Lemma 10, any two profiles sharing the same sorted vector of total priority ranks are equally ex-post fair.

• **Step 3:** All priority profiles in the support of \( g^{STB} \) are equally ex-post fair.

The argument is the same as in Step 2.

By steps 1 to 3, we have that any profile in the support of \( g^{FTB} \) is more ex-post fair than any profile in the support of \( g^{STB} \), the desired result.

**Proof of Proposition 11/ (i).** Take any \( k \in \{1, \ldots, m\} \). By Corollary 4, we have that \( E_{DAk}^{dom}(g^{STB}) = \min\{kq, n\} \). There remains to show that \( E_{DAk}^{dom}(\tilde{g}^{SPR}) = \min\{kq, n\} \). There are two cases to consider: either sets of \( k \) schools are oversupplied, or not.

• **Case 1:** Oversupply: \( kq \geq n \).

This case happens when \( k = m \) and \( mq = n \), which is not ruled out by our assumptions \((mq \leq n)\). By the argument used in Case 1 of the proof of Corollary 4, we have \( E_{DAk}^{dom}(g^{SPR}) = n \).

• **Case 2:** Short-supply: \( kq < n \).

Take any \( F^{SPR} \) in the support of \( g^{SPR} \). For each student \( t \) with a top-priority at some school \( s \) under \( F^{SPR} \), \( t \) has a dominant strategy if and only if \( s \) is one of \( t \)'s \( k \) most-preferred schools. For each such student, this occurs with probability \( k/m \).

If students have at most one top-priority in \( F^{SPR} \), then \( mq \) students have a top-priority. In that case, the expected number of students with a dominant strategy is \( mq * k/m = kq \), the desired result.

There remains to show that all students have at most one top-priority in \( F^{SPR} \). Consider any student \( t \) who has top-priority at a school, say \( s_1 \). We show that the rank of \( t \) is between \( q + 1 \) and \( n \) for all other schools \((s_2 \text{ to } s_m)\). Student \( t \) has at \( s_1 \) a rank between \( 1 \) and \( q \). Let \( x = \lfloor n/m \rfloor \). By the construction of \( F^{SPR} \), \( t \) has at \( s_2 \) a rank at least equal to \( 1 + x \). As we assume \( mq \leq n \) we have that \( x \geq q \) and therefore \( 1 + x \) is larger than \( q \). At school \( s_m \), \( t \) has a rank at most equal to \( q + (m - 1)x \). As \( x = \lfloor n/m \rfloor \) we have that \( q + (m - 1)x \leq n \), the desired result.

(ii). The proof that \( \tilde{g}^{SPR} \) is more ex-post fair than \( g^{STB} \) when \( \lfloor n/m \rfloor \geq 1 \) is similar to the proof of part (ii) of Proposition 10 and is therefore omitted.

Using Lemma 10, we use three steps to prove the second part of Proposition 11(ii) (i.e., that \( g^{SPR} \) is more ex-post fair than \( g^{FTB} \) when \( \lfloor n/m \rfloor \geq 1 \) and \( kq < n \)). In Step 1, we show that inequality (27) is true for all \( b \in \{1, \ldots, kq\} \). Based on Step 1, we show in Step 2 that (27) is also true for all \( b \in \{kq + 1, \ldots, n\} \). Together, Steps 1 and 2 show that \( F^{SPR} \) is at least as ex-post fair as \( F^{FTB} \). Step 3 shows that if \( kq < n \), the inequality in (27) is strict for \( b^* = kq \). Therefore, \( F^{SPR} \) is more ex-post fair than \( F^{FTB} \).

**Step 1:** for all \( b \in \{1, \ldots, kq\} \), \( \sum_{i=1}^{b} \hat{r}_i^{FTB} \leq \sum_{i=1}^{b} \hat{r}_i^{SPR} \).
The proof of Step 1 relies on the fact that the construction of $F^{SPR}$ is identical to the construction of the first $kq$ priorities in $F^{FTB}$. If $kq = n$, then $F^{SPR}$ is identical to $F^{FTB}$. Thus, it is sufficient to prove

$$\sum_{i=1}^b r_i^{FTB_{kq}} \leq \sum_{i=1}^b r_i^{FTB_n}, \quad \text{for all } b \in \{1, \ldots, kq\}, \tag{28}$$

where $r_i^{FTB_{kq}}$ is the ordered vector of total ranks under $F^{FTB_{kq}} = F^{FTB}$, and $r_i^{FTB_n}$ is the ordered vector of total ranks under $F^{FTB_n}$, the version of $F^{FTB}$ where all $n$ students have a high priority.

To do so, we show that for all $kq$,

$$\sum_{i=1}^b r_i^{FTB_{kq}} \leq \sum_{i=1}^b r_i^{FTB_{kq+1}}, \quad \text{for all } b \in \{1, \ldots, kq\}, \tag{29}$$

where $r_i^{FTB_{kq+1}}$ is the ordered vector of total ranks under $F^{FTB_{kq+1}}$, the version of $F^{FTB}$ where $kq + 1$ students have a high priority (and the rest of the students have a low priority). The inequalities in (28) then follow from the inequalities in (29) by induction.

The inequalities in (29) are established by considering three cases. We first characterize the total rank of students in $F^{FTB_{kq}}$ as a function of their position in the priority profile. We then use this characterization to show that (29) is true (Case 1) when neither $kq/m$ nor $(kq + 1)/m$ are integers, (Case 2) when $kq/m$ is an integer but $(kq + 1)/m$ is not, and (Case 3) when $kq/m$ is not integer but $(kq + 1)/m$ is. This completes the proof of Step 1, as these are all possible cases of (29).

Charaterization: the total rank of a student in $F^{FTB_{kq}}$ as a function of her position in the priority profile.

