Club Formation Games with Farsighted Agents

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Abstract

Modeling club structures as bipartite networks, we formulate the problem of club formation as a game of network formation and identify those club networks that are stable if agents behave farsightedly in choosing their club memberships. Using the farsighted core as our stability notion, we show that if agents’ payoffs are single-peaked and agents agree on the peak club size (i.e., agents agree on the optimal club size) and if there sufficiently many clubs to allow for the partition of agents into clubs of optimal size, then a necessary and sufficient condition for the farsighted core to be nonempty is that agents who end up in smaller-than-optimal size clubs have no incentive to switch their memberships to already existing clubs of optimal size. In contrast, we show via an example that if there are too few clubs relative to the number of agents, then the farsighted core may be empty. Contrary to prior results in the literature involving myopic behavior, our example shows that overcrowding and farsightedness lead to instability in club formation.

1 Introduction

The study of club formation has a long history in economics going back to Buchanan (1965). Here we offer a new approach to the study of clubs. In particular, modeling club structures as bipartite networks, we formulate the problem of club formation as a game of network formation and identify those club networks that are stable if agents behave farsightedly in choosing their club memberships. Thus we bring together two strands of the literature: club theory† and the theory of social and economic networks initiated by Kirman (1983).

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†See Demange and Wooders (2005), Part II for surveys of club theory from several perspectives.
Unlike the random graph theoretic approach taken by Kirman (1983), here we follow an approach similar to that taken by Jackson and Wolinsky (1996) in their study of networks and focus exclusively on strategic considerations in club network formation. The basic setup of our model is closely related to the model of Konishi, Le Breton and Weber (1997). They examine, however, free mobility equilibrium of a local public goods economy (an assignment of players to clubs, locations, or jurisdictions that partitions the population and has the property that no individual can gain by either moving to any other existing club, or creating his own club). The partition derived from the players’ strategy choices is thus stable against unilateral deviations by individuals.\(^2\)

In contrast to the prior literature on clubs, we allow strategic coalitional moves and permit agents to be farsighted.\(^3\) Using the farsighted core introduced in Page and Wooders (2004) as our stability notion, we show that if agents’ payoffs are single-peaked and agents agree on the club size at which payoffs peak (i.e., agents agree on the optimal club size) and if there are sufficiently many clubs (i.e., sufficiently many club types or club locations) to allow for the partition of agents into clubs of optimal size, then a necessary and sufficient condition for the farsighted core to be nonempty is that agents who end up in smaller-than-optimal size clubs (i.e., the left-over agents) have no incentive to switch their memberships to already existing clubs of optimal size.\(^4\) We note that in this case, the outcome of farsighted behavior corresponds to outcomes of myopic behavior as in Arnold and Wooders (2005) and the set of outcomes in the farsighted core correspond to the ‘Nash club equilibrium outcomes’.

The coincidence of outcomes of farsighted behavior and myopic behavior does not extend to all cases, however. We demonstrate via an example that if there are too few clubs relative to the number of agents so that on average clubs must be larger than optimal size, then the farsighted core may be empty. This emptiness problem is caused by the fact that farsighted agents, unlike myopic agents, might switch their club memberships to already overcrowded clubs, temporarily making themselves worse off, if in the end switching induces an out migration that makes them better off. We note that the Arnold and Wooders club formation model agents behave myopically in choosing their club memberships and will switch memberships if and only if switching makes them strictly better off next period. Thus, in their model, since agents are assumed to be unwilling to make themselves temporarily worse off, even if doing so induces payoff improving future out migrations, fewer membership defections are possible. As a result, Arnold and Wooders are able to show that

\(^2\)In a similar set up, Conley and Konishi (2002) analyze migration proof equilibrium, which are stable only against credible deviations on the part of a coalition. A coalitional deviation to another jurisdiction is credible if no outsiders to the coalition will want to follow the deviators and, within the deviating group, no player can gain by a further deviation. Conley and Konishi consider only the case where the number of possible clubs is unconstraining.

\(^3\)Our approach differs from the cooperative/price-taking approach in much of the literature on clubs (again see Part II of Demange and Wooders) in that coalitions behave strategically.

\(^4\)Stated loosely, a club network is contained in the farsighted core if no group of agents has an incentive to alter their club memberships, taking into account club membership changes that might take place in the future.
even when there are too few clubs, a club structure in which all clubs are of nearly equal size is immune to coalitional defections. Using their terminology, Arnold and Wooders are able to show that if agents are myopic and if there is overcrowding, a Nash club equilibrium always exists. In contrast, our analysis suggests that in general overcrowding and farsightedness may lead to instability in club formation.

We shall proceed as follows. In Section 2, we introduce the notion of a club network and state the assumptions of our model. In Section 3, we define the farsighted dominance relation over the feasible set of club networks, and we define the farsighted path dominance relation. In Section 4, we define the abstract club network formation game with respect to the farsighted path dominance relation and we define the farsighted core of the club network formation game. Finally, in Section 4, we state our main result giving necessary and sufficient conditions for nonemptiness of the farsighted core for the case in which there are sufficiently many clubs.

2 Clubs Networks

We begin by introducing the notion of a club network. Using bipartite networks we are able to represent in a very compact and precise way the totality of any given club structure.

