Top Trading Cycles in Prioritized Matching: An Irrelevance of Priorities in Large Markets

Yeon-Koo Che* Olivier Tercieux†

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Abstract

We study top trading cycles in a two-sided matching environment (Abdulkadiroglu and Sonmez (2003)) under the assumption that individuals' preferences and objects' priorities are drawn *iid* uniformly. We show that the *number of individuals/objects assigned* at each round follows a simple Markov chain and we explicitly derive the transition probabilities. This Markov property is used to shed light on the role priorities play in TTC. We show that, as the market grows large, the effect of priorities in TTC disappears, leading in the limit to an assignment that entails virtually the same amount of justified envy as does RSD.

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1 Introduction

Top Trading Cycles (TTC) algorithm, introduced by Abdulkadiroglu and Sonmez (2003) in a prioritirized resource allocation, has been an influential method for achieving efficient outcomes particularly in school choice environments. For instance, TTC was used until recently in New Orleans school systems for assigning students to public high schools and

*Department of Economics, Columbia University, USA. Email: yeonkooche@gmail.com.
†Department of Economics, Paris School of Economics, France. Email: tercieux@pse.ens.fr.

recently San Francisco school system announced plans to implement a top trading cycles mechanism. A generalized version of TTC is also used for kidney exchange among donor-patient pairs with incompatible donor kidneys (see Sonmez and Unver (2011)).

The TTC assigns agents efficiently based on their preferences and is strategyproof, providing them with the incentive to report truthfully as dominant strategy. These qualities are not unique to TTC, however. Serial dictatorship (SD), where the agents take turns in claiming objects one at a time according to a serial order, and its random variant, Random Serial Dictorship (RSD), where SD is run with randomly chosen serial order, guarantee Pareto efficiency and strategyproofness. Instead, the unique distinction of TTC lies with its use of agents' priorities in resource allocation. For example, Boston public schools system prioritizes a student based on his/her sibling or walkzone status; New York city public schools use a student's academic performances, the borough of her residence, and registration in a school's annual information session, among other things, for the same purpose, depending on the school's types. Housing allocation and organ exchanges also have similar priority system based on a number of factors. Respecting agents' priorities—defined formally as ensuring that each agent weakly prefers her assignment to any other potential choice (e.g., school seat, housing, an organ) either unassigned or assigned to an agent with lower priority—is an important desideratum. This is also desirable from the fairness standpoint, for it eliminates justified envy (Balinski and Sönmez (1999) and Abdulkadiroglu and Sonmez (2003))—namely, an agent envying another despite having higher priority at the latter's assignment.

Unlike SD or RSD which completely ignores agents' priorities, TTC explicitly uses them for allocation: at each round, agents enjoying the highest priorities for objects available at that round may trade their priorities to obtain their preferred objects. In particular, the agent with the highest priority for an object may claim that object, if she so chooses, ahead of all other agents who may also desire it. This feature makes one hopeful that TTC may do well in respecting priorities, at least among those that satisfy efficiency and strategyproofness. Indeed, Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) demonstrates that among this class of mechanisms, TTC is *justified envy minimal*—namely no other efficient and strategyproof mechanism reduces the set of agents with justified envy. By contrast, the SD is not justified envy minimal in the same sense, regardless of how the serial order is chosen. Still, a direct comparison between TTC and RSD in terms of justified envy is difficult, since one can find profiles of priorities and preferences for which TTC admit more (expected) incidences of justified envy than does RSD.[1] Nevertheless, Abdulkadiroglu,

[1]Consider an instance with three students and three schools each with a single seat. Preferences of individuals $i \in \{i_1, i_3\}$ are $o_1 \succ_i o_2 \succ_i o_3$ while preferences for individual i_2 are $o_2 \succ_{i_2} o_1 \succ_{i_2} o_3$. Objects'

Che, Pathak, Roth, and Tercieux (2017) demonstrate that, for each profile of preferences, TTC admits fewer incidences of justified *on average* than RSD, when average is taken with respect to all possible profiles of priorities; or equivalently, TTC admits probilistically less justified envy than RSD, when the agents' priorities are drawn uniform randomly.

Just *how well* does TTC do in respecting priorities? Namely what is the "quantitative" significance of the benefit TTC yields in respecting priorities say over and above RSD? The aforementioned characterization is silent on this question. A main contribution of this paper is to demonstrate that, at least for some canonical environment, the answer to this question is in the negative. Consider a classic one-to-one matching environment with n agents and n objects, where the agents' preferences and their priorities for objects are drawn iid uniformly. We show that as the economy grows large with $n \to \infty$, the outcome of TTC in terms of the *joint* distribution of preference ranks enjoyed by the agents *and* their realized priorities becomes asymptotically equivalent to that under the RSD.[2] This means, among other things, that TTC does virtually no better in respecting priorities than does RSD. More precisely, the proportion of agents with justified envy (or those whose priorities are respected) under TTC becomes indistinguishable from that under RSD, both from the average and probabilistic senses.

The reason for this striking result is explained by the particular way in which TTC uses agents' priorities for allocating objects. If an object is assigned via a *short cycle*—or a cycle in which an agent points to an object and the object in turn points to that agent—, then the acquiring agent is likely to have high priority, so it is impossible that somebody else's priority for that object is violated. The matters are quite different, however, if an object is assigned via a *long cycle*, or a cycle of length more than 2. In that case, the acquiring agent has no *a priori* reason to have a higher priority compared with any other agent who may also like that object; the only distinction is that the former has a priority that she can exchange eventually with somebody who has the high priority for the object in question. Hence, any object o assigned via a long cycle is assigned uniform-randomly across

priorities in turn are given by $i_2 \succ_{o_1} i_3 \succ_{o_1} i_1$ and $i_1 \succ_{o_2} i_3 \succ_{o_2} i_2$. We don't need to specify the priority ranking for object o_3. If we run TTC, we end up with i_1 and i_2 getting their top choices (trading their priorities) while i_3 gets o_3. There are two blocking pairs (i_3, o_1) and (i_3, o_2). Now, let us consider RSD. If the realization of the serial order has i_3 in last position one gets exactly the same outcome as with TTC and so two blocking pairs. More generally, one can easily check that for any realized serial order there are either one or two blocking pairs. So, in particular, the expected number of blocking pairs under RSD is actually smaller than the number of blocking pairs under TTC.

[2]The well-known equivalence result of Carroll (2014) means that the *marginal* distribution of the ranks enjoyed by agents are identical between the two mechanisms. As will be noted below, however, the *joint* distribution of agents' preference ranks *and* their priorities are distinct in the finite economy.

all agents whose priority ranks are below the priority rank $R^*(o)$ of the agent o points to at the round of its assignment. Our irrelevance result is a consequence of the findings that, as the market grows large, (i) the proportion of objects assigned via short cycles out of all objects assigned vanishes in probability and (ii) $R^*(o)/n$'s tend to zero for all objects o's, except for a proportion vanishing in probability. In other words, TTC allocates virtually all objects via long cycles rather than short cycles and all objects point to virtually top (more precisely sublinear) priority agents when they are assigned, and these are why the priorities become virtually irrelevant in a large market.

We argue that the irrelevance is robust to the introduction of correlation in agents' preferences. Irrelevance also extends to many-to-one matching but only when the number of copies per each object type grows sufficiently slowly as the economy grows large. Importantly, irrelevance does *not* extend when the number of copies for each object type grows fast, in which case TTC performs significantly better than RSD in respecting priorities, as pointed out by Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017). Hence, our ultimate contribution is to clarify the circumstances in which TTC's use of priorities are relevant in the large market.

While our irrelevance result appears intuitive, its proof requires a deep and precise characterization of how the TTC allocates objects in our random model. A crucial step is to establish a Markov property: *the number of objects assigned at any round of TTC follows a simple Markov chain, with the number depending only on the number of agents and objects at the beginning of that round in a well-specified manner.* We further exhibit the formula for the transition probabilities governing the Markov chain. The Markov characterization allows us to show that TTC algorithm terminates in the number of rounds which is sublinear in n. With the number of objects assigned via short cycles further shown never to exceed two per round, this implies that the proportion of objects assigned via short cycles vanishes in probability, leading ultimately to the irrelevance claim stated above.

We view the Markov characterization of TTC as our second main contribution, of independent value beyond the particular application explored in the current paper. It is of interest since it can lead to a precise understanding of the outcome of TTC, in terms of the distribution of ranks that agents enjoy and the ranks that objects enjoy. While the former is known from the analysis of RSD due to its equivalence,[3] the current analysis may shed additional light on the agents' welfare more directly based on the formula derived in Theorem 1. More importantly, the distribution of ranks enjoyed by the objects (e.g., school)

[3]See Knuth (1996) for the rank distribution of agents under RSD and see Pathak and Sethuraman (2011) for the equivalence which generalizes that of Knuth (1996) and Abdulkadiroglu and Sönmez (1998).

has not been known before. We provide a characterization of the ranks that lead to the irrelevance result mentioned above, as well as the asymptotic instability of TTC in the large market studied in Che and Tercieux (2015). Also, the Markov characterization is highly nontrivial. The difficulty stems from the fact that the preferences of agents and those of objects remaining after the first round of TTC need not be uniform, with their distributions affected nontrivially by the realized event of the first round of TTC, and the nature of the conditioning is difficult to analyze.[4] A remarkable implication of our chracterization is that even though the exact composition of cycles are subject to the conditioning issue, the number of agents/objects assigned in each round follows a Markov chain, and is thus free from the conditioning issue.

The current paper is related to several strands of literature. First, the Markov characterization of TTC is related closely to a similar Markov characterization of the Shapley-Scarf TTC derived in Frieze and Pittel (1995). The Shapley-Scarf TTC has agents endowed with *property rights* over objects, *exactly one object for each agent*, and are allowed to trade their rights along cycles in successive rounds. Despite the close resemblance, the two mechanisms are distinct in terms of the distribution of agents that objects point to. The associated ("pointing") map from objects to agents is always bijective in the Shapley-Scarf TTC since alternative agents own distinct objects but not bijective in our TTC. This difference leads to different probablistic structures in the associated composite map—agents pointing to objects which in turn point to agents—in our random economy, and required different arguments albeit following a similar approach. The concepts of random rooted forests and random composite maps prove crucial in our analysis, which to our knowledge have never been applied in economics. We believe they may constitute a useful tool box in other economic applications.

Second, the irrelevance result is closely related to the equivalence across a class of random allocations recognized by a number of authors (Knuth (1996), Abdulkadiroglu

[4]To see this, assume that the set of agents and objects have the same size n and that they are indexed from $1, \ldots, n$. Observe first that in Round 1 of TTC, each pair of an individual and an object has probability $1/n^2$ to form a cycle of order 2. Since there are n^2 such pairs, at Round 1, the expected number of cycles of order 2 is 1. Now, to see where the conditioning issue comes from, consider the event that at Round 1 of TTC, each object points to the individual with the same index while each individual with index $k \leq n-1$ points to the object with index $k+1$. Finally assume that individual n points to object n. Given this event, observe that at Round 1 a single cycle clears and it only involves the individual and the object with index n. Thus, conditionally on this event, the expected number of cycles of order 2 in Round 2 is much smaller than 1. Indeed, in Round 2, only individual $n-1$ can be part of a cycle of order 2 and the only way for this to happen is for individual $n-1$ to point to object $n-1$. This occurs with probability $1/(n-1)$ and so the expected number of cycles of order 2 goes to 0 as n grows.

and Sönmez (1998), Pathak and Sethuraman (2011), Carroll (2014), Bade (2016)). These authors show that a class of random allocations, including TTC and RSD, implement the identical (probabilistic) assignment of agents to objects. However, they are silent on the *joint* disribution of agents' preference ranks *and* their priorities under alternative mechanisms. As illustrated, the joint distribution of allocation matters for the extent to which agents' priorities are respected and to which agents justifiably envy others. As we will shortly illustrate, the alternative mechanisms are not equivalent in this regard for a finite market. Nevertheless, the equivalence is restored as the market grows large. Our result therefore can be seen as the strengthening of equivalence (to include priority rank distribution), for the iid preferences case.

Third, the current paper is related to the literature studying the tradeoff between efficiency and stability of mechanisms particularly in the school choice context. The tradeoff was first recognized by Roth (1982), and was confirmed by Abdulkadiroglu and Sonmez (2003) in the school choice context, by Abdulkadiroglu, Pathak, and Roth (2009) in the context of indifferences. Che and Tercieux (2015) show that the tradeoff remains significant even quantatively in a large market with sufficient correlation in agents' preferences. Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) show the sense in which TTC minimizes justified envy/maximizes priority-respecting in the class of Pareto-efficient and strategyproof mechanisms in one-to-one matching setting. Together with the irrelevance result of the current paper, this result yields the sense in which the irrelevance is driven by the Pareto efficiency and strategyproofness rather than the feature of TTC itself.