Consider the case $m = 5$ and $kq = 14$. In this case, $FTB^{kq}$ takes the following form

$$\begin{align*}
t_1 & \quad \overline{t_{13}} & \quad t_11 & \quad t_9 & \quad t_7 \\
t_2 & \quad t_{14} & \quad t_{12} & \quad t_{10} & \quad t_8 \\
t_3 & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
t_4 & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
t_5 & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
t_6 & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
\end{align*} \tag{30}$$

Observe that all students appear once and only once either (a) among the 2 first ranks at one of the schools, or (b) among ranks 3 to 6 at school $s_1$.

For any $k$, $q$, and $m$, let $\tau^{kq} := \lfloor kq/m \rfloor$ and $\rho^{kq} := kq - m\tau$. Informally, $\tau^{kq}$ is the “translation factor” used in the construction of $FTB^{kq}$, whereas $\rho^{kq}$ is the remainder of $kq/m$. In general, in $FTB^{kq}$, all students appear once and only once either (a’) in the first $\tau^{kq}$ priority rank of some school, or (b’) in the $\tau^{kq} + 1$ to $\rho^{kq}$ first rank of the first
school. The equivalent of (30) for a general $FTB^{kq}$ pattern can be represented as follows.

\[
D_1 \subset T \text{ with } \#D_1 = \tau^{kq} \quad D_2 \subset T \text{ with } \#D_2 = \tau^{kq} \quad \ldots \quad D_m \subset T \text{ with } \#D_m = \tau^{kq} \\
R \subset T \text{ with } \#R = \rho^{kq} \quad \vdots \quad \vdots \quad \vdots
\]

(31)

where for any two sets of students $\tilde{T}, \hat{T} \in \{D_1, \ldots, D_m, R\}$, we have $\tilde{T} \cap \hat{T} = \emptyset$.

Observe that $kq = m(\tau^{kq} + 1) - x^{kq}$, where $x^{kq}$ is the smallest integer such that $(kq + x^{kq})/m$ is an integer. For example, when $m = 5$ and $kq = 14$, we have $kq = 5 \times (2 + 1) - 1$, and $x^{kq} = 1$. Using this formula for $kq$ and replacing in the definition of $\rho^{kq}$, we have $\rho^{kq} = m(\tau^{kq} + 1) - x^{kq} - m\tau^{kq}$ which implies $\rho^{kq} = m - x^{kq}$. In particular, when $m = 5$ and $kq = 14$, we have $\rho^{kq} = (m - 1)$.

Next, we compute the total rank associated with the position of a student in (31). Consider again $m = 5$ and $kq = 14$, which implies $x^{kq} = 1$ and $\rho^{kq} = (m - 1)$. In this case, the total rank associated with the position of a student in (31) is

\[
\begin{array}{cccccc}
    r & r + m - 1 & r + 2(m - 1) & r + 3(m - 1) & r + 4(m - 1) & \\
    r + 2m & r + m - 1 + m & r + 2(m - 1) + m & r + 3(m - 1) + m & r + 4(m - 1) + m & \\
    r + 3m & r + 2m & r + 3(m - 1) & r + 4(m - 1) & \\
    r + 4m & r + 3m & r + 4m &  & \\
    r + 5m & r + 4m &  &  & \\
\end{array}
\]

(32)

where $r = 25$.

Now, consider the case $m = 5$ and $kq = 13$, which implies $x = 2$ and $\rho^{kq} = (m - 2)$. In this case, the total rank associated with the position of a student in (31) becomes

\[
\begin{array}{cccccc}
    r & r + (m - 2) & r + 2(m - 2) & r + 3(m - 2) & r + 4(m - 2) & \\
    r + 2m & r + (m - 2) + m & r + 2(m - 2) + m & r + 3(m - 2) + m & r + 4(m - 2) + m & \\
    r + 3m & r + 2m & r + 3(m - 2) & r + 4(m - 2) & \\
    r + 4m & r + 3m & r + 4m &  & \\
    r + 5m & r + 4m &  &  & \\
\end{array}
\]

(33)

In general, the total rank associated with the position of a student in (31) is

\[
\begin{array}{cccccccc}
    r & r + \rho^{kq} & r + 2\rho^{kq} & r + 3\rho^{kq} & \ldots & r + (m - 1)\rho^{kq} & \rho^{kq} & \vdots \\
    r + m & r + \rho^{kq} + m & r + 2\rho^{kq} + m & \vdots & \vdots & \vdots & & \\
    r + (\tau^{kq} - 1)m & r + \rho^{kq} + (\tau^{kq} - 1)m & r + 2\rho^{kq} + (\tau^{kq} - 1)m & \vdots & \vdots & \vdots & & \\
    r + \tau^{kq}m & r + \rho^{kq} + \tau^{kq}m & r + 2\rho^{kq} + \tau^{kq}m & \vdots & \vdots & \vdots & & \\
    r + (\tau^{kq} + \rho^{kq} - 1)m & & & & & & & \\
\end{array}
\]

(34)

52
where \( r = \hat{r}_1^{mk} \) where

\[
\hat{r}_1^{mk} = 1 + \sum_{i=1}^{m-1} (1 + \tau^{mk}_i),
\]

\[
= 1 + (m - 1) + \tau^{mk} \sum_{i=1}^{m-1} i,
\]

\[
= m + \tau^{mk} (m-1) m / 2.
\]

Case 1: Neither \( kq/m \) nor \( (kq+1)/m \) are integers.

The proof for this case and the next rely on the following lemma, which shows that it is sufficient to prove the existence of a particular increasing injection between the vector of total ranks under \( kq \) and the vector of total ranks under \( (kq+1) \).