Let \( N \) be a finite set of agents consisting of two or more agents with typical element denoted by \( i \), and let \( C \) be a finite set of club types - or alternatively, a set of club labels or club locations - with typical element denoted by \( c \).

**Definition 1 (Club Networks)**

A club network \( g \) is a nonempty subset of \( N \times C \) such that \((i, c) \in g\) if and only if agent \( i \) is a member of club \( c \).

Given club network \( g \),

\[
g(c) := \{ i \in N : (i, c) \in g \}
\]

(i.e., the section of \( g \) at \( c \)) is the set of members of club \( c \) in network \( g \subseteq N \times C \), while the set

\[
g(i) := \{ c \in C : (i, c) \in g \}
\]

(i.e., the section of \( g \) at \( i \)) is the set of clubs to which agent \( i \) belongs in network \( g \subseteq N \times C \).

**Example 1** To illustrate, suppose there are five agents \( N = \{i_1, i_2, i_3, i_4, i_5\} \) and two clubs \( C = \{c_1, c_2\} \). Further, suppose that \( c_1 \) denotes the chess club while \( c_2 \) denotes the fencing club. Club network \( g_0 \) depicted in Figure 1 represents one possible club network.
In club network $g_0$ the chess club has three members

$$g_0(c_1) = \{i_2, i_3, i_4\} ,$$

while the fencing club has two members

$$g_0(c_2) = \{i_1, i_5\} .$$

Note that in club network $g_0$ each agent is a member of one and only one club. Thus, for example

$$g_0(i_5) = \{c_2\} ,$$

that is, agent $i_5$ is a member of the fencing club, but is not a member of the chess club. Below we will formalize the single club membership property of this example in an assumption that we will maintain throughout the paper.

The collection of all club networks given $N$ and $C$ is given by the collection of all nonempty subsets of $N \times C$, denoted by $P(N \times C)$. We shall denote by $|g(c)|$ the number of members of club $c$ (i.e., the club size) in network $g$ and by $|g(i)|$ the number of clubs to which $i$ belongs in network $g$. In Example 1, the chess club has three members, that is $|g_0(c_1)| = 3$, and agent $i_5$ belongs to one club - the fencing club - and thus $|g_0(i_5)| = 1$.

We shall maintain the following assumptions throughout:

A-1 (single club membership) The feasible set of club networks, $\mathcal{K} \subset P(N \times C)$, is given by

$$\mathcal{K} \subseteq \{g \in P(N \times C) : |g(i)| = 1 \text{ for all } i \in N\} .$$

Thus, in each feasible club network $g \in \mathcal{K}$ each agent is a member of one and only one club. Again note that club network $g_0$ in Example 1 satisfies the single club membership assumption [A-1]. Also note that under assumption [A-1] the collection $\{g(c) : c \in C\}$ forms a partition of the set of agents.
A-2 (identical payoff functions depending on club size) Agents have identical payoff functions, $u(\cdot)$, and payoffs are a function of club size only. In Example 1, agent $i_5$ is a member of the fencing club, that is, $g_0(i_5) = \{c_2\}$, and this club has a membership set given by

$$g_0(g_0(i_5)) := g_0^2(i_5) = \{i_1, i_5\}.$$  

Thus, in network $g_0$ agent $i_5$ has a payo$ff given by

$$u(|g_0(g_0(i_5))|) = u(|g_0^2(i_5)|) = u(|\{i_1, i_5\}|) = u(2).$$

In general, given any club network $g$, $|g^2(i)|$ denotes the total number of club members in the club to which agent $i$ belongs.

A-3 (single-peaked payoffs) There exists a club size $s^*$ with $1 \leq s^* < |N|$ such that payoffs are increasing in club size up to club size $s^*$ and decreasing thereafter.

A-4 (free mobility) Each agent can move freely and unilaterally from one club to another. This means that an agent can drop his membership in any given club and join any other club without bargaining with or seeking the permission of any agent or group of agents. In this sense our model of club formation as a game over club networks is noncooperative. The assumption of free mobility is quite common in models of noncooperative network formation (see, for example, Bala and Goyal (2000)), as well as in the club literature (see, for example, Demange (2005) and the references contained therein).

Example 2 It is important to note that our assumptions do not rule out the possibility that some clubs have no members (i.e., are empty). Thus, in some feasible club networks $g \in \mathbb{K}$, it may be the case that $g(c) = \emptyset$ for some club type $c \in C$. If club $c$ has no members, then $|g(c)| = |\emptyset| = 0$. Figure 2 depicts just such a situation.

![Club Network $g_1$](Figure 2: Club Network $g_1$)

In moving from club network $g_0$ in Example 1 to club network $g_1$ above, agents $i_1$ and $i_5$ have freely and unilaterally dropped their memberships in the fencing club and joined the chess club. Thus, in club network $g_1$ the fencing club $c_2$ has no members.\(^5\)

\(^5\)While we assume that in moving from club network $g_0$ to club network $g_1$, agents $i_1$ and $i_5$ act...
3 Dominance Relations Over Club Networks

Under the assumption of free mobility agents can alter any existing club network by simply switching their memberships. Such membership changes however can trigger further membership changes by other agents which in the end leave some or all of the agents who initially switched not better off and possibly worse off. Here we