Finally, the current paper studies the large literature studying TTC (Shapley and Scarf (1974), Ma (1995), Abdulkadiroglu and Sonmez (2003)). Leshno and Lo (2017) studies TTC in a large market but with a very different asymptotics where the number of object types is finite while there are a continuum of copies/seats for each object type has and a continuum of agents with finite preference types. This distinction makes the analysis largely unrelated.

The remainder of the paper is organized as follows. Section 2 illustrates the main irrelevance result in an example. Section 3 introduces the formal model and preliminary tools for analysis. Section 4 provide the Markov characterization. Section 5 presents the irrelevance result and its implications. Section 6 discusses robustness and limitation of our results. We argue that the irrelevance is robust to the introduction of correlation in agents' preferences.

2 Example

Suppose there are two agents, 1 and 2, and two objects, a and b, and the ordinal preference ranking of each entity, an agent or an object, over the entities on the other side is iid uniform. By the equivalence result of Carroll (2014), the agents' assignment probabilities are exactly the same under TTC and RSD for each profile of preferences. This does not mean, however, equivalence from the perspective of the objects. In particular, one can see that the preference ranks enjoyed by the objects differ between the two mechanisms.

To see this precisely, we compute the expected rank enjoyed by the object under RSD and TTC. This is simple for RSD. Given the uniform iid assumption, an object has the equal chance of matching with its first- and second-best agents in RSD. Hence, the expected rank an object enjoys under RSD is:

$$\mathrm{RANK}_{RSD} = \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 2 = 3/2.$$

For TTC, there are two cases. With probability 1/2, the agents prefer different objects. In this case, Pareto efficiency (satisfied by TTC) dictates that the agents obtain their preferred objects. This means that from the objects' perspectives, the assignment is completely (uniform) random. With the remaining probability 1/2, the agents prefer the same object, say a. In this case, a matches with its top choice agent, via a "short" cycle it forms with that agent. Object b matches with the left-over agent, so its assignment is (uniform) random. Since an object has probability 1/2 of being the commonly preferred item, with probability 1/4 it matches with its top choice and with the remaining probability 3/4, it matches randomly, or its top choice with probability 1/2. In sum, under TTC, an object enjoys the expected rank:

$$\mathrm{RANK}_{TTC} = (\tfrac{1}{4} + \tfrac{3}{4} \cdot \tfrac{1}{2}) \cdot 1 + (1 - \tfrac{1}{4} - \tfrac{3}{4} \cdot \tfrac{1}{2}) \cdot 2 = 11/8,$$

which is less than 3/2.

This difference in assignment gives rise to the difference in the likelihood of there being justified envy. Under TTC, no justified envy arises here; whenever an agent is assigned its less preferred object (the second case above), its preferred object matches with its top choice. But under RSD, when both agents prefer the same object, that object does not match its preferred agent. There is 1/4 chance of there being one agent who has justified envy.

Ultimately, the difference between the two mechanisms is traced to the fact that TTC uses agents' priorities (or objects' ranking of agents) to organize trades, in fact more precisely to their use in short cycles. The main observation we make below is that increasingly few objects as a proportion of the entire set of objects are assigned via short cycles as the market grows large. And this implies that priorities become increasingly irrelevant as the market grows large.

3 Model

We consider a market consisting of a set I of **agents** and a set O of **objects**. We do not require $|I| = |O|$, so the market could be unbalanced. A **matching** is a map $\mu : I \to O \cup \{\o\}$ such that $i \neq i' \Rightarrow \mu(i) \cap \mu(i') \setminus \{\o\} = \emptyset$, where $\o$ means not being matched.

In Section 5, we will consider a large market, and for that purpose, we will assume $|I| = |O| = n$ and consider the outcome as $n \to \infty$. The preferences of each agent i is a permutation P_i of O interpreted as her preference ranking over objects (in the descending order), and the preference of each object o, or agents' priorities for o is a permuation, denoted $\succ_o$, of I again interpreted as its preference ranking over agents (in descendign order). This assumes that all partners on the other side are acceptable. Let $(P, \succ) := (P_i, \succ_o)_{i \in I, o \in O}$ be a profile of preferences. Throughout, we shall consider a **random market** $(I, O, \tilde{P}, \tilde{\succ})$ in which the preference of each entity, agent or object, is drawn *iid* uniformly. Another way to interpret our random economy is that we are taking average over all priorities and preferences, say for the purpose of evaluating the relative incidence of justified envy under alternative mechanisms. From this perspective, randomness is simply an analytical device to compute the average performance of interest (e.g. incidence of justified envy).

Let ω be a (realized) **state** which consists of a profile $(P, \succ)$ of preferences on both sides and a (realized) random variable θ. A **mechanism** is a mapping from each state to a matching. Given our random economy, randomness may arise from the random structure generating preferences or the randomness in θ, as is the case with RSD. In RSD, a serial order—a permutation of I—is chosen at random and, following that order, each agent claims the most preferred remaining object and exits the market, starting with the agent at the top of the serial order. The resulting matching defines the mechanism, where the random serial order constitutes θ.

Our main interest is with TTC, which proceeds in multiple rounds as follows. In Round $t = 1, \ldots$, each individual $i \in I$ points to his most preferred object (if any). Each object $o \in O$ points to the individual who has the highest priority at that object. Since the number of individuals and objects are finite, the directed graph thus obtained has at least one cycle. Every individual who belongs to a cycle is assigned the object he is pointing at. All assigned individuals and objects are then removed. The algorithm terminates when all individuals have been assigned; otherwise, it proceeds to Round $t + 1$. This mechanism terminates in finite rounds. Indeed, there are finite individuals, and at least one individual is removed at the end of each round. The TTC mechanism selects a matching via this algorithm for all possible profiles of agents' preferences and their priorities.

As noted, both RSD and TTC mechanisms are **Pareto efficient**; that is, for each profile P of agents' preferences, the matching produced by either mechanism cannot be improved upon by a different matching that makes all agents weakly better off and some strictly better off. Note that Pareto efficiency is defined only taking the agents' welfare into account. Both are **strategyproof**; namely, each agent has a dominant strategy of reporting her perference truthfully, in the sense that a shift to truth-telling from any report of preference yields a first-order stochastically dominating shift of allocation in terms of her preference ranking.

In fact, the two mechanisms are identical from the agents' perspectives. The random matchings two mechanisms induce for each profile P of agents' preferences give rise to an identical lottery for each agent (Carroll (2014)). As noted in Section 2, the equivalence does not extend to the object side. To gain more precise comparison of the mechanisms in this regard, we need to understand the probabilistic structure of allocation in TTC more precisely.

4 Markov Chain Property of TTC

The main challenge in analyzing TTC is to deal with the conditioning issue: the preferences of the agents and objects remaining in any round $t \geq 2$ depend nontrivially on the exact history of the TTC process up to the previous round. The condition prevents us from invoking the oft-used *principle of deferred decision*, whereby one views each agent as *drawing* preferences of the remaining objects at random in each round, instead of having drawn preferences for objects in the beginning.

To illustrate the conditioning issue, suppose there are three agents, 1,2 and 3, and three objects, a, b, and c. Suppose agent 3 matches with object c in Round 1, and two agents, 1 and 2, and two objects, a and b remain in Round 2. Whether a given agent say 1 can point to either object depends on what she pointed to in Round 1. If she had pointed to c, then she can point to a or b at random, much in the way prescribed by the principle of deferred decision. But if she had pointed to say b, then she cannot *redraw* her target of pointing; she must continue to point to b. In particular, this means that a simple structure such as that of Markov cannot exist at the individual level of agents/objects.

Nonetheless, a Markov structure exists at a more aggregate level, with respect to the *total* numbers of agents and objects that are assigned in each round of TTC.

THEOREM 1. *Suppose any round of TTC begins with n agents and o objects remaining in the market. Then, the probability that there are $m \le \min\{o, n\}$ agents assigned at the end of that round is*

$$p_{n,o;m} = \left(\frac{m}{(on)^{m+1}}\right)\left(\frac{n!}{(n-m)!}\right)\left(\frac{o!}{(o-m)!}\right)(o+n-m).$$

Thus, denoting n_i and o_i the number of individuals and objects remaining in the market at any round i, the random sequence (n_i, o_i) is a Markov chain.

PROOF. See Appendix A. □

This theorem means that the numbers of agents and objects that are assigned in each round of TTC follow a simple Markov chain depending *only* on the numbers of agents and objects at the beginning of that round. It also characterizes the probability structure of the Markov chain. This implies that there are no conditioning issues at least with respect to the *total* numbers of agents and objects that are assigned in each round of TTC. Namely, one does not need to keep track of the precise history leading up to a particular economy at the beginning of a round, as far as the numbers of objects assigned in that round is concerned.

One can combine Theorem 1 with some existing result to yield fuller understanding of TTC process. For instance, as shown in Appendix B, one can compute the expectation and the variance of the number of agents matched at a given stage of TTC given the the remaining number of individuals and objects at the beginning of that round.

Next, it is instructive to compare our result with Frieze and Pittel (1995)'s analyis of Shapley-Scarf TTC. They obtain a similar Markov chain result for Shapley-Scarf TTC. Our result allows us to compare the two Markov chains. Specifically, we can order the two chains in terms of likelihood ratio order. To see this, let us recall the transition probabilities of the Markov chain obtained by Frieze and Pittel (1995):

$$\hat{p}_{n;m} = \frac{n!}{n^m(n-m)!}\frac{m}{n}$$

By Theorem 1, we obtain (assuming $n = o$):

$$\begin{aligned} p_{n;m} \quad : \quad &= p_{n,n;m} = \left(\frac{m}{(n)^{2(m+1)}}\right)\left(\frac{n!}{(n-m)!}\right)^2(2n-m) \\ &= \left(\frac{n!}{n^m(n-m)!}\right)^2\left(\frac{m(2n-m)}{n^2}\right). \end{aligned}$$

Let us compare the two distributions in terms of likelihood ratio order. Fix $n \geq 1$ and any $m' \geq m$. It is easy to check that

$$\frac{\hat{p}_{n,m'}}{\hat{p}_{n,m}} = \frac{n^m (n-m)!}{n^{m'}(n-m')!} \frac{m'}{m}$$

while

$$\frac{p_{n,m'}}{p_{n,m}} = \left(\frac{n^m (n-m)!}{n^{m'}(n-m')!} \right)^2 \frac{m'}{m} \frac{2n-m'}{2n-m}.$$

Now, observe that

$$\begin{aligned} \left(\frac{\hat{p}_{n,m'}}{\hat{p}_{n,m}} \right)^{-1} \frac{p_{n,m'}}{p_{n,m}} &= \left(\frac{1}{n^{m'-m}} \right) \left(\frac{(n-m)!}{(n-m')!} \right) \frac{(2n-m')}{(2n-m)} \\ &= \frac{(n-m)(n-m-1)...(n-m'+1)}{n^{m'-m}} \frac{2n-m'}{2n-m} \leq 1. \end{aligned}$$

This proves that for any n, the distribution $\hat{p}_{n,\cdot}$ dominates $p_{n,\cdot}$ in terms of likelihood ratio order. One can prove an interesting implication of this result: for each $t \geq 1$, the probability that TTC stops before Round t is smaller than the probability that Shapley-Scarf TTC stops before Round t. Put in another way, the random round at which TTC stops is (first order) stochastically dominated by that at which the Shapley-Scarf TTC stops.

Interestingly, Frieze and Pittel (1995) show that, in the case of balanced market with $|I| = |O| = n$ the number of rounds required for Shapley-Scarf TTC to conclude is on average in the order of $\sqrt{n}$, where n is the number of agents/objects (see Theorem 1 of Frieze and Pittel (1995)).[5] Our TTC on average takes longer to complete. Nevertheless, we can show that our TTC terminates not much more slowly:

Proposition 1. *Assume $|I| = |O| = n$. Let T denote the number of rounds required for TTC to conclude. Then, $\frac{T}{n} \xrightarrow{p} 0$.*

Proof. See Appendix C. □

5 Irrelevance of Priorities in Large Markets

As highlighted before, the key observation for the irrelevance result is that any object o assigned via a long cycle is assigned uniform-randomly across agents below the rank R_o^* of agent that o points to at the time it is assigned. The proof will be completed by establishing

[5] The expected number of the rounds is: $\sqrt{\frac{8}{\pi} n} - \frac{3}{\pi} \log(n) + O(1)$.

that all objects, except for a fraction vanishing in probability, are assigned via long cycles and have the rank R_o^* sublinear in n. With this goal in mind, we define two (random) sets of objects. First, let $\hat{O}$ denote the set of objects that are assigned via long cycles (as defined earlier). Second, let $\tilde{O}$ denote the set of objects which point to agents these objects rank sublinearly when they are assigned. Specifically, for a fixed $\varepsilon > 0$, let

$$\tilde{O} := \{o \in O | R_o^* \leq \log^{1+\varepsilon}(n)\},$$

where R_o^* is the priority rank order of the individual that an object o points to when it is assigned in TTC. Fix $\varepsilon > 0$ and let $\bar{O} := \hat{O} \cap \tilde{O}$.