**Lemma 11.** Consider any two lists of (not necessarily ordered) total ranks \( r^{mk} \) and \( r^{mk+1} \) containing respectively \( kq \) and \( kq+1 \) total ranks. If there exist an injection \( \alpha : \{1, \ldots, kq\} \to \{1, \ldots, kq+1\} \) such that for all \( i \in \{1, \ldots, kq\} \),

(a) \( \hat{r}_i^{mk} \leq \hat{r}_\alpha^{mk+1} \), and

(b) \( r_i^{mk} < r_{\alpha(i)}^{mk+1} \), for \( h \in \{1, \ldots, kq + 1\} \) such that \( \alpha(i) \neq h \) for all \( i \in \{1, \ldots, kq\} \), then for all \( b \in \{1, \ldots, kq\} \),

\[
\sum_{i=1}^{b} \hat{r}_i^{mk} \leq \sum_{i=1}^{b} \hat{r}_i^{mk+1}.
\]

Proof. In order to derive a contradiction, suppose that for some \( c \in \{1, \ldots, kq\} \), we have \( \hat{r}_c^{mk} > \hat{r}_{\alpha(i)}^{mk+1} \). This implies \( \hat{r}_c^{mk} > \hat{r}_j^{mk+1} \) for all \( j \in \{1, \ldots, c\} \). That is, there exists at most \( (kq+1) - c \) indices \( I \subset \{1, \ldots, kq + 1\} \) such that \( \hat{r}_i^{mk+1} > \hat{r}_c^{mk} \) for all \( i \in I \) (i.e., \#I \leq (kq+1) - c \). But by (a) and (b), there must exist at least \( (kq+1) - c + 1 \) such indices, a contradiction. \( \blacksquare \)

We now show how such an injection can be constructed when neither \( kq/m \) nor \( (kq+1)/m \) are integers. Consider the case \( kq = 4 \) and \( m = 3 \). In this case, the priorities of the \( kq \) and \( kq+1 \) first students in \( FTB^{mk} \) and \( FTB^{mk+1} \) are as follows

<table>
<thead>
<tr>
<th>( F_1^{mk} )</th>
<th>( F_2^{mk} )</th>
<th>( F_3^{mk} )</th>
<th>( F_1^{mk+1} )</th>
<th>( F_2^{mk+1} )</th>
<th>( F_3^{mk+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>( t_4 )</td>
<td>( t_3 )</td>
<td>( t_1 )</td>
<td>( t_5 )</td>
<td>( t_4 )</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>( t_1 )</td>
<td>( t_4 )</td>
<td>( t_2 )</td>
<td>( t_1 )</td>
<td>( t_5 )</td>
</tr>
<tr>
<td>( t_3 )</td>
<td>( t_2 )</td>
<td>( t_1 )</td>
<td>( t_3 )</td>
<td>( t_2 )</td>
<td>( t_1 )</td>
</tr>
<tr>
<td>( t_4 )</td>
<td>( t_3 )</td>
<td>( t_2 )</td>
<td>( t_4 )</td>
<td>( t_3 )</td>
<td>( t_2 )</td>
</tr>
</tbody>
</table>

In this case, injection \( \check{\alpha} \) with \( \check{\alpha}(1) = 1, \check{\alpha}(2) = 2, \check{\alpha}(3) = 4 \) and \( \check{\alpha}(4) = 5 \) satisfies conditions (a) and (b) in Lemma 11 (with \( h = 3 \)).

As (35) illustrates, students receiving the first \( kq \) priorities in \( FTB^{mk} \) can in general be divided into two groups:

\( (kq+1) - c \) such indices are required to satisfy (a). Furthermore, there must exist one additional index \( h \in \{1, \ldots, kq + 1\} \) such that \( r_{\hat{h}}^{mk+1} > r_c^{mk} \) and \( \alpha(i) \neq h \) for all \( i \in \{c, \ldots, kq\} \) in order to satisfy (b).
(A) The “diagonal” students: students $t_1$ to student $t_{\tau_{kq + \rho_{kq}}}$ (students $t_1$ and $t_2$ in bold in (35)), and

(B) The “corner” students: students $t_{\tau_{kq + \rho_{kq} + 1}}$ to $t_{kq}$ (students $t_3$ and $t_4$ underlined in (35)).

Students receiving the first $kq + 1$ priorities in $FTB^{kq+1}$ can in general be divided into three groups:

(A') The “first $kq$ diagonal” students: students $t_1$ to student $t_{\tau_{kq + \rho_{kq} + \rho_{kq} + 1}}$ (students $t_1$ and $t_2$ in bold in (35)),

(B') The “last diagonal” student: student $t_{\tau_{kq + \rho_{kq} + 1} + \rho_{kq} + \rho_{kq} + 1}$ (student $t_3$ in (35)),

(C') The “corner” students: students $t_{\tau_{kq + \rho_{kq} + 1} + \rho_{kq} + 1}$ to $t_{kq + 1}$ (students $t_4$ and $t_5$ underlined in (35)).

When neither $kq/m$ nor $(kq + 1)/m$ are integers, $\tau_{kq} = \tau_{kq + 1}$ and $\rho_{kq + 1} = \rho_{kq} + 1$. Thus, groups (A) and (A') and groups (B) and (C') contain the same number of students. Therefore, it is possible to define an injection $\alpha^*$ that matches diagonal students in $kq$ with the first diagonal students in $kq + 1$ and corner students in $kq$ with corner students in $kq + 1$. Formally,

- $\alpha^*(i) = i$ for all $i \in \{1, \ldots, \tau_{kq} + \rho_{kq}\}$, and
- $\alpha^*(i) = i + 1$ for all $i \in \{\tau_{kq} + \rho_{kq} + 1, \ldots, kq\}$.

Clearly, $\alpha^*$ satisfies condition (a) in Lemma 11. There remains to prove that $\alpha^*$ also satisfies condition (b), i.e., that

$$r_{\tau_{kq + \rho_{kq} + 1} + \rho_{kq} + 1}^{kq + 1} > r_i^{kq + 1}$$

for all $i \in \{1, \ldots, kq\}$. Because $\alpha^*$ satisfied (a) in Lemma 11, this is equivalent to proving

$$r_{\tau_{kq + \rho_{kq} + 1} + \rho_{kq} + 1}^{kq + 1} > r_{\alpha^*(i)}^{kq + 1}$$

for all $i \in \{1, \ldots, kq\}$,

which by definition of $\alpha^*$ is the same as

$$r_{\tau_{kq + \rho_{kq} + 1} + \rho_{kq} + 1}^{kq + 1} > r_i^{kq + 1} \quad \forall i \in \{1, \ldots, kq\} \setminus \{\tau_{kq + 1} + \rho_{kq + 1}\}. \quad (36)$$

Based on the first column of (34), we have

$$r_{\tau_{kq + \rho_{kq} + 1} + \rho_{kq} + 1}^{kq + 1} > r_i^{kq + 1} \quad \forall i \in \{1, \ldots, \tau_{kq + 1} + \rho_{kq + 1} - 1\}.$$

Also, if student $s_{j^*}$ is the student with priority rank $\tau_{kq + 1}$ at school $m$, we have

$$r_{j^*}^{kq + 1} > r_i^{kq + 1} \quad \forall i \in \{\tau_{kq + 1} + \rho_{kq + 1} + 1, \ldots, kq + 1\}.$$
Hence, in order to prove (36), is it sufficient to prove that
\[ r_{r^{kq+1} + \rho^{kq+1}} > r_{j^*}. \] (37)

By (34), inequality (37) is equivalent to
\[
\begin{align*}
  r + (\tau^{kq+1} + \rho^{kq+1} - 1)m &> r + (m - 1)\rho^{kq+1} + (\tau^{kq} - 1)m \\
  \tau^{kq+1}m + \rho^{kq+1}m - m &> \rho^{kq+1}m - \rho^{kq+1} + \tau^{kq+1}m - m \\
  0 &> -\rho^{kq+1},
\end{align*}
\]
which is true because when neither \( kq/m \) nor \((kq + 1)/m \) are integers, \( \rho^{kq+1} > 0 \).

Case 2: \( kq/m \) is an integer but \((kq + 1)/m \) is not

The proof is similar to the proof of Case 1. Consider the case \( kq = 6 \) and \( m = 3 \). In this case, the priorities of the \( kq \) and \( kq + 1 \) first students in \( FTB^{kq} \) and \( FTB^{kq+1} \) are as follows

\[
\begin{array}{c|c|c}
F_{1}^{kq} & F_{2}^{kq} & F_{3}^{kq} \\
\hline
\tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 \\
\tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 \\
\tilde{t}_3 & \tilde{t}_4 & \tilde{t}_5 \\
\tilde{t}_4 & \tilde{t}_5 & \tilde{t}_6 \\
\tilde{t}_5 & \tilde{t}_6 & \tilde{t}_1 \\
\tilde{t}_6 & \tilde{t}_1 & \tilde{t}_2 \\
\end{array}
&&
\begin{array}{c|c|c}
F_{1}^{kq+1} & F_{2}^{kq+1} & F_{3}^{kq+1} \\
\hline
\tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 \\
\tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 \\
\tilde{t}_3 & \tilde{t}_4 & \tilde{t}_5 \\
\tilde{t}_4 & \tilde{t}_5 & \tilde{t}_6 \\
\tilde{t}_5 & \tilde{t}_6 & \tilde{t}_1 \\
\tilde{t}_6 & \tilde{t}_1 & \tilde{t}_2 \\
\end{array}
\] (38)

In this case, injection \( \tilde{\alpha} \) with \( \tilde{\alpha}(1) = 1, \tilde{\alpha}(2) = 2, \tilde{\alpha}(3) = 4, \tilde{\alpha}(4) = 5, \alpha(5) = 6 \) and \( \alpha(6) = 7 \) satisfies conditions (a) and (b) in Lemma 11 (with \( h = 3 \)). In general, injection \( \alpha^* \) defined in Case 1 also satisfies (a) and (b) in Lemma 11 when \( kq/m \) is an integer but \((kq + 1)/m \) is not, the proof being identical to the proof in Case 1.

Case 3: \( kq/m \) is not integer but \((kq + 1)/m \) is.

Let \( \tilde{r}_{i}^{kq} \) be the total rank of the \( i^{th} \)-student with lowest total rank under pattern \( FTB^{kq+1} \). We need to show that
\[
\sum_{i=1}^{b} \tilde{r}_{i}^{kq} \leq \sum_{i=1}^{b} \tilde{r}_{i}^{kq+1}, \quad \text{for all } b \in \{1, \ldots, kq\}. \] (39)

Now, consider the \( kq + 1 \)-student pattern \( \overline{FT}^{kq} \) constructed from the FTB \((kq)\)-students pattern by adding student \( kq + 1 \) at rank \( kq + 1 \) at all schools. For example, if \( kq = 5 \) and \( m = 3 \), pattern \( \overline{FT}^{kq} \) is

\[
\begin{array}{c|c|c|c}
\overline{F}_{1}^{kq} & \overline{F}_{2}^{kq} & \overline{F}_{3}^{kq} \\
\hline
\tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 \\
\tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 \\
\tilde{t}_3 & \tilde{t}_4 & \tilde{t}_5 \\
\tilde{t}_4 & \tilde{t}_5 & \tilde{t}_6 \\
\tilde{t}_5 & \tilde{t}_6 & \tilde{t}_1 \\
\tilde{t}_6 & \tilde{t}_1 & \tilde{t}_2 \\
\end{array}
\] (40)
The proof of this case relies on a series of observations about \( FTB^{kq}, \bar{F}^{kq} \) and \( FTB^{kq+1} \).