The significance of this set is that any object in this set is uniform-randomly assigned across individuals whom the object ranks below R_o^* (i.e., ranks larger than R_o^*). This in turn means that the rank enjoyed by each object $o \in \bar{O}$ is uniformly distributed across $\{\lceil \log^{1+\varepsilon}(n) \rceil + 1, ..., n\}$ conditional on the rank R_o being higher than $\log^{1+\varepsilon}(n)$. To precisely characterize the implication of this observation for the joint distribution of ranks, we define a few pieces of notation. First we let R_o (resp. R_i) denote the rank enjoyed by object o (resp. enjoyed by individual i) under TTC. We also let an arbitrary vector $(x_k)_{k \in K}$ be denoted by $\mathbf{x}_K$. For instance, $\mathbf{R}_O$ stands for $\{R_o\}_{o \in O}$. We are now ready to present the corner stone for our irrelevance result, the proof of which can be found in Appendix D.

PROPOSITION 2. *Fix any $I' \subseteq I$ and $O'' \subseteq O$. For any $O' \subseteq O''$, given $\bar{O} = O''$, the distribution of the collection $\{\mathbf{R}_{O'}, \mathbf{R}_{I'}\}$ is stochastically dominated by the collection $\{\{Y_o\}_{o \in O'}, \mathbf{R}_{I'}\}$ where $\{Y_o\}_{o \in O'}$ is a collection of iid random variables where each Y_o follows the uniform distribution over $\{\lceil \log^{1+\varepsilon}(n) \rceil + 1, ..., n\}$. Formally, for any $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}_{I'}$,*

$$\Pr\left\{\mathbf{R}_{O'} \leq \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \bar{O} = O''\right\} \geq \prod_{o \in O'} \Pr\left\{Y_o \leq \ell_o\right\} \times \Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \bar{O} = O''\right\}.$$

In addition, for any $O' \subseteq O''$, given $\bar{O} = O''$, the distribution of the collection $\{\mathbf{R}_{O'}, \mathbf{R}_{I'}\}$ stochastically dominates the collection $\{\{X_o\}_{o \in O'}, \mathbf{R}_{I'}\}$ where $\{X_o\}_{o \in O'}$ is a collection of iid random variables where each X_o follows the uniform distribution over $\{1, ..., n\}$. Formally, for any $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}_{I'}$,

$$\Pr\left\{\mathbf{R}_{O'} \leq \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \bar{O} = O''\right\} \leq \prod_{o \in O'} \Pr\left\{X_o \leq \ell_o\right\} \times \Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \bar{O} = O''\right\}.$$

Roughly speaking, the proposition asserts that the distribution of the rank enjoyed by each object within $\bar{O}$ is "squeezed" (accoring to first-order stochastic dominance) in between uniform from $\{\lceil \log^{1+\varepsilon}(n) \rceil + 1, ..., n\}$ from above and uniform from $\{1, ..., n\}$ from below, independently of the distribution of the ranks enjoyed by the agents and the ranks enjoyed by the other objects in the set $\bar{O}$. Since $\log^{1+\varepsilon}(n)/n \to 0$ as $n \to \infty$, the joint distribution converges to the iid uniform distribution as $n \to \infty$.

We next prove that this set $\bar{O}$ eventually comprises the entire proportion of the objects as $n \to \infty$.

PROPOSITION 3. $\frac{|\bar{O}|}{n} \xrightarrow{p} 1$ *as* $n \to \infty$.

PROOF. It suffices to prove that $\frac{|\hat{O}|}{n} \xrightarrow{p} 1$ and $\frac{|\tilde{O}|}{n} \xrightarrow{p} 1$. We first prove former. Corollary S1 of our Supplementary Appendix proves that at each round of TTC, irrespective of the history, the expected number of objects matched via short cycles is smaller than 2. Thus, denoting by $\hat{o}_t$ the number of objects involved in short cycles at Step t of TTC, we must have $\mathbb{E}[\hat{o}_t\,|T] \leq 2$. Hence,

$$\begin{aligned} \frac{1}{n}\mathbb{E}[|\hat{O}|] &= 1 - \frac{1}{n}\mathbb{E}\left[\sum_{t=1}^{T} \hat{o}_t\right] \\ &= 1 - \frac{1}{n}\mathbb{E}_T\left[\mathbb{E}\left[\sum_{t=1}^{T} \hat{o}_t\,|\right]\right] \\ &\geq 1 - 2\mathbb{E}_T\left[\frac{T}{n}\right] \to 1, \end{aligned}$$

where the convergence result comes from Proposition 1. Since $\frac{1}{n}\mathbb{E}[|\hat{O}|] \to 1$ implies $\frac{|\hat{O}|}{n} \xrightarrow{p} 1$, we are done.

We next prove that $\frac{|\tilde{O}|}{n} \xrightarrow{p} 1$. To this end, we define a new mechanism TTC*, which operates exactly like TTC, except that, in each round, objects in each cycle are assigned to the agents that the objects point to (rather than the other way around). Clearly, in each round of TTC*, the same cycles as those in the corresponding round of TTC are formed, and the same associated set of agents and objects are assigned and removed. One crucial difference, though, is that the assignment is Pareto efficient *from the perspective of objects*. Proposition 1 of Che and Tercieux (2017), applied to the object side, then implies the result. □

We are almost ready for the main theorem. In the sequel, if W is a random variable defined on $\{1, ..., n\}$ then we let $\bar{W}$ be equal to $\frac{1}{n}W$. Denote by $O^n := \{o_1, ..., o_n\}$ and $I^n := \{i_1, ..., i_n\}$ respectively the set of objects and the set of individuals in an n-economy. Let $\{V_{o_1}, ..., V_{o_n}, V_{i_1}, ..., V_{i_n}\}$ be a sequence of collections of $2n$ random variables, each random variable taking values in $[0, 1]$. We need to define a notion of convergence for a vector whose length increases as $n \to \infty$:

Definition 1. *A random vector* $\{\bar{R}_{o_1}, ..., \bar{R}_{o_n}, \bar{R}_{i_1}, ..., \bar{R}_{i_n}\}$ ***converges in distribution to*** $\{V_{o_1}, ..., V_{o_n}, V_{i_1}, ..., V_{i_n}\}$ *as* $n \to \infty$ *if for any integer* K*, any* $x \in [0,1]^K$ *and any sequence* $\{y^n\}$ *with values in* $[0,1]^n$*, we have*

$$\lim_{n\to\infty} |F^n(x, y^n) - G^n(x, y^n)| = 0$$

where F^n *is the cdf of* $\{\bar{R}_{o_1}, ..., \bar{R}_{o_K}, \bar{R}_{i_1}, ..., \bar{R}_{i_n}\}$ *while* G^n *is the cdf of* $\{V_{o_1}, ..., V_{o_K}, V_{i_1}, ..., V_{i_n}\}$*.*

Recall that $\bar{X}_{o_1}, ..., \bar{X}_{o_n}$ is a collection of iid random variables each being $U\{1, ..., n\}/n$. Since RSD ignores priorities, Carroll (2014)'s equivalence result implies that the distribution of ranks of objects and individuals under RSD is exactly $\{\bar{X}_{o_1}, ..., \bar{X}_{o_n}, \bar{R}_{i_1}, ..., \bar{R}_{i_n}\}$.

As we already mentioned, $\frac{|\bar{O}|}{n} \xrightarrow{p} 1$. Hence, for any given integer K (which does not depend on n), $\Pr\left\{\{o_1, ..., o_K\} \subseteq \bar{O}\right\}$ converges to 1. In addition, from the above proposition, we directly obtain that for any $I' \subseteq I$ and for any $O' \subseteq O$, for any $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}_{I'}$,

$$\Pr\left\{\mathbf{R}_{O'} \leq \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid O' \subseteq \bar{O}\right\} \geq \prod_{o\in O'} \Pr\left\{Y_o \leq \ell_o\right\} \times \Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid O' \subseteq \bar{O}\right\}$$

and, in addition,

$$\Pr\left\{\mathbf{R}_{O'} \leq \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid O' \subseteq \bar{O}\right\} \leq \prod_{o\in O'} \Pr\left\{X_o \leq \ell_o\right\} \times \Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid O' \subseteq \bar{O}\right\}.$$

We are now ready to present our main result, suggesting the irrelevance of priorities in TTC: the limit distribution of ranks enjoyed by the objects as $n \to \infty$ is uniform just like RSD.

Theorem 2. $\{\bar{R}_{o_1}, ..., \bar{R}_{o_n}, \bar{R}_{i_1}, ..., \bar{R}_{i_n}\}$ *converges in distribution to* $\{\bar{X}_{o_1}, ..., \bar{X}_{o_n}, \bar{R}_{i_1}, ..., \bar{R}_{i_n}\}$ *as* $n \to \infty$*.*

This result shows that the joint distribution of ranks under TTC converges in distribution to the joint distribution of ranks of RSD, showing that as the market grows, objects' priorities become irrelevant.

The irrelevance results given in Theorem 2 and Proposition 2 have a number of important corollaries. First, the empirical cumulative distribution function of ranks of objects $\{\bar{R}_{o_1}, ..., \bar{R}_{o_n}\}$ converges in probability to the cumulative distribution of the uniform distribution. This is fairly intuitive: Theorem 2 suggests that the collection of $\{\bar{R}_{o_1}, ..., \bar{R}_{o_n}\}$ converges in distribution to a collection of iid uniform random variables over $[0, 1]$. Hence, using some version of the LLN yields the result.

Corollary 1. *Fix any* $x \in [0,1]$. *We must have* $\frac{1}{n}\sum_{o\in O}\mathbf{1}\{\bar{R}_o \le x\} \xrightarrow{p} x$.

Proof. See Appendix E. $\square$

Our result also has natural implications when comparing TTC and RSD in terms of justified envy. Indeed, the next result shows that relative to the size of the market, the incidences of justified envy under TTC and RSD become indistinguishable.

Corollary 2. *Fix a pair* (i,o). *The difference between the probability that* (i,o) *blocks TTC and the probability that* (i,o) *blocks RSD goes to* 0. *Hence, the difference in expected fraction of blocking pairs under TTC and RSD converges to* 0.[6]

Proof. See Appendix F. $\square$

6 Discussion

TTC achieves Pareto efficient assignment of agents to objects, through the trading of their priorities at objects. A recent paper by Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) develops the sense in which this use of priorities in TTC minimizes justified envy: in the one-to-one matching, no efficient and strategyproof mechanism can further reduce justified envy over TTC. While this does not mean that for any given priorities TTC entails strictly less justified envy than other efficient mechanism, such as Serial Dictatorship, Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) further prove that TTC does admit fewer justified envy than RSD on average when the average is taken over all possible priorities, or equivalently when priorities are drawn at random.

Strikingly, however, the current paper suggests that the envy-minimizing benefit of TTC vanishes in a large market; simply put, the priorities in TTC have virtually no effect on the outcome, compared with RSD, in a sufficiently large market. We must caution, however, that this result need not be the consequence of the specific feature of TTC. Rather, the envy-minimality of TTC suggests that, at least in the environment we considered, the vanishing role of priorities is likely to be the consequence of Pareto efficiency and strategyproofness; namely, these two requirements prove too strong for priorities to have any residual effect on the outcome. Moreover, the irrelevance of priorities rests on the canonical one-to-one matching environment. As we argue below, in the many-to-one matching environment, the irrelevance result extends under certain asymptotics but not under others. In this sense,

[6] The fraction of blocking pair corresponds to the total number of blocking pairs divided by the total number of possible pairs $|I| \times |O|$.

one must not view our contribution as establishing general irrelevance of priorities under TTC but as clarifying more precisely than before about the types of large economies in which TTC's use of priorities matters.

We now discuss the robustness of our result.

6.1 Correlated preferences:

If agents' preferences are perfectly correlated, the TTC leaves no justified envy, as it implements the unique matching that is both Pareto efficient and (perfectly) stable. By contrast, RSD will entail a significant amount of justified envy even in the limit. But this case is extreme. A more interesting case is when the preferences are not perfectly correlated. Suppose for instance that agents' preferences are represented by a cardinal utility function, $u_i(o) = u_o + \xi_{io}$, where u_o is a common utility from object o common for all agents and ξ_{io} is a idiosyncratic utility from o drawn iid for each agent i. In particular, the support of common utility is finite and sufficiently far apart from each other that the objects are effectively "tiered": all agents prefer top tier objects (with the highest value of u_o), and they all prefer the second tier objects next, and so on. In this case, the TTC are effectively partitioned into multiple stages: in stage 1, all agents point to objects in tier 1, and once all tier 1 objects are assigned, stage 2 begins in which the remaing agents point to tier 2 objects, and they are assigned, etc. Since agents' priorities at each object is iid, each stage can be separated as a distinct TTC market, for which our asymptotic irrelevance result would apply. Hence, in this environment, the irrelevance result extends.