Relationships between \( \bar{F}^{kq}, \text{ and } FTB^{kq} \text{ and } FTB^{kq+1} \)

Observation 1:

\[
\sum_{i=1}^{b} \bar{r}_{i}^{kq} = \sum_{i=1}^{b} \bar{r}_{i}^{F^{kq}}, \quad \text{for all } b \in \{1, \ldots, kq\}. \tag{41}
\]

Proof. This follows directly from the construction of \( \bar{F}^{kq} \) from \( FTB^{kq} \). ■

Observation 2:

\[
\sum_{i=1}^{kq+1} \bar{r}_{i}^{F^{kq}} = \sum_{i=1}^{kq+1} \bar{r}_{i}^{kq+1}.
\]

Proof. This follows from the fact that the \( kq+1 \) first students receive all \( kq+1 \) first ranks at all schools in both \( \bar{F}^{kq} \) and in \( FTB^{kq+1} \). ■

Properties of \( \bar{F}^{kq} \)

In the current case, \( \rho^{kq} = m - 1 \). Hence, the equivalent of (34) for \( \bar{F}^{kq} \) is

\[
\begin{array}{cccccc}
    r & r + m - 1 & r + 2m - 2 & \ldots & r + (m - 1)m - (m - 1) \\
    r + m & r + 2m - 1 & r + 3m - 2 & \ldots & r + mm - (m - 1) \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    r + (\tau^{kq} - 1)m & r + \tau^{kq}m - 1 & r + (\tau^{kq} + 1)m - 2 & \ldots & r + (\tau^{kq} + m - 2)m - (m - 1) \\
    r + \tau^{kq}m & - & - & \ldots & - \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    r + (\tau^{kq} + m - 3)m & - & - & \ldots & - \\
    r + (\tau^{kq} + m - 2)m & - & - & \ldots & - \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    (kq + 1)m & - & - & \ldots & - \\
\end{array}
\]

(43)

where, \( (kq + 1)m \), the rank of student \( kq + 1 \), was simply added in position \( (kq + 1) \) at school \( s_1 \). In (43), \( r := \bar{r}_{1}^{F^{kq}} = m + \tau^{kq} \frac{(m-1)m}{2} \).

Observation 3:

\[
\bar{r}_{i}^{kq} < \bar{r}_{i+1}^{kq}, \quad \text{for all } i \in \{1, \ldots, kq + 1\}. \tag{44}
\]
Proof. Observe that, once \( r \) is removed from every total rank, the list of total ranks in (43) can be re-written as

\[
0, \\
m - 1, m, \\
2m - 2, 2m - 1, 2m, \\
\
\vdots \\
(m - 1)m - (m - 1), (m - 1)m - (m - 2), \ldots, (m - 1)m \\
mm - (m - 1), mm - (m - 2), \ldots, mm \\
\vdots \\
(\tau_{kq} + m - 4)m - (m - 1), (\tau_{kq} + m - 4)m - (m - 2), (\tau_{kq} + m - 4)m - (m - 3), (\tau_{kq} + m - 4)m, \\
(\tau_{kq} + m - 3)m - (m - 1), (\tau_{kq} + m - 3)m - (m - 2), (\tau_{kq} + m - 3)m, \\
(\tau_{kq} + m - 2)m - (m - 1), (\tau_{kq} + m - 2)m, \\
(kq + 1)m \\
\text{(45)}
\]

In (45), the total ranks in each row are arranged in increasing order. Thus, the smallest and largest elements of each rank are

\[
0, \\
m - 1, m, \\
2m - 2, 2m, \\
\vdots \\
(m - 1)m - (m - 1), (m - 1)m \\
mm - (m - 1), mm \\
\vdots \\
(\tau_{kq} + m - 4)m - (m - 1), (\tau_{kq} + m - 4)m, \\
(\tau_{kq} + m - 3)m - (m - 1), (\tau_{kq} + m - 3)m, \\
(\tau_{kq} + m - 2)m - (m - 1), (\tau_{kq} + m - 2)m, \\
(kq + 1)m \\
\text{(46)}
\]

Observe that the difference between the smallest total rank in one row and the largest total rank in the preceding row is always at least \( m - (m - 1) = 1 \). Together with the fact that the total ranks in each rows are arranged in increasing order, this proves (44). 

\[\tag{47} \hat{\alpha}^{\text{F}_{kq}}_{i + m} \geq \hat{\alpha}^{\text{F}_{kq}}_i + m, \quad \text{for all } i \in \{1, \ldots, kq + 1 - m\}.\]

Proof. By the argument in the proof of Observation 3, the total ranks in (45) are arranged in increasing order. Then (51) follows from the fact that no row in (45) contains more than \( m \) total ranks.\footnote{When \( \tau_{kq} \leq m - 2 \), all rows in (45) contain at most \( \tau_{kq} + 1 \) total ranks with \( \tau_{kq} + 1 < m \).}
Properties of \(FTB^{kq+1}\)

Applying (34) to the case \(\tau^{kq} + 1 = \frac{kq+1}{m}\), we obtain

\[
\begin{align*}
\hat{r}' &= \hat{r}' + m \\
\hat{r}' &= \hat{r}' + m \\
\vdots \\
\hat{r}' + \tau^{kq}m &= \hat{r}' + \tau^{kq}m \\
\hat{r}' + \tau^{kq}m &= \hat{r}' + \tau^{kq}m \\
\end{align*}
\]

(48)

where above, \(r' := \hat{r}_1^{kq+1}\), and

\[
\hat{r}_1^{kq+1} = 1 + \sum_{i=1}^{m-1} (1 + (1 + \tau^{kq})i),
\]

\[
= 1 + (m - 1) + (1 + \tau^{kq}) \sum_{i=1}^{m-1} i,
\]

\[
= m + (1 + \tau^{kq}) \frac{(m - 1)m}{2}.
\]

From (48), it can bee seen that

\[
\hat{r}_b^{kq+1} = r' + ym,
\]

(49)

where

\[
r' = m + (\tau^{kq} + 1) \frac{(m - 1)m}{2}
\]

and

\[
y = \begin{cases} 
0 & \text{if } b \leq m, \\
1 & \text{if } b \in \{m + 1, \ldots, 2m\}, \\
2 & \text{if } b \in \{2m + 1, \ldots, 3m\}, \\
\vdots \\
\tau^{kq} & \text{if } b \in \{\tau^{kq}m + 1, \ldots, kq + 1\}.
\end{cases}
\]

Observation 5:

\[
\hat{r}_{x+1}^{kq+1} = \ldots = \hat{r}_{x+a}^{kq+1} = \ldots = \hat{r}_{x+m}^{kq+1}, \quad \text{for all } x \in \{1, \ldots, \tau^{kq}\} \text{ and all } a \in \{1, \ldots, m\}.
\]

(50)

Proof. This follows directly from (49).