6.2 Many-to-one matching:

Our model has considered an one-to-one matching. While one-to-one matching serves as a good baseline model, many real-world situations involve many-to-one matchings. School choice or housing allocation typically involves multiple seats or mulitiple identical units available for assignment. We believe that our result of asymptotic irrelevance extends to many-to-one matching setting as long as the number of copies per object type grows sufficiently slowly compared with the number of object types. Such asymptotics, which one may call a "small school" model, has been adopted by a number of authors, such as Kojima and Pathak (2009), Ashlagi, Kanoria, and Leshno (2017), Che and Tercieux (2017) and Che and Tercieux (2015), fits well settings such as medical matching (where about 20,000 doctors apply to about 3,000-4,000 hospitals), and NYC public high school matching in which the number of programs (about 800) exceeds the number of students admitted by each program (about 100).

At the same time, the irrelevance result does not extend to the other commonly used asymptotics, in which the number of copies per object type grows sufficiently fast compared with the number of object types. Such asymptotics, which one may call a "large school" model, has been adopted by many authors such as Abdulkadiroglu, Che, and Yasuda (2015), Azevedo and Leshno (2016), Che, Kim, and Kojima (2013) and Leshno and Lo (2017), and fits well with the school choice in many US cities in which a handful of schools admit each hundreds of students. To see that our irrelevance result does not extend to this environment, suppose that the number of object types is fixed at some finite number, as the number of students and the copies of each object type grows large. In that case, the proportion of agents that are assigned via short cycles under TTC does not vanish even in probability. Since the agents assigned via short cycles under TTC tend to have exhibit less justified envy than under RSD, the amount of justified envy is smaller under TTC than under RSD (see Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017)).[7]

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[7]More precisely, the expected number of agents each agent justifiably envies is strictly smaller under TTC than under RSD in the limit economy when the probability measure over types has full support.

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A Proof of Theorem 1

Given the nature of conditioning mentioned earlier, it is crucial for our purpose to keep track of the agents and objects that *can* draw their partners at random and those who *cannot* in each round of TTC. This requires us to investigate the probabilistic structure known as random rooted forests.

To begin, consider any two finite sets I and O, with cardinalities $|I| = n, |O| = o$. A **bipartite digragh** $G = (I \times O, E)$ consists of vertices I and O on two separate sides and directed edges $E \subset (I \times O) \cup (O \times I)$, comprising ordered pairs of the form (i, o) or (o, i) (corresponding to edge originating from i and pointing to o and an edge from o to i, respectively). A **rooted tree** is a bipartite digraph where all vertices have out-degree 1 except the root which has out-degree 0.[8] A **rooted forest** is a bipartite graph which consists of a collection of disjoint rooted trees. A **spanning rooted forest over** $I \cup O$ is a forest comprising vertices $I \cup O$. From now on, a spanning forest will be understood as being over $I \cup O$.

[8] Sometimes, a tree is defined as an acyclic undirected connected graph. In such a case, a tree is rooted when we name one of its vertex a "root." Starting from such a rooted tree, if all edges now have a direction leading toward the root, then the out-degree of any vertex (except the root) is 1. So the two definitions are actually equivalent.

We begin by noting that TTC induces a random sequence of spanning rooted forests. Indeed, one could see the beginning of the first round of TTC as a situation where we have the trivial forest consisting of $|I| + |O|$ trees with isolated vertices. In the above examples, there are 6 separate trees: $\{1\}, \{2\}, \{3\}, \{a\}, \{b\}, \{c\}$. Within this step, each vertex in I will randomly point to a vertex in O and each vertex in O will randomly point to a vertex in I. Say in the above example that agents 1 and 3 point to c, and agent 2 points to a, and all objects point to 3. Note that once we delete the realized cycles ($3 - c$ in the example), we again get a spanning rooted forest. So we can think again of the beginning of the second round of TTC as a situation where we start with a spanning rooted forest where the agents and objects remaining from the first round form this spanning rooted forest, where the roots consist of those agents and objects that had pointed to the entities that were cleared via cycles. In the above example, the spanning rooted forest in the begining of Round 2 has three rooted trees: $\{1\}, \{2 \to b\}, \{a\}$. Here again objects that are roots randomly point to a remaining individual and individuals that are roots randomly point to a remaining object. Once cycles are cleared we again obtain a forest and the process goes on like this.

A.1 Markov Properties of Spanning Rooted Forests

Formally, the random sequence of forests, $F_1, F_2, ...$, is defined as follows. First, we let F_1 be a trivial unique forest consisting of $|I| + |O|$ trees with isolated vertices, forming their own roots. For any $i = 2, ...$, we first create a random directed edge from each root of F_{i-1} to a vertex on the other side, and then delete the resulting cycles (these are the agents and objects assigned in round $i - 1$) and F_i is defined to be the resulting rooted forest.

For any rooted forest F_i, let $N_i = I_i \cup O_i$ be its vertex set and $k_i = (k_i^I, k_i^O)$ be the vector denoting the numbers of roots on both sides, and use (N_i, k_i) to summarize this information. And let $\mathcal{F}_{N_i,k_i}$ denote the set of all rooted forests having N_i as the vertex set and k_i as the vector of its root numbers.

Lemma 1. *Given $(N_j, k_j), j = 1, ..., i$, every (rooted) forest of $\mathcal{F}_{N_i,k_i}$ is equally likely.*

Proof. We prove this result by induction on i. Since for $i = 1$, by construction, the trivial forest is the unique forest which can occur, this is trivially true for $i = 1$. Fix $i \geq 2$, and assume our statement is true for $i - 1$.

Fix $N_i = I_i \cup O_i \subset N_{i+1} = I_{i+1} \cup O_{i+1}$, and k_i and k_{i+1}. For each forest $F \in \mathcal{F}_{N_{i+1},k_{i+1}}$, we consider a possible pair (F', ϕ) that could have given rise to F, where $F' \in \mathcal{F}_{N_i,k_i}$ and ϕ maps the roots of F' in I_i to its vertices in O_i as well as the roots of F' in O_i to its vertices in I_i. In words, such a pair (F', ϕ) corresponds to a set N_i of agents and objects remaining at the beginning of round i of TTC, of which k_i^I agents of I_i and k_i^O objects have lost their favorite parties (and thus they must repoint to new partners in N_i under TTC in round i), and the way in which they repoint to the new partners under TTC in round i causes a new forest F to emerge at the beginning of round $i+1$ of TTC. There are typically multiple such pairs that could give rise to F.

We start by showing that each forest $F \in \mathcal{F}_{N_{i+1},k_{i+1}}$ arises from the same number of such pairs—i.e., that the number of pairs (F', ϕ), $F' \in \mathcal{F}_{N_i,k_i}$, causing F to arise does not depend on the particular $F \in \mathcal{F}_{N_{i+1},k_{i+1}}$. To this end, for any given $F \in \mathcal{F}_{N_{i+1},k_{i+1}}$, we construct all such pairs by choosing a quadruplet (a, b, c, d) of four non-negative integers with $a + c = k_i^I$ and $b + d = k_i^O$,

(i) choosing c old roots from I_{i+1}, and similarly, d old roots from O_{i+1},

(ii) choosing a old roots from $I_i \backslash I_{i+1}$ and similarly, b old roots from $O_i \backslash O_{i+1}$,

(iii) choosing a partition into cycles of $N_i \backslash N_{i+1}$, each cycle of which contains at least one old root from (ii),[9]

(iv) choosing a mapping of the $k_{i+1}^I + k_{i+1}^O$ new roots to $N_i \backslash N_{i+1}$.[10]

Clearly, the number of pairs (F', ϕ), $F' \in \mathcal{F}_{N_i,k_i}$, satisfying the above restrictions depends only on $|I_i|$, $|O_i|$, k_i, k_{i+1}, and $|N_{i+1}| - |N_i|$.[11] We denote the number of such pairs by $\beta(|I_i|, |O_i|, k_i; |N_{i+1}| - |N_i|, k_{i+1})$. Let $\phi_i = (\phi_i^I, \phi_i^O)$ where ϕ_i^I is the random mapping from the roots of F_i in I_i to O_i and ϕ_i^O is the random mapping from the roots of F_i in O_i to I_i. Let $\phi = (\phi^I, \phi^O)$ be a generic mapping of that sort. Since, conditional on $F_i = F'$, the mappings ϕ_i^I and ϕ_i^O are uniform, we get

$$\Pr(F_{i+1} = F | F_i = F') = \frac{1}{|O_i|^{k_i^I}} \frac{1}{|I_i|^{k_i^O}} \sum_{\phi} \Pr(F_{i+1} = F | F_i = F', \phi_i = \phi). \qquad (1)$$

[9]Within round i of TTC, one cannot have a cycle creating only with nodes that are not roots in the forest obtained at the beginning of round i. This is due to the simple fact that a forest is an acyclic graph. Thus, each cycle creating must contain at least one old root. Given that, by definition, these roots are eliminated from the set of available nodes in round $i+1$, these old roots that each cycle must contain must be from (ii).

[10]Since, by definition, any root in $F \in \mathcal{F}_{N_{i+1},k_{i+1}}$ does not point, this means that, in the previous round, this node was pointing to another node which was eliminated at the end of that round.

[11]Recall that by definition of TTC, whenever a cycle creates, the same number of individuals and objects must be eliminated in this cycle. Hence, $|O_{i+1}| - |O_i| = |I_{i+1}| - |I_i|$ and $|N_{i+1}| - |N_i| = 2|I_{i+1}| - |I_i|$.

Therefore, we obtain

$$\begin{aligned}
&\Pr(F_{i+1} = F|(N_1, k_1), ..., (N_i, k_i)) \\
=&\sum_{F' \in \mathcal{F}_{N_i,k_i}} \Pr(F_{i+1} = F, F_i = F'|(N_1, k_1), ..., (N_i, k_i)) \\
=&\sum_{F' \in \mathcal{F}_{N_i,k_i}} \Pr(F_{i+1} = F|(N_1, k_1), ..., (N_i, k_i), F_i = F') \Pr(F_i = F'|(N_1, k_1), ..., (N_i, k_i)) \\
=&\frac{1}{|\mathcal{F}_{N_i,k_i}|} \sum_{F' \in \mathcal{F}_{N_i,k_i}} \Pr(F_{i+1} = F|F_i = F') \\
=&\frac{1}{|\mathcal{F}_{N_i,k_i}|} \sum_{F' \in \mathcal{F}_{N_i,k_i}} \frac{1}{|O_i|^{k_i^I}} \frac{1}{|I_i|^{k_i^O}} \sum_{\phi} \Pr(F_{i+1} = F|F_i = F', \phi_i = \phi) \\
=&\frac{1}{|\mathcal{F}_{N_i,k_i}|} \frac{1}{|O_i|^{k_i^I}} \frac{1}{|I_i|^{k_i^O}} \sum_{F' \in \mathcal{F}_{N_i,k_i}} \sum_{\phi} \Pr(F_{i+1} = F|F_i = F', \phi_i = \phi) \\
=&\frac{1}{|\mathcal{F}_{N_i,k_i}|} \frac{1}{|O_i|^{k_i^I}} \frac{1}{|I_i|^{k_i^O}} \beta(|I_i|, |O_i|, k_i; |N_{i+1}| - |N_i|, k_{i+1}),
\end{aligned} \tag{2}$$

where the third equality follows from the induction hypothesis and the Markov property of $\{F_j\}$, the fourth follows from (1), and the last follows from the definition of β and from the fact that the conditional probability in the sum of the penultimate line is 1 or 0, depending upon whether the forest F arises from the pair $(F^{'}, \phi)$ or not. Note that this probability is independent of $F \in \mathcal{F}_{N_{i+1},k_{i+1}}$. Hence,

$$\begin{aligned}
&\Pr(F_{i+1} = F|(N_1, k_1), ..., (N_i, k_i), (N_{i+1}, k_{i+1})) \\
=&\frac{\Pr(F_{i+1} = F|(N_1, k_1), ..., (N_i, k_i))}{\Pr(F_{i+1} \in \mathcal{F}_{N_{i+1},k_{i+1}}|(N_1, k_1), ..., (N_i, k_i))} \\
=&\frac{\Pr(F_{i+1} = F|(N_1, k_1), ..., (N_i, k_i))}{\sum_{\tilde{F} \in \mathcal{F}_{N_{i+1},k_{i+1}}} \Pr(F_{i+1} = \tilde{F}|(N_1, k_1), ..., (N_i, k_i))} = \frac{1}{|\mathcal{F}_{N_{i+1},k_{i+1}}|},
\end{aligned} \tag{3}$$

which proves that, given (N_j, k_j), $j = 1, ..., i$, every rooted forest of $\mathcal{F}_{N_i,k_i}$ is equally likely. □

The next lemma then follows easily.

LEMMA 2. *Random sequence (N_i, k_i) forms a Markov chain.*

PROOF. By (2) we must have

$$\begin{aligned}\Pr((N_{i+1},k_{i+1})|(N_1,k_1),....,(N_i,k_i)) &= \sum_{F\in\mathcal{F}_{N_{i+1},k_{i+1}}} \Pr(F_{i+1}=F|(N_1,k_1),....,(N_i,k_i)) \\ &= \sum_{F\in\mathcal{F}_{N_{i+1},k_{i+1}}} \frac{1}{|\mathcal{F}_{N_i,k_i}|}\frac{1}{|O_i|^{k_i^I}}\frac{1}{|I_i|^{k_i^O}}\beta(|I_i|,|O_i|,k_i;|N_{i+1}|-|N_i|,k_{i+1}).\end{aligned}$$

Observing that the conditional probability depends only on (N_{i+1},k_{i+1}) and (N_i,k_i), the Markov chain property is established. $\square$

The proof of Lemma 2 reveals in fact that the conditional probability of (N_{i+1},k_{i+1}) depends on N_i) only through its cardinalities $(|I_i|,|O_i|)$, leading to the following conclusion. Let $n_i:=|I_i|$ and $o_i:=|O_i|$.

COROLLARY 3. *Random sequence* $\{(n_i,o_i,k_i^I,k_i^O)\}$ *forms a Markov chain.*

PROOF. By symmetry, given $(n_1,o_1,k_1^I,k_1^O),...,(n_i,o_i,k_i^I,k_i^O)$, the set (I_i,O_i) is chosen uniformly at random among all the $\binom{n}{n_i}\binom{o}{o_i}$ possible sets. Hence,

$$\begin{aligned}&\Pr((n_{i+1},o_{i+1},k_{i+1}^I,k_{i+1}^O)|(n_1,o_1,k_1^I,k_1^O),....,(n_i,o_i,k_i^I,k_i^O)) \\ =&\sum_{(I_i,O_i):|I_i|=n_i,|O_i|=o_i} \Pr\{(n_{i+1},o_{i+1},k_{i+1}^I,k_{i+1}^O)|(n_1,o_1,k_1^I,k_1^O),....,(n_i,o_i,k_i^I,k_i^O),(I_i,O_i)\} \\ &\qquad\times \Pr\left\{(I_i,O_i)\,|\,(n_1,o_1,k_1^I,k_1^O),...,(n_i,o_i,k_i^I,k_i^O)\right\} \\ =&\left(\sum_{(I_i,O_i):|I_i|=n_i,|O_i|=o_i} \Pr\{(n_{i+1},o_{i+1},k_{i+1}^I,k_{i+1}^O)|(n_1,o_1,k_1^I,k_1^O),....,(I_i,O_i,k_i^I,k_i^O)\}\right)\frac{1}{\binom{n}{n_i}\binom{o}{o_i}} \\ =&\left(\sum_{\substack{(I_i,O_i):|I_i|=n_i,|O_i|=o_i\\(I_{i+1},O_{i+1}):|I_{i+1}|=n_{i+1},|O_{i+1}|=o_{i+1}}} \Pr\{(I_{i+1},O_{i+1},k_{i+1}^I,k_{i+1}^O)|(n_1,o_1,k_1^I,k_1^O),....,(I_i,O_i,k_i^I,k_i^O)\}\right) \\ &\qquad\times\frac{1}{\binom{n}{n_i}\binom{o}{o_i}} \\ =&\frac{1}{\binom{n}{n_i}\binom{o}{o_i}}\sum_{\substack{(I_i,O_i):|I_i|=n_i,|O_i|=o_i\\(I_{i+1},O_{i+1}):|I_{i+1}|=n_{i+1},|O_{i+1}|=o_{i+1}}} \Pr\{(I_{i+1},O_{i+1},k_{i+1}^I,k_{i+1}^O)|(I_i,O_i,k_i^I,k_i^O)\},\end{aligned}$$

where the second equality follows from the above reasoning and the last equality follows from the Markov property of $\{(I_i,O_i,k_i^I,k_i^O)\}$. The proof is complete by the fact that the last line, as shown in the proof of Lemma 2, depends only on $(n_{i+1},o_{i+1},k_{i+1}^I,k_{i+1}^O),(n_i,o_i,k_i^I,k_i^O)$). $\square$

Theorem 1 makes use of a few additional results.

A.2 Useful Computational Lemmas

The first lemma characterizes the number of spanning rooted forests.

LEMMA 3 (Jin and Liu (2004)). *Let $V_1 \subset I$ and $V_2 \subset O$ where $|V_1| = \ell$ and $|V_2| = k$. The number of spanning rooted forests having k roots in V_1 and ℓ roots in V_2 is $f(n, o, k, \ell) := o^{n-k-1} n^{o-\ell-1}(\ell n + ko - k\ell)$.*

For the next result, consider agents I' and objects O' such that $|I'| = |O'| = m > 0$. We say a mapping $f = h \circ g$ is a **bipartite bijection**, if $g : I' \to O'$ and $h : O' \to I'$ are both bijections. A **cycle** of a bipartite bijection is a cycle of the induced digraph. Note that a bipartite bijection consists of disjoint cycles. A **random bipartite bijection** is a (uniform) random selection of a bipartite bijection from the set of all bipartite bijections. The following result will prove useful for a later analysis.

LEMMA 4. *Fix sets I' and O' with $|I'| = |O'| = m > 0$, and a subset $K \subset I' \cup O'$, containing $a \geq 0$ vertices in I' and $b \geq 0$ vertices in O'. The probability that each cycle in a random bipartite bijection contains at least one vertex from K is*

$$\frac{a+b}{m} - \frac{ab}{m^2}.$$

PROOF. We begin with a few definitions. A **permutation** of X is a bijection $f : X \to X$. A **cycle** of a permutation is a cycle of the digraph induced by the permutation. A permutation consists of disjoint cycles. A **random permutation** chooses uniform randomly a permutation f from the set of all possible permutations. Our proof will invoke following result:

FACT 1 (Lovasz (1979) Exercise 3.6). *The probability that each cycle of a random permutation of a finite set X contains at least one element of a set $Y \subset X$ is $|Y|/|X|$.*

To begin, observe first that a bipartite bijection $h \circ g$ induces a permutation of set I'. Thus, a random bipartite bijection defined over $I' \times O'$ induces a random permutation of I'. To compute the probability that each cycle of a random bipartite bijection $h \circ g$ contains at least one vertex in $K \subset I' \times O'$, we shall apply Fact 1 to this induced random permutation of I'.

Indeed, each cycle of a random bipartite bijection contains at least one vertex in $K \subset I' \times O'$ if and only if each cycle of the induced random permutation of I' contains either a vertex in $K \cap I'$ or a vertex in $I' \setminus K$ that points to a vertex in $K \cap O'$ in the original random bipartite bijection. Hence, the relevant set $Y \subset I'$ for the purpose of applying Fact 1 is a random set that contains $|K \cap I'| = a$ vertices of the former kind and Z vertices of the latter kind.

The number Z is random and takes a value z, $\max\{b-a, 0\} \leq z \leq \min\{m-a, b\}$, with probability:

$$\Pr\{Z = z\} = \frac{\binom{m-a}{z}\binom{a}{b-z}}{\binom{m}{b}}.$$

This formula is explained as follows. $\Pr\{Z = z\}$ is the ratio of the number of bipartite bijections having exactly z vertices in $I' \setminus K$ pointing toward $K \cap O'$ to the total number of bipartite bijections.

Note that since we consider bipartite bijections, the number of vertices in I' pointing to the vertices in $K \cap O'$ must be equal to b. Focusing first on the numerator, we have to compute the number of bipartite bijections having exactly z vertices in $I' \setminus K$ pointing toward $K \cap O'$ and the remaining $b - z$ vertices pointing to the remaining $K \cap O'$. There are $\binom{m-a}{z}\binom{a}{b-z}$ ways one can choose z vertices from $I' \setminus K$ and $b - z$ vertices from $K \cap I'$. Thus, the total number of bipartite bijections having exactly z vertices in $I' \setminus K$ that point to $K \cap O'$ is $\binom{m-a}{z}\binom{a}{b-z}\upsilon$, where υ is the total number of bipartite bijections in which the b vertices thus chosen point to the vertices in $K \cap O'$. This gives us the numerator. As for the denominator, the total number of bipartite bijections having b vertices in I' pointing to $K \cap O'$ is $\binom{m}{b}$ (the number of ways b vertices are chosen from I'), multiplied by υ (the number of bijections in which the b vertices thus chosen point to the vertices in $K \cap O'$). Hence, the denominator is $\binom{m}{b}\upsilon$. Thus, we get the above formula.

Recall our goal is to compute the probability that each cycle of the random permutation induced by the random bipartite bijection contains at least one vertex in the random set Y, with $|Y| = a + Z$, where $\Pr\{Z = z\} = \frac{\binom{m-a}{z}\binom{a}{b-z}}{\binom{m}{b}}$. Applying Fact 1, then the desired probability is

$$\begin{aligned}
\mathbb{E}\left[\frac{|Y|}{|I'|}\right] =& \sum_{z=\max\{b-a,0\}}^{\min\{m-a,b\}} \Pr\{Z=z\}\frac{a+z}{m} \\
=& \frac{a}{m} + \sum_{z=\max\{b-a,0\}}^{\min\{m-a,b\}} \Pr\{Z=z\}\frac{z}{m} \\
=& \frac{a}{m} + \sum_{z=\max\{b-a,0\}}^{\min\{m-a,b\}} \frac{\binom{m-a}{z}\binom{a}{b-z}}{\binom{m}{b}}\left(\frac{z}{m}\right) \\
=& \frac{a}{m} + \left(\frac{m-a}{m\binom{m}{b}}\right) \sum_{z=\max\{b-a,1\}}^{\min\{m-a,b\}} \binom{a}{b-z}\binom{m-a-1}{z-1} \\
=& \frac{a}{m} + \left(\frac{m-a}{m\binom{m}{b}}\right)\binom{m-1}{b-1} \\
=& \frac{a}{m} + \frac{b(m-a)}{m^2} \\
=& \frac{a+b}{m} - \frac{ab}{m^2},
\end{aligned}$$

where the fifth equality follows from Vandermonde's identity. □

A.3 Proof of Theorem 1

Given Lemmas 1, 2 and Corollary 3, the proof of Theorem 1 is complete with the following lemma:

LEMMA 5. *The random sequence (n_i, o_i) is a Markov chain, with transition probability given by*

$$\begin{aligned}
p_{n,o;m} :=& \Pr\{n_i - n_{i+1} = o_i - o_{i+1} = m | n_i = n, o_i = o\} \\
=& \left(\frac{m}{(on)^{m+1}}\right)\left(\frac{n!}{(n-m)!}\right)\left(\frac{o!}{(o-m)!}\right)(o+n-m).
\end{aligned}$$

PROOF. We first compute the probability of transition from (n_i, o_i, k_i^I, k_i^O) such that $k_i^I + k_i^O = \kappa$ to $(n_{i+1}, o_{i+1}, k_{i+1}^I, k_{i+1}^O)$ such that $k_{i+1}^I = \lambda^I$ and $k_{i+1}^O = \lambda^O$:

$$\begin{aligned}
&\mathbf{P}(n, o, \kappa; m, \lambda^I, \lambda^O) \\
:=& \Pr\left\{n_i - n_{i+1} = o_i - o_{i+1} = m, k_{i+1}^I = \lambda^I, k_{i+1}^O = \lambda^O \mid n_i = n, o_i = o, k_i^I + k_i^O = \kappa\right\}.
\end{aligned}$$

This will be computed as a fraction $\frac{\Theta}{\Upsilon}$. The denominator Υ counts the number of rooted forests in the bipartite digraph with k_i^I roots in I_i and k_i^O roots in O_i where $k_i^I + k_i^O = \kappa$, multiplied by the ways in which k_i^I roots of I_i could point to O_i and k_i^O roots of O_i could point to I_i.[12] Hence, letting $f(n, o, k^I, k^O)$ denote the number of rooted forests in a bipartite digraph (with n and o vertices on both sides) containing k^I and k^O roots on both sides.

$$\begin{aligned}
\Upsilon =& \sum_{(k^I,k^O):k^I+k^O=\kappa} o^{k^I} n^{k^O} f(n, o, k^I, k^O) \\
=& \sum_{k^I+k^O=\kappa} o^{k^I} n^{k^O} \binom{n}{k^I}\binom{o}{k^O} o^{n-k^I-1} n^{o-k^O-1}(nk^O + ok^I - k^I k^O) \\
=& \sum_{k^I+k^O=\kappa} \binom{n}{k^I}\binom{o}{k^O} o^{n-1} n^{o-1}(nk^O + ok^I - k^I k^O) \\
=& o^n n^o \left(2\binom{n+o-1}{\kappa-1} - \binom{n+o-2}{\kappa-2}\right).
\end{aligned}$$

The first equality follows from the fact that there are $o^{k^I} n^{k^O}$ ways in which k^I roots in I_i point to O_i and k^O roots in O_i could point to I_i. The second equality follows from Lemma 3. The last uses Vandermonde's identity.