Observation 6:

\[
\hat{r}_{i+m}^{kq+1} = \hat{r}_{i}^{kq+1} + m, \quad \text{for all } i \in \{1, \ldots, kq + 1 - m\}.
\]

(51)

Proof. This follows directly from (49).

58
The proof now follows in two substeps. In Substep 1, we show that for all \(x \in \{1, \ldots, \tau^{kq}\},\)
\[
\sum_{i=1}^{x} \hat{r}_i^{kq} \leq \sum_{i=1}^{x} \hat{r}_i^{kq+1}.
\] (52)

Then, in Substep 2, we show that for all \(b \neq x\),
\[
\sum_{i=1}^{b} \hat{r}_i^{kq} \leq \sum_{i=1}^{b} \hat{r}_i^{kq+1}.
\] (53)

Substep 1: proof of (52).

In order to derive a contradiction, assume that there is a \(\tilde{x} \in \{1, \ldots, \tau^{kq}\}\) for which
\[
\sum_{i=1}^{\tilde{x}} \hat{r}_i^{kq} > \sum_{i=1}^{\tilde{x}} \hat{r}_i^{kq+1}.
\] (54)

For any \(x \in \{1, \ldots, \tau^{kq}\},\) let \(N^F_x := \{xm - (m - 1), \ldots, xm\}\) be the set of the \(m\) students whose total ranks lies between the \(xm - (m - 1)\)-th and the \(xm\)-th position in the ordered vector of total rank \(\hat{r}^F\) for pattern \(F\). Inequality (54) implies that for some \(x' \leq \tilde{x}\) we have
\[
\sum_{i \in N^{F+1}_x} \hat{r}_i^{kq} > \sum_{i \in N^{F+1}_x} \hat{r}_i^{kq+1}.
\] (55)

Notice that, by (51) (Observation 6) we have that
\[
\sum_{i \in N^{kq+1}_x} \hat{r}_i^{kq+1} = m^2 + \sum_{i \in N^{kq+1}_x} \hat{r}_i^{kq+1}.
\]
Also, by (47) (Observation 4), we have that
\[
\sum_{i \in N^{kq}_x} \hat{r}_i^{kq} \geq m^2 + \sum_{i \in N^{kq}_x} \hat{r}_i^{kq}.
\]

More generally, together with (47) and (51), (55) implies that
\[
\sum_{i \in N^{F+1}_x} \hat{r}_i^{kq} > \sum_{i \in N^{F+1}_x} \hat{r}_i^{kq+1} \text{ for all } x'' \geq x'.
\] (56)

Therefore, together with (56), (54) implies that
\[
\sum_{i=1}^{kq+1} \hat{r}_i^{kq} > \sum_{i=1}^{kq+1} \hat{r}_i^{kq+1},
\]
which contradicts (42) (Observation 2).

Substep 2: proof of (53).

Take any $b' = xm + a$ with $a \in \{1, 2, \ldots, m\}$ and with $x \in \{0, 1, \ldots, \tau^{kq}\}$. We must show that

$$\sum_{i=1}^{b'} \hat{r}_{i}^{kq} \leq \sum_{i=1}^{b'} \hat{r}_{i}^{kq+1},$$

$$\sum_{i=1}^{xm} \hat{r}_{i}^{kq} + \left( \hat{r}_{xm+1} + \cdots + \hat{r}_{xm+a} \right) \leq \sum_{i=1}^{xm} \hat{r}_{i}^{kq+1} + \left( \hat{r}_{xm+1} + \cdots + \hat{r}_{xm+a} \right).$$

If it was the case that

$$\sum_{i=1}^{b'} \hat{r}_{i}^{kq} > \sum_{i=1}^{b'} \hat{r}_{i}^{kq+1},$$

then by (52) we have that

$$\hat{r}_{xm+1} + \cdots + \hat{r}_{xm+a} > \hat{r}_{xm+1} + \cdots + \hat{r}_{xm+a}.$$

By (50) (Observation 5) and (44) (Observation 3), last inequality implies that $\hat{r}_{xm+a} > \hat{r}_{xm+a}$ and also

$$\hat{r}_{xm+a+1} + \cdots + \hat{r}_{xm+m} > \hat{r}_{xm+a+1} + \cdots + \hat{r}_{xm+m}$$

and therefore

$$\sum_{i=1}^{xm+m} \hat{r}_{i}^{kq} > \sum_{i=1}^{xm+m} \hat{r}_{i}^{kq+1},$$

contradicting (52). Notice that this argument also applies to the particular case $b' = a$. For the particular case $b' = \tau^{kq} + a$, we have a contradiction with (42). The desired result follows then from (41).

Step 2: for all $b \in \{kq + 1, \ldots, n\}$, $\sum_{i=1}^{b} \hat{r}_{i}^{FTB} \leq \sum_{i=1}^{b} \hat{r}_{i}^{SPR}$.