The numerator Θ counts the number of ways in which m agents are chosen from I_i and m objects are chosen from O_i to form a bipartite bijection each cycle of which contains at least one of κ old roots, and for each such choice, the number of ways in which the remaining vertices form a spanning rooted forest and the λ^I roots in I_{i+1} point to objects in $O_i \setminus O_{i+1}$ and λ^O roots in O_{i+1} point to agents in $O_i \setminus O_{i+1}$. To compute this, we first compute

$$\alpha(n, o, \kappa; m, \lambda^I, \lambda^O) = \sum_{(k^I,k^O):k^I+k^O=\kappa} \beta(n, o, k^I, k^O; m, \lambda^I, \lambda^O),$$

where β is defined in the proof of Lemma 1. In words, α counts, for any F with $n-m$ agents and $o-m$ objects and roots λ^I and λ^O on both sides, the total number of pairs (F', ϕ) that could have given rise to F, where F' has n agents and o objects with κ roots and ϕ maps the roots to the remaining vertices. Following the construction in the beginning of

[12] Given that we have $n_i = n$ individuals, $o_i = o$ objects and $k_i^I + k_i^O = \kappa$ roots at the beginning of step i under TTC, one may think of this as the total number of possible bipartite digraph one may obtain via TTC at the end of step i when we let k_i^I roots in I_i point to their remaining most favorite object and k_i^O roots in O_i point to their remaining most favorite individual.

the proof of Lemma 1, the number of such pairs is computed as

$$\begin{aligned}
&\alpha(n,o,\kappa;m,\lambda^I,\lambda^O)\\
:=&\sum_{a+b+c+d=\kappa}\binom{n-m}{c}\binom{o-m}{d}\binom{m}{a}\binom{m}{b}\left(\frac{a+b}{m}-\frac{ab}{m^2}\right)(m!)^2 m^{\lambda^I+\lambda^O}\\
=&(m!)^2 m^{\lambda^I+\lambda^O}\times\left(\sum_{a+b+c+d=\kappa}\binom{n-m}{c}\binom{o-m}{d}\binom{m-1}{a-1}\binom{m}{b}\right.\\
&\left.+\sum_{a+b+c+d=\kappa}\binom{n-m}{c}\binom{o-m}{d}\binom{m}{a}\binom{m-1}{b-1}-\sum_{a+b+c+d=\kappa}\binom{n-m}{c}\binom{o-m}{d}\binom{m-1}{a-1}\binom{m-1}{b-1}\right)\\
=&(m!)^2 m^{\lambda^I+\lambda^O}\left(2\binom{n+o-1}{\kappa-1}-\binom{n+o-2}{\kappa-2}\right).
\end{aligned}$$

The first equality follows from Lemma 4, along with the fact that there are $(m!)^2$ possible bipartite bijections between $n-m$ agents and $o-m$ objects, and the fact that there are $m^{\lambda^I}m^{\lambda^O}$ ways in which new roots λ^I agents and λ^O objects) could have pointed to $2m$ cyclic vertices (m on the individuals' side and m on the objects' side), and the last equality follows from Vandermonde's identity.

The numerator Θ is now computed as:

$$\begin{aligned}
\Theta &= \binom{n}{m}\binom{o}{m}f(n-m,o-m,\lambda^I,\lambda^O)\alpha(n,o,\kappa;m,\lambda^I,\lambda^O)\\
&= \binom{n}{m}\binom{o}{m}f(n-m,o-m,\lambda^I,\lambda^O)(m!)^2 m^{\lambda^I+\lambda^O}\left(2\binom{n+o-1}{\kappa-1}-\binom{n+o-2}{\kappa-2}\right).\\
&= \left(\frac{n!}{(n-m)!}\right)\left(\frac{o!}{(o-m)!}\right)m^{\lambda^I+\lambda^O}f(n-m,o-m,\lambda^I,\lambda^O)\left(2\binom{n+o-1}{\kappa-1}-\binom{n+o-2}{\kappa-2}\right).
\end{aligned}$$

Collecting terms, let us compute

$$\mathbf{P}(n,o,\kappa;m,\lambda^I,\lambda^O)=\frac{1}{o^n n^o}\left(\frac{n!}{(n-m)!}\right)\left(\frac{o!}{(o-m)!}\right)m^{\lambda^I+\lambda^O}f(n-m,o-m,\lambda^I,\lambda^O).$$

A key observation is that this expression does not depend on κ, which implies that (n_i,o_i) forms a Markov chain.

Its transition probability can be derived by summing the expression over all possible (λ^I,λ^O)'s:

$$p_{n,o;m}:=\sum_{0\le\lambda^I\le n-m,0\le\lambda^O\le o-m}P(n,o,\kappa;m,\lambda^I,\lambda^O).$$

To this end, we obtain:

$$
\begin{aligned}
&\sum_{0\le\lambda^I\le n-m}\sum_{0\le\lambda^O\le o-m} m^{\lambda^I}m^{\lambda^O}f(n-m,o-m,\lambda^I,\lambda^O)\\
=&\sum_{0\le\lambda^I\le n-m}\sum_{0\le\lambda^O\le o-m} m^{\lambda^I}m^{\lambda^O}\binom{n-m}{\lambda^I}\binom{o-m}{\lambda^O}\times\\
&\qquad (o-m)^{n-m-\lambda^I-1}(n-m)^{o-m-\lambda^O-1}((n-m)\lambda^O+(o-m)\lambda^I-\lambda^I\lambda^O)\\
=&m\left(\sum_{0\le\lambda^I\le n-m}\binom{n-m}{\lambda^I}m^{\lambda^I}(o-m)^{n-m-\lambda^I}\right)\left(\sum_{1\le\lambda^O\le o-m}\binom{o-m-1}{\lambda^O-1}m^{\lambda^O-1}(n-m)^{o-m-\lambda^O}\right)\\
+&m\left(\sum_{1\le\lambda^I\le n-m}\binom{n-m-1}{\lambda^I-1}m^{\lambda^I-1}(o-m)^{n-m-\lambda^I}\right)\left(\sum_{0\le\lambda^O\le o-m}\binom{o-m}{\lambda^O}m^{\lambda^O}(n-m)^{o-m-\lambda^O}\right)\\
-&m^2\left(\sum_{1\le\lambda^I\le n-m}\binom{n-m-1}{\lambda^I-1}m^{\lambda^I-1}(o-m)^{n-m-\lambda^I}\right)\left(\sum_{1\le\lambda^O\le o-m}\binom{o-m-1}{\lambda^O-1}m^{\lambda^O-1}(n-m)^{o-m-\lambda^O}\right)\\
=&mo^{n-m}n^{o-m-1}+mo^{n-m-1}n^{o-m}-m^2o^{n-m-1}n^{o-m-1}\\
=&mo^{n-m-1}n^{o-m-1}(n+o-m),
\end{aligned}
$$

where the first equality follows from Lemma 3, and the third follows from the Binomial Theorem.

Multiplying the term $\frac{1}{o^n n^o}\left(\frac{n!}{(n-m)!}\right)\left(\frac{o!}{(o-m)!}\right)$, we get the formula stated in the Lemma. □

B Number of agents matched at each stage of TTC

Consider an arbitrary mapping, $g: I \to O$ and $h: O \to I$, defined over our finite sets I and O. Note that such a mapping naturally induces a bipartite digraph with vertices $I \cup O$ and directed edges with the number of outgoing edges equal to the number of vertices, one for each vertex. In this digraph, $i \in I$ points to $g(i) \in O$ while $o \in O$ points to $h(o) \in I$. Such a mapping will be called a bipartite mapping. A **cycle** of a bipartite mapping is a cycle in the induced bipartite digraph, namely, distinct vertices $(i_1, o_1, ..., i_{k-1}, o_{k-1}, i_k)$ such that $g(i_j) = o_j, h(o_j) = i_{j+1}, j = 1, ..., k-1, i_k = i_1$. A **random bipartite mapping** selects a composite map $h \circ g$ uniformly from a set $\mathcal{H} \times \mathcal{G} = I^O \times O^I$ of all bipartite mappings. Note that a random bipartite mapping induces a random bipartite digraph consisting of vertices $I \cup O$ and directed edges emanating from vertices, one for each vertex. We say that a vertex in a digraph is **cyclic** if it is in a cycle of the digraph.

The following lemma states the number of cyclic vertices in a random bipartite digraph induced by a random bipartite mapping.

LEMMA 6 (Jaworski (1985), Corollary 3). *The number q of the cyclic vertices in a random bipartite digraph induced by a random bipartite mapping $g : I \to O$ and $h : O \to I$ has an expected value of*

$$\mathbb{E}[q] := 2 \sum_{i=1}^{\min\{o,n\}} \frac{(o)_i (n)_i}{o^i n^i},$$

and a variance of

$$8 \sum_{i=1}^{\min\{o,n\}} \frac{(o)_i (n)_i}{o^i n^i} i - \mathbb{E}[q] - \mathbb{E}^2[q],$$

where $(x)_j := x(x-1)\cdots(x-j-1)$.

It is clear that at the beginning of the first round of TTC, if there are n agents and o objects in the economy, the distribution of the number of individuals and objects assigned is the same as that of q. Appealing to Theorem 1 we can further obtain that *for any* round of TTC which begins with n agents and o objects remaining in the market, the number of individuals and objects assigned has the same distribution as q. Hence, the first and second moments of the number of individuals/objects matched at that round corresponds exactly to those in the above lemma. Jaworski (1985) also shows that asymptotically (as o and n grow) the expectation of q is $\sqrt{2\pi \frac{no}{n+o}}(1 + o(1))$ while its variance is $(4 - \pi)\frac{2no}{n+o}(1 + o(1))$. Given the number n of individuals and o of objects available at the beginning of Stage t of TTC, if we denote X_t the number of agents and objects matched at that stage, we have that $\mathbb{E}[\frac{X_t}{\sqrt{2\pi \frac{no}{n+o}}}]$ converges to 1 as n grows while the variance of $\frac{X_t}{\sqrt{2\pi \frac{no}{n+o}}}$ converges to the constant $\frac{4-\pi}{\pi}$.

C Proof of Proposition 1

We establish several lemmas before proving the lemma. Let $\{n_t\}$ be the (random) sequence corresponding to the number of individuals at Step t of TTC. By our main result, this is a markov chain. Let c_t be the number of cyclic vertices on the individual side obtained in the graph of TTC at Step t so that $n_{t+1} = n_t - c_t$ for each $t \geq 1$. In general, $n_t = n - \sum_{k=1}^{t-1} c_k$. Thus, $\mathbb{E}[n_t] = n - \sum_{k=1}^{t-1} \mathbb{E}[c_k]$.

The following claim shows that if we start from any Step t_0 of TTC where $n_{t_0} \geq \delta n$, then with a significant probability, after a number of steps linear in $\sqrt{n}$ we will end up with an arbitrarily small fraction of agents remaining in the market.

LEMMA 7. *Consider any Step $t_0 \geq 1$ of TTC. Fix any $\delta > 0$ and let $c := \frac{1}{\sqrt{\pi\delta}}$. There is $\gamma > 0$ such that $\lim \Pr\{n_{t_0+c\sqrt{n}} \leq \delta n \,|n_{t_0} \geq \delta n\} > \gamma$ where γ does not depend on t_0.*[13]

PROOF. In the sequel, we condition w.r.t. the event that $n_{t_0} \geq \delta n$. By the markov chain property, we can do as if this we were starting the process at n_{t_0} (and so we ignore this conditioning in the notations). Proceed by contradiction and assume that there is $\delta > 0$ such that $\lim \Pr\{n_{t_0+c\sqrt{n}} > \delta n\} = 1$. Note that the event $\{n_{t_0+c\sqrt{n}} > \delta n\}$ implies that $n_t > \delta n$ for any $t_0 \leq t \leq t_0 + c\sqrt{n}$. Thus, for each $t_0 \leq t \leq t_0 + c\sqrt{n}$, the probability of $\{n_t > \delta n\}$ goes to 1 as n goes to infinity. In addition, for each $t_0 \leq t \leq t_0 + c\sqrt{n}$, $\Pr\{n_t > \delta n\} \geq \Pr\{n_{t_0+c\sqrt{n}} > \delta n\} \to 1$ and so, since the lower bound on $\Pr\{n_t > \delta n\}$ does not depend on t, $\Pr\{n_t > \delta n\}$ goes to 1 uniformly across $t_0 \leq t \leq t_0 + c\sqrt{n}$. Now, by definition,

$$\mathbb{E}[c_t] = \mathbb{E}[c_t \,|n_t > \delta n] \Pr\{n_t > \delta n\} + \mathbb{E}[c_t \,|n_t \leq \delta n] \Pr\{n_t \leq \delta n\}$$

and so

$$\frac{\mathbb{E}[c_t]}{\mathbb{E}[c_t \,|n_t > \delta n]} = \Pr\{n_t > \delta n\} + \frac{\mathbb{E}[c_t \,|n_t \leq \delta n]}{\mathbb{E}[c_t \,|n_t > \delta n]} \Pr\{n_t \leq \delta n\}.$$

Thus, using the fact that $\Pr\{n_t > \delta n\}$ converges to 1 uniformly across any $t_0 \leq t \leq t_0+c\sqrt{n}$, we obtain that $\frac{\mathbb{E}[c_t]}{\mathbb{E}[c_t|n_t>\delta n]}$ converges to 1 uniformly across $t_0 \leq t \leq t_0 + c\sqrt{n}$. So we must have that for any $\varepsilon > 0$, there is $N > 0$ and for any $n > N$,

$$\mathbb{E}[c_t] \geq \left(1 - \frac{\varepsilon}{2}\right) \mathbb{E}[c_t \,|n_t > \delta n] \geq (1-\varepsilon)\sqrt{\pi\delta n} = (1-\varepsilon)\sqrt{\pi\delta}\sqrt{n},$$

for any $t_0 \leq t \leq t_0 + c\sqrt{n}$, where the last inequality uses the fact that $\lim \frac{\mathbb{E}[c_t|n_t>\delta n]}{\sqrt{\pi\delta n}} \geq 1$ (Jaworski, Theorem 9).[14] Importantly, note that the N exhibited above does not depend on the specific $t_0 \leq t \leq t_0 + c\sqrt{n}$.

[13] If $\Pr\{n_{t_0+c\sqrt{n}} \leq \delta n \,|n_{t_0} \geq \delta n\}$ does not converge when n grows, we take a convergent subsequence.

[14] Note that by $\frac{\mathbb{E}[c_t|n_t>\delta n]}{\sqrt{\pi\delta n}} \geq \frac{\mathbb{E}[c_t|n_t=\delta n]}{\sqrt{\pi\delta n}}$ and by the markov property, the latter term does not depend on t. Hence, $\frac{\mathbb{E}[c_t|n_t>\delta n]}{\sqrt{\pi\delta n}}$ converges uniformly across t.

Thus, for any $\varepsilon > 0$, there is N such that for any $n > N$, we have

$$\begin{aligned}
\mathbb{E}[n_{t_0+c\sqrt{n}}] &= \mathbb{E}\left[n_{t_0} - \sum_{k=t_0}^{t_0+c\sqrt{n}-1} c_k\right] \\
&\leq n - \sum_{k=t_0}^{t_0+c\sqrt{n}-1} \mathbb{E}[c_k] \\
&\leq n - \left(c\sqrt{n}\right)(1-\varepsilon)\sqrt{\pi\delta}\sqrt{n} \\
&= n - (1-\varepsilon)\, n = \varepsilon n.
\end{aligned}$$

Otherwise stated, $\lim \mathbb{E}[n_{c\sqrt{n}}/n] = 0$. This in turn implies that $\lim \Pr\{n_{c\sqrt{n}} \leq \delta n\} = 1$, a contradiction with our assumption that $\lim \Pr\{n_{c\sqrt{n}} > \delta n\} = 1$.

To recap, we obtain that there is $\gamma > 0$ such that $\lim \Pr\{n_{t_0+c\sqrt{n}} \leq \delta n \,|n_{t_0} \geq \delta n\} > \gamma$. That γ does not depend on the specific starting date t_0 comes from the markov property of the random process $\{n_t\}$. $\square$

Lemma 8. *Fix any $\delta > 0$ and let $c := \frac{1}{\sqrt{\pi\delta}}$. For any $\xi > 0$, for any $k \in \mathbb{N}$ large enough, $\lim \Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \xi$.*

Proof. We know by the previous claim that there is $\gamma > 0$ such that for n large enough, $\Pr\{n_{c\sqrt{n}} \leq \delta n\} > \gamma$. First, note that $\Pr\{n_{2c\sqrt{n}} \leq \delta n\} > \gamma + (1-\gamma)\gamma$. Indeed, because $\{n_t\}$ is a decreasing sequence, $\{n_{c\sqrt{n}} \leq \delta n\}$ implies $\{n_{2c\sqrt{n}} \leq \delta n\}$. Hence, we have

$$\begin{aligned}
\Pr\{n_{2c\sqrt{n}} &\leq \delta n\} \\
&= \Pr\{n_{c\sqrt{n}} \leq \delta n\} \Pr\{n_{2c\sqrt{n}} \leq \delta n \,\big|\{n_{c\sqrt{n}} \leq \delta n\}\} + \Pr\{n_{c\sqrt{n}} > \delta n\} \Pr\{n_{2c\sqrt{n}} \leq \delta n \,\big|\{n_{c\sqrt{n}} > \delta n\}\} \\
&= \Pr\{n_{c\sqrt{n}} \leq \delta n\} + \Pr\{n_{c\sqrt{n}} > \delta n\} \Pr\{n_{2c\sqrt{n}} \leq \delta n \,\big|\{n_{c\sqrt{n}} > \delta n\}\}
\end{aligned}$$

Applying Claim 1 for $t_0 = 1$, we know that, for n large enough, $\Pr\{n_{c\sqrt{n}} \leq \delta n\} > \gamma$. In addition, applying Claim 1 for $t_0 = c\sqrt{n}$, we know that, for n large enough, $\Pr\{n_{2c\sqrt{n}} \leq \delta n \,\big|\{n_{c\sqrt{n}} > \delta n\}\} > \gamma$. Thus, we obtain $\Pr\{n_{2c\sqrt{n}} \leq \delta n\} \geq \gamma + (1-\gamma)\gamma$, as claimed.

Similar reasoning yields that for each $k \in \mathbb{N}$, there is N large enough so that

$$\Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \sum_{\ell=1}^{k} (1-\gamma)^{\ell-1}\gamma = 1 - (1-\gamma)^k.$$

Note that the right-hand side is equal to the cumulative distribution at k of a geometric distribution with parameter γ. Clearly, this goes to 1 as k increases and so our argument is completed. Thus, if we fix any $\xi > 0$, we can find k large enough so that $\Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \xi$ for any n large enough, as was to be proved. $\square$

We are now ready to prove Proposition 1.

Proof. Fix any $\alpha > 0$ and $\xi < 1$, we claim that there is n large enough so that $\Pr\{\frac{T}{n} \leq \alpha\} > \xi$. Consider any $\delta \in (0, \alpha)$ and fix $k \in \mathbb{N}$ and $c = \frac{1}{\sqrt{\pi\delta}}$ in order to have $\lim \Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \xi$ which is well-defined by Claim 2. Note that $\{n_{kc\sqrt{n}} \leq \delta n\}$ implies that $T \leq kc\sqrt{n} + \delta n$. Because, $\delta < \alpha$, the term on the right-hand side of the inequality is smaller than αn when n is large enough. Thus, for n large enough, we obtain $\Pr\{n_{kc\sqrt{n}} \leq \delta n\} \leq \Pr\{T \leq kc\sqrt{n} + \delta n\} \leq \Pr\{\frac{T}{n} \leq \alpha\}$. Now, because $\lim \Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \xi$, we obtain that for n large enough, $\Pr\{\frac{T}{n} \leq \alpha\} > \xi$, as claimed. $\square$

D Proof of Proposition 2

We start with the following lemma.

Lemma 9. *Fix any $O'' \subseteq O$. For any $O' \subseteq O''$, for any $\boldsymbol{\ell}_I := (\ell_i)_{i \in I}, \boldsymbol{\ell}_{O'} := (\ell_o)_{o \in O'}$ and $\mathbf{R}^*_{O'} := (r_o)_{o \in O'}$,*

$$\Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \bar{O} = O''\} = 0$$

if $\ell_o \leq r_o$ for some $o \in O'$ and is a strictly positive number which does not depend on $\boldsymbol{\ell}_{O'}$ otherwise.

Proof. In the sequel, to save on notations, we let $\mathcal{E}$ be $\{\bar{O} = O''\}$. We first note that

$$\Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} = 0$$

if for some $o \in O'$, $\ell_o \leq r_o$. Indeed, by definition, o points to r_o when involved in a cycle. In addition, $o \in O' \subseteq \bar{O}$ implies that object o is assigned via a long cycle, hence, the individual he is matched to must have a priority rank strictly greater than r_o.

Now, for any $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}'_{O'}$ satisfying $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}'_{O'} \gg \mathbf{r}_{O'}$, we argue that

$$\Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} = \Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}'_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}.$$

Indeed, fix a profile of preferences and priorities yielding $\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$. For each object $o \in O'$, let i be the individual with rank ℓ'_o. Swap i and $k := \text{TTC}(o)$ in o's priority ordering. Clearly, k has rank ℓ'_o at o. In addition, since for each object o, r_o (the individual o points to when involved in a cycle under the original profile) has a priority rank higher than both i and k at the original profile (recall that by assumption

$\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}'_{O'} \gg \mathbf{r}_{O'}$), each step of the TTC algorithm remains unchanged after the swaps. Hence, $\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}'_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ is obtained. Thus, we have an injection from the set of profiles of preferences and priorities yielding $\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ to the one yielding $\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}'_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$. Given the iid distribution of priority order, it follows that

$$\Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \leq \Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}'_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}.$$

A similar reasoning shows that the inequality holds in the other direction as well. $\square$

Now, we can complete the proof of Proposition 2. Here again, in the sequel, to save on notations, we let $\mathcal{E}$ be $\{\bar{O} = O''\}$. By the above lemma, for any $O_1, O_2 \subseteq O'$ disjoint, whenever well-defined, $\Pr\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ is a positive number which does not depend on $\boldsymbol{\ell}_{O_2}$.[15] Hence, $\Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\}$ can be written as

$$\begin{aligned} & \sum_{\boldsymbol{\ell}'_{O_2}} \Pr\left\{\mathbf{R}_{O_2} = \boldsymbol{\ell}'_{O_2} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \Pr\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}'_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \quad (4) \\ = \;& \Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \sum_{\boldsymbol{\ell}'_{O_2}} \Pr\left\{\mathbf{R}_{O_2} = \boldsymbol{\ell}'_{O_2} \mid \mathbf{J}_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \\ = \;& \Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \end{aligned}$$

where $\boldsymbol{\ell}_{O_2}$ is an arbitrary profile under which the above conditional probability is well-defined. Hence, conditional on $\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'}\}$ and $\mathcal{E}$, the joint distribution of $\mathbf{R}_{O_1}$ and $\mathbf{R}_{I'}$ does not depend on the specific realization of $\mathbf{R}_{O_2}$. This implies first that (setting $O_1 = \emptyset$)

$$\Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} = \Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\}.$$

Note that, using Equation (4) for $I' = \emptyset$, we also obtain that

$$\Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} = \Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \quad (5)$$

Now, pick an arbitrary $o \in O_1$. We have

[15] Indeed, by the above lemma, $\Pr\{\mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ does not depend on $\boldsymbol{\ell}_{O_2}$ as long as it is strictly positive. In addition, provided that the conditional distribution is well-defined (i.e., $\boldsymbol{\ell}_{O_2} \gg \mathbf{r}_{O_2}$), $\Pr\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'}, \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ is equal to 0 if $\ell_o < j(o)$ for some $o \in O_1$. In this case, this remains equal to 0 irrespective of $\boldsymbol{\ell}_{O_2}$. Finally, if $\boldsymbol{\ell}_{O_1} \gg \mathbf{r}_{O_1}$ then the above lemma implies that $\Pr\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'}, \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ is a strictly positive number which does not depend on $\boldsymbol{\ell}_{O_2}$.

$$\begin{aligned}\Pr\left\{\mathbf{R}_{O_1}=\boldsymbol{\ell}_{O_1}\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\} &= \Pr\left\{\mathbf{R}_{O_1\setminus\{o\}}=\boldsymbol{\ell}_{O_1}\mid R_o=\ell_o,\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}\Pr\left\{R_o=\ell_o\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}\\ &= \Pr\left\{\mathbf{R}_{O_1\setminus\{o\}}=\boldsymbol{\ell}_{O_1}\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}\Pr\left\{R_o=\ell_o\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}\end{aligned}$$

where the last equality comes from Equation (5) above. Now, applying the argument inductively, we obtain

$$\Pr\left\{\mathbf{R}_{O_1}=\boldsymbol{\ell}_{O_1}\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}=\prod_{o\in O_1}\Pr\left\{R_o=\ell_o\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}.$$

Put in another way, conditional on $\mathbf{R}^*_{O'}=\mathbf{r}_{O'}$ and $\mathcal{E}$, $\{R_o\}_{o\in O'}$ is a collection of mutually independent random variables (not necessarily identically distributed). In addition, conditional on $\{\mathbf{R}^*_{O'}=\mathbf{r}_{O'}\}$ and $\mathcal{E}$, for each $o\in O'$, R_o is stochastically dominated by the uniform distribution over $\{\lceil\log^{1+\varepsilon}(n)\rceil+1,...,n\}$. Indeed, the above lemma implies that for any $o\in O'$, $\Pr\left\{R_o=\ell_o\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}=0$ if $\ell_o\le j_o$ and is constant over all possible ℓ_o such that $\ell_o>j_o$. Thus, in the latter case, $\Pr\left\{R_o=\ell_o\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}=\frac{1}{n-j_o}$. Put in another way, given $\{\mathbf{R}^*_{O'}=\mathbf{r}_{O'}\}$ and $\mathcal{E}$, for $o\in O'$, R_o follows a uniform distribution over $\{j_o+1,...,n\}$. Since $o\in O'\subseteq\bar{O}\subseteq\tilde{O}$, we must have $j_o<\log^{1+\varepsilon}(n)$ and so R_o is stochastically dominated by the uniform distribution over $\{\lceil\log^{1+\varepsilon}(n)\rceil+1,...,n\}$. To recap, conditional on $\{\mathbf{R}^*_{O'}=\mathbf{r}_{O'}\}$ and $\mathcal{E}$, $\{R_o\}_{o\in O'}$ is a collection of independent random variables that is stochastically dominated by the collection of $|O'|$ iid random variables distributed according to a uniform distribution over $\{\lceil\log^{1+\varepsilon}(n)\rceil+1,...,n\}$, i.e.,

$$\begin{aligned}\Pr\left\{\mathbf{R}_{O'}\le\boldsymbol{\ell}_{O'}\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\} &= \prod_{o\in O'}\Pr\left\{R_o\le\ell_o\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}\\ &\ge \prod_{o\in O'}\Pr\left\{Y_o\le\ell_o\right\}.\end{aligned}\tag{6}$$