In order to derive a contradiction, assume that, for some $b^* \in \{kq + 1, \ldots, n\}$, $\sum_{i=1}^{b^*} \hat{r}_{i}^{FTB} > \sum_{i=1}^{b^*} \hat{r}_{i}^{SPR}$. We know that

$$\sum_{i=1}^{n} \hat{r}_{i}^{FTB} = \sum_{i=1}^{n} \hat{r}_{i}^{SPR}. \quad (57)$$

Hence, proving

$$\sum_{i=b^*+1}^{n} \hat{r}_{i}^{FTB} \geq \sum_{i=b^*+1}^{n} \hat{r}_{i}^{SPR}. \quad (58)$$
Thus, we have for any \( z \) line, and, the double plain lines in the case \( t \) minus the rank of orthonal symmetry around rank \( n/2 \). That is, the rank of \( t \) at school \( s \) in \( F^X \) equals \( n + 1 \) minus the rank of \( t \) at school \( s \) in \( F^X \). These constructions are illustrated below for the case \( kq = 3 \), where the orthogonal symmetry is taken with respect to the dashed line, and, the double plain lines in \( FTB \) and \( FTB \) represents the separation between the students with high and low priorities.

\[
\begin{array}{ccc}
F_1^{FTB} & F_2^{FTB} & F_3^{FTB} \\
\hline
t_1 & t_3 & t_2 \\
t_2 & t_1 & t_3 \\
t_3 & t_2 & t_1 \\
\hline \\
\uparrow kq & \downarrow (n - kq) \\
\hline
t_4 & t_8 & t_7 \\
t_5 & t_4 & t_8 \\
t_6 & t_5 & t_4 \\
t_7 & t_6 & t_5 \\
t_8 & t_7 & t_6 \\
\end{array}
\quad
\begin{array}{ccc}
F_1^{FTB} & F_2^{FTB} & F_3^{FTB} \\
\hline
t_1 & t_3 & t_2 \\
t_2 & t_1 & t_3 \\
t_3 & t_2 & t_1 \\
\hline \\
\uparrow kq & \downarrow (n - kq) \\
\hline
t_4 & t_8 & t_7 \\
t_5 & t_4 & t_8 \\
t_6 & t_5 & t_4 \\
t_7 & t_6 & t_5 \\
t_8 & t_7 & t_6 \\
\end{array}
\quad
\begin{array}{ccc}
F_1^{SPR} & F_2^{SPR} & F_3^{SPR} \\
\hline
t_1 & t_7 & t_5 \\
t_2 & t_8 & t_6 \\
t_3 & t_1 & t_7 \\
t_4 & t_2 & t_8 \\
\hline \\
t_5 & t_3 & t_1 \\
t_6 & t_4 & t_2 \\
t_7 & t_5 & t_3 \\
t_8 & t_6 & t_4 \\
\end{array}
\quad
\begin{array}{ccc}
F_1^{SPR} & F_2^{SPR} & F_3^{SPR} \\
\hline
t_1 & t_7 & t_5 \\
t_2 & t_8 & t_6 \\
t_3 & t_1 & t_7 \\
t_4 & t_2 & t_8 \\
\hline \\
t_5 & t_3 & t_1 \\
t_6 & t_4 & t_2 \\
t_7 & t_5 & t_3 \\
t_8 & t_6 & t_4 \\
\end{array}
\tag{59}
\quad
\begin{array}{ccc}
F_1^{SPR} & F_2^{SPR} & F_3^{SPR} \\
\hline
t_1 & t_7 & t_5 \\
t_2 & t_8 & t_6 \\
t_3 & t_1 & t_7 \\
t_4 & t_2 & t_8 \\
\hline \\
t_5 & t_3 & t_1 \\
t_6 & t_4 & t_2 \\
t_7 & t_5 & t_3 \\
t_8 & t_6 & t_4 \\
\end{array}
\tag{60}

By definition of \( \bar{X} \), we have \( r_{i}^{\bar{X}} = m(n+1) - r_{i}^{X} \), which implies \( \bar{r}_{i}^{\bar{X}} = m(n+1) - \bar{r}_{(n+1)-i}^{X} \). Thus, we have for any \( z \in \{1, \ldots, n-1\} \) that

\[
\sum_{i=1}^{z} \bar{r}_{i}^{FTB} = zm(n + 1) - \sum_{i=n-z}^{n} \bar{r}_{i}^{FTB} \quad \text{and} \quad \sum_{i=1}^{z} \bar{r}_{i}^{SPR} = zm(n + 1) - \sum_{i=n-z}^{n} \bar{r}_{i}^{SPR}
\]

which implies that (58) is equivalent to

\[
(n + 1 - b^*)m(n + 1) - \sum_{i=1}^{(n+1)-b^*} \bar{r}_{i}^{FTB} \geq (n + 1 - b^*)m(n + 1) - \sum_{i=1}^{(n+1)-b^*} \bar{r}_{i}^{SPR}
\]

\[
\sum_{i=1}^{(n+1)-b^*} \bar{r}_{i}^{FTB} \leq \sum_{i=1}^{(n+1)-b^*} \bar{r}_{i}^{SPR}. \quad \tag{61}
\]

But observe that because \( b^* > kq \), we have \( (n + 1 - b^*) \leq n - kq \). That is, in \( FTB \), the sum in (61) is over students with high priorities only. But then, (61) follows from Step 1, and we have the desired contradiction.

To see why, observe that, from the point of view of ex-post fairness, \( FTB \) is equivalent to the \( FTB \) pattern where exactly \( (n - kq) \) students have high priority, and \( kq \) students
have low priority. This can be seen by observing that the pattern $\tilde{FTB}$ obtained from $FTB$ by yet another orthogonal symmetry, this time around the $m/2$’s column yields the same distribution of total ranks as $FTB$, and is indeed of the form of an $FTB$ pattern with $(n - kq)$ high priority students. This is illustrated below, where the orthogonal symmetry is with respect to the second column.

\[
\begin{array}{ccc}
F_{1}^{FTB} & F_{2}^{FTB} & F_{3}^{FTB} \\
\uparrow (n - kq) & \downarrow kq \\
t_{8} & t_{7} & t_{6} \\
t_{7} & t_{6} & t_{5} \\
t_{6} & t_{5} & t_{4} \\
t_{5} & t_{4} & t_{3} \\
t_{4} & t_{3} & t_{2} \\
t_{3} & t_{2} & t_{1} \\
t_{2} & t_{1} & t_{3} \\
t_{1} & t_{3} & t_{2} \\
\end{array}
\begin{array}{ccc}
F_{1}^{\tilde{FTB}} & F_{2}^{\tilde{FTB}} & F_{3}^{\tilde{FTB}} \\
\uparrow (n - kq) & \downarrow kq \\
t_{6} & t_{7} & t_{8} \\
t_{5} & t_{6} & t_{7} \\
t_{4} & t_{5} & t_{6} \\
t_{3} & t_{4} & t_{5} \\
t_{2} & t_{3} & t_{4} \\
t_{1} & t_{2} & t_{3} \\
\end{array}
\]