Now, for any $\boldsymbol{\ell}_{O'},\boldsymbol{\ell}_{I'}$,

$$\begin{aligned}&\Pr\left\{\mathbf{R}_{O'}\le\boldsymbol{\ell}_{O'},\mathbf{R}_{I'}\le\boldsymbol{\ell}_{I'}\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}\\ =\ &\Pr\left\{\mathbf{R}_{O'}\le\boldsymbol{\ell}_{O'}\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}\Pr\left\{\mathbf{R}_{I'}\le\boldsymbol{\ell}_{I'}\mid\mathbf{R}_{O'}\le\boldsymbol{\ell}_{O'},\mathbf{J}_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}\\ \ge\ &\prod_{o\in O'}\Pr\left\{Y_o\le\ell_o\right\}\Pr\left\{\mathbf{R}_{I'}\le\boldsymbol{\ell}_{I'}\mid\mathbf{R}^*_{O'}=\mathbf{r}_{O'},\mathcal{E}\right\}.\end{aligned}$$

where the inequality comes from the Equation (6) together with the fact that the distribution of $\mathbf{R}_{I'}$ does not depend on the specific realization of $\mathbf{R}_{O'}$, as we already claimed.

Hence, we obtain

$$\begin{aligned}
& \Pr\{\mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathcal{E}\} \\
= & \sum_{\mathbf{r}_{O'}} \Pr\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'} \mid \mathcal{E}\} \Pr\{\mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \\
\ge & \sum_{\mathbf{r}_{O'}} \Pr\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'} \mid \mathcal{E}\} \prod_{o \in O'} \Pr\{Y_o \le \ell_o\} \Pr\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \\
= & \prod_{o \in O'} \Pr\{Y_o \le \ell_o\} \sum_{\mathbf{r}_{O'}} \Pr\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'} \mid \mathcal{E}\} \Pr\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \\
= & \prod_{o \in O'} \Pr\{Y_o \le \ell_o\} \Pr\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathcal{E}\}
\end{aligned}$$

as claimed.

Finally, note further that, conditional on $\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'}\}$ and $\mathcal{E}$, $\{R_o\}_{o \in O'}$ stochastically dominates the collection of $|O'|$ iid random variables $X_1, ..., X_{|O'|}$ where the distribution of X_o is uniform over $\{1, ..., n\}$. Using a similar argument as above, we obtain that, conditional on $\mathcal{E}$, $\{R_o\}_{o \in O'}$ stochastically dominates the collection of $|O'|$ iid random variables distributed according to a uniform distribution over $\{1, ..., n\}$ and we can easily complete the proof of the second part of the proposition.

E Proof of Corollary 1

Fix $x \in [0, 1]$. Note that, by the above result, given $\{\bar{O} = O''\}$, the collection $\{\mathbf{1}\{\bar{R}_o \le x\}\}_{o \in \bar{O}}$ is stochastically dominated by $\{\mathbf{1}\{\bar{Y}_o \le x\}\}_{o \in \bar{O}}$ where $\bar{Y}_o$ is $\frac{1}{n} U\{\lceil \log^{1+\varepsilon}(n) \rceil + 1, ..., n\}$ which converges in distribution to $U[0, 1]$. Similarly, given $\{\bar{O} = O''\}$, the collection $\{\mathbf{1}\{R_o \le x\}\}_{o \in \bar{O}}$ stochastically dominates the collection $\{\mathbf{1}\{\bar{X}_o \le x\}\}_{o \in \bar{O}}$ where $\bar{X}_o$ is $\frac{1}{n} U\{1, ..., n\}$ which converges in distribution to $U[0, 1]$.

Now, fix any $\delta > 0$ and let us further condition w.r.t. the event that $|\bar{O}| \ge (1-\delta)n$. Note that the probability of this event goes to 1 as n grows. Now, conditional on $|\bar{O}| \ge (1-\delta)n$

and $\{\bar{O}=O''\}$, we have,

$$\begin{aligned}
\frac{1}{n}\sum_{o\in O}\mathbf{1}\left\{\bar{R}_o\le x\right\} &= \frac{1}{n}\left(\sum_{o\in\bar{O}}\mathbf{1}\left\{\bar{R}_o\le x\right\}+\sum_{o\in O\backslash\bar{O}}\mathbf{1}\left\{\bar{R}_o\le x\right\}\right)\\
&\le_{st} \frac{\left|\bar{O}\right|}{n}\frac{1}{\left|\bar{O}\right|}\sum_{o\in\bar{O}}\mathbf{1}\left\{\bar{R}_o\le x\right\}+\frac{\left|O\backslash\bar{O}\right|}{n}\\
&\le_{st} (1-\delta)\frac{1}{\left|\bar{O}\right|}\sum_{o\in\bar{O}}\mathbf{1}\left\{\bar{R}_o\le x\right\}+\delta\\
&\le_{st} (1-\delta)\frac{1}{\left|\bar{O}\right|}\sum_{o\in\bar{O}}\mathbf{1}\{\bar{Y}_o\le x\}+\delta \xrightarrow{p} (1-\delta)x+\delta
\end{aligned}$$

where the convergence result is by the LLN. Similarly, we must have that conditional on the above events,

$$\frac{1}{n}\sum_{o\in O}\mathbf{1}\left\{\bar{R}_o\le x\right\}\ge_{st}(1-\delta)\frac{1}{\left|\bar{O}\right|}\sum_{o\in\bar{O}}\mathbf{1}\{\bar{X}_o\le x\}\xrightarrow{p}(1-\delta)x.$$

Hence, conditional on $\left|\bar{O}\right|\ge(1-\delta)n$ and $\{\bar{O}=O''\}$, we must have that with probability going to 1, $\frac{1}{n}\sum_{o\in O}\mathbf{1}\left\{\bar{R}_o\le x\right\}$ falls in $[(1-\delta)x,(1-\delta)x+\delta]$. This must also be true if we only condition w.r.t. $\left|\bar{O}\right|\ge(1-\delta)n$. Since $\left|\bar{O}\right|\ge(1-\delta)n$ is a large probability event, we must have that, unconditionally, with probability going to 1, $\frac{1}{n}\sum_{o\in O}\mathbf{1}\left\{\bar{R}_o\le x\right\}$ falls in $[(1-\delta)x,(1-\delta)x+\delta]$. Since $\delta>0$ is arbitrary, this implies that

$$\frac{1}{n}\sum_{o\in O}\mathbf{1}\left\{\bar{R}_o\le x\right\}\xrightarrow{p}x.$$

F Proof of Corollary 2

Denote $\text{Rank}_i(o)$ (resp., $\text{Rank}_o(i)$) for the rank of object o (individual i) in i's preferences (o's priority ordering). Let us denote by E the joint event $\{o\in\bar{O}$ and $\text{Rank}_o(i)>R_o^*\}$ and let us first show that

$$\Pr\left\{R_o>\ \text{Rank}_o(i)\,|E,R_i>\ \text{Rank}_i(o)\right\}=\Pr\left\{R_o<\ \text{Rank}_o(i)\,|E,R_i>\ \text{Rank}_i(o)\right\}=\frac{1}{2}.$$

Consider the event $\{R_i>\text{Rank}_i(o),R_o>\text{Rank}_o(i),o\in\bar{O},\text{Rank}_o(i)>R_o^*\}$. Pick any preference profile under which this event is true. Let k be the individual with rank R_o (i.e., $\text{TTC}(o)=k$). Since $o\in\bar{O}\subseteq\hat{O}$, $R_o^*<R_o$. In addition, by assumption, we must have $R_o^*<\text{Rank}_o(i)$. Hence, both k and i have a priority ranking at o worse than that of R_o^*. Now, let us swap k and i in object o's priority ordering. Because both k and i have a priority ranking at o worse than that of R_o^*, this has no impact on the outcome of TTC. Thus, we must have $\{R_i > \text{Rank}_i(o), R_o < \text{Rank}_o(i), o \in \bar{O}, \text{Rank}_o(i) > R_o^*\}$. Thus, we have an injection from the set of profiles of preferences and priorities yielding $\{R_i > \text{Rank}_i(o), R_o > \text{Rank}_o(i), o \in \bar{O}, \text{Rank}_o(i) > R_o^*\}$ to the one yielding $\{R_i > \text{Rank}_i(o), R_o < \text{Rank}_o(i), o \in \bar{O}, \text{Rank}_o(i) > R_o^*\}$, showing that

$$\Pr\{R_o > \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\} \le \Pr\{R_o < \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\}.$$

Clearly, a symmetric reasoning shows that

$$\Pr\{R_o > \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\} \ge \Pr\{R_o < \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\}$$

and so we can conclude that

$$\Pr\{R_o > \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\} = \Pr\{R_o < \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\} = \frac{1}{2}.$$

Using a similar reasoning one can show that

$$\Pr\{R_o > \text{ Rank}_o(i)\,|E\} = \Pr\{R_o < \text{ Rank}_o(i)\,|E\} = \frac{1}{2}.$$

Hence, we conclude that

$$\Pr\{R_o > \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\} = \Pr\{R_o > \text{ Rank}_o(i)\,|E\}$$

put in another way, the probability that $R_o > \text{Rank}_o(i)$ does not depend on the realization of event $\{R_i > \text{Rank}_i(o)\}$.

To complete the proof, let us consider the probability that (i, o) blocks TTC given that event E holds. This is,

$$\begin{aligned}
& \Pr\{R_o > \text{ Rank}_o(i), R_i > \text{ Rank}_i(o)\,|E\} \\
= \; & \Pr\{R_i > \text{ Rank}_i(o)\,|E\} \Pr\{R_o > \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\} \\
= \; & \Pr\{R_i > \text{ Rank}_i(o)\,|E\} \Pr\{R_o > \text{ Rank}_o(i)\,|E\}
\end{aligned}$$

where the last equality holds by our argument above.

Now, we first claim that

$$|\Pr\{R_o > \text{ Rank}_o(i), R_i > \text{ Rank}_i(o)\} - \Pr\{R_i > \text{ Rank}_i(o)\} \Pr\{R_o > \text{ Rank}_o(i)\}|$$

goes to 0 as the market grows large. We have shown that conditional on E, this difference is just equal to 0. Hence, to show this convergence result, it is enough to prove that the probability of the joint event $E = \{o \in \bar{O}, \text{Rank}_o(i) > R_o^*\}$ goes to 1. Indeed, we already know that the probability of $\{o \in \bar{O}\}$ goes to 1. In addition, $\text{Rank}_o(i)$ follows a uniform distribution over $\{1, ..., |I|\}$. Hence, $\Pr\{\text{Rank}_o(i) > \log^{1+\varepsilon}(n)\}$ goes to 1. We also know that $\Pr\{R_o^* < \log^{1+\varepsilon}(n)\}$ goes to 1. Thus, $\Pr\{\text{Rank}_o(i) > R_o^*\}$ goes to 1 as well and so $\Pr(E)$ goes to 1.

Second, we know that $\text{Rank}_o(i)$ is a uniform distribution over $\{1, ..., |I|\}$ and let us observe the realization of $\text{Rank}_o(i)$ has no impact on the distribution of R_o. Now, Proposition 2 showing that $\bar{R}_o$ converges in distribution to $U[0,1]$ gives us that $\Pr\{R_o > \text{ Rank}_o(i)\}$ goes to $\frac{1}{2}$. Taken together the above two points yield

$$\left| \Pr\{R_o > \text{ Rank}_o(i), R_i > \text{ Rank}_i(o)\} - \frac{1}{2}\Pr\{R_i > \text{ Rank}_i(o)\} \right|$$

goes to 0 as the market grows large. This completes the proof of the first part of the statement of Corollary 2 since $\frac{1}{2}\Pr\{R_i > \text{ Rank}_i(o)\}$ is equal to the probability that (i, o) blocks RSD (recall the equivalence result by Carroll (2014)).

Finally, we can easily obtain that the difference between the expected fractions of blocking pairs under TTC and that under RSD converges to 0.