(62)

**Step 3:** If $kq < n$ then $\sum_{i=1}^{kq} \hat{r}_{i}^{FTB} < \sum_{i=1}^{kq} \hat{r}_{i}^{SPR}$.

Given that in $F^{FTB}$ the $kq$ smallest priority ranks are attributed at all schools to the same $kq$ students, we have for the case $b^* = kq$ that

$$\sum_{i=1}^{kq} \hat{r}_{i}^{FTB} = \sum_{i=1}^{kq} \hat{r}_{i}^{STB}.$$  

There remains to show for $F^{SPR}$ that, when $kq < n$, there is a student $t$ that has a priority rank smaller than $kq$ at some school and a priority rank larger than $kq$ at another school. If this is the case, we have

$$\sum_{i=1}^{kq} \hat{r}_{i}^{SPR} > \sum_{i=1}^{kq} \hat{r}_{i}^{STB}.$$  

There are two cases

- **Case 1:** $\lfloor n/m \rfloor < kq$.

  Consider student $t_n$ who has priority rank $n$ at school $s_1$. Given that $kq < n$, $t_n$ has a priority rank larger than $kq$ at $s_1$. The priority rank of $t_n$ at school $s_2$ is equal to $n + \lfloor n/m \rfloor$ modulo $n$, which is equal to $\lfloor n/m \rfloor$. This case being such that $\lfloor n/m \rfloor < kq$, $t_n$ has a priority rank smaller than $kq$ at $s_2$, the desired result.

- **Case 2:** $\lfloor n/m \rfloor \geq kq$.

  Consider student $t_1$ who has the priority rank 1 at school $s_1$. Given that $q \geq 1$, $t_1$ has a priority rank smaller than $kq$ at $s_1$. The priority rank of $t_1$ at school $s_2$ is equal to $1 + \lfloor n/m \rfloor$ modulo $n$. This case being such that $\lfloor n/m \rfloor \geq kq$, $t_1$ has a priority rank larger than $kq$ at $s_2$ if $1 + \lfloor n/m \rfloor \leq n$. By assumption $m \geq 2$ and therefore $\lfloor n/m \rfloor < n$, implying that $1 + \lfloor n/m \rfloor \leq n$, the desired result.
Figure 6: Screen-shot of the proof of concept for a decision support app.
B Decision support app

A screenshot of the app can be found in Figure 6. This is an example of the message received by a student who, given the preference she entered in the app, has a dominant strategy ($k = 5$).

**List the schools in the order of your preferences**

You listed the following preference for schools (in actual applications, this would be inputed by the user):

```
2 10 1 4 7 6 5 8
```

**Which schools should you report and how should you rank them?**

Because you can only report 5 schools, it is important that you report the right combination of schools.

If you only report schools that are in high demand, or schools at which you have a low priority, you may be rejected from all the schools you report and end up unassigned.

This decision support tool will help you determine the schools you may want to report, and how you should rank them.

**Our advice**

(Dominant Strategy) Given the preferences you inputed and the priorities at schools, you have a clear best strategy.

Listing your 5 most-preferred schools in the order of your preference will guarantee you a better assignment than with any other ranking you could submit.

We strongly recommend that you report the following preference list:

```
2 10 1 4 7
```

This is an example of the message received by a student who, given the preference she entered in the app, does not have a single safe set ($k = 5$).

**List the schools in the order of your preferences**

You listed the following preference for schools (in actual applications, this would be inputed by the user):

```
5 1 7 3 9 6 2 4
```

**Which schools should you report and how should you rank them?**

Because you can only report 5 schools, it is important that you report the right combination of schools.
If you only report schools that are in high demand, or schools at which you have a low priority, you may be rejected from all the schools you report and end up unassigned.

This decision support tool will help you determine the schools you may want to report, and how you should rank them.

**Our advice**

(Not a single safe set) In your case, it is especially important that you be selective in choosing the schools that you report.

Given the preferences you inputed and the priorities at schools, there is no combination of schools that can guarantee that you would not end up unassigned.

In particular, if you report

$$5 1 7 3 9$$

there is a chance you could end up unassigned.

The risk that you end up unassigned is particularly high if you only list schools that are in high demand.

We strongly recommend that you include some schools that are in low demand in your report in order to protect yourself as much as possible from ending up unassigned.

Finally, this is an example of the message received by a student who, given the preference she entered in the app, does not have a dominant strategy but does have safe sets ($k = 5$)

**List the schools in the order of your preferences**

You listed the following preference for schools (in actual applications, this would be inputed by the user):

$$7 8 2 4 3 1 5 9$$

**Which schools should you report and how should you rank them?**

Because you can only report 5 schools, it is important that you report the right combination of schools.

If you only report schools that are in high demand, or schools at which you have a low priority, you may be rejected from all the schools you report and end up unassigned.

This decision support tool will help you determine the schools you may want to report, and how you should rank them.

**Our advice**

(Safe sets, but no dominant strategy) Given the preferences you inputed and the priorities at schools, you do not have a clear best strategy.

In particular, if you report
there is a chance you could end up unassigned.
However, some of the list of schools you can report can protect your from ending up unassigned.
One of the best strategies for you is to report the following preference list:

7 2 3 1 5

This strategy will guarantee that you are at worst assigned to school 5.
No other strategy can guarantee you a better worst case assignment.
The following are other rankings that guarantee that you will be assigned and that you may be interested in reporting:

c(7, 8, 3, 1, 5), c(8, 2, 3, 1, 5)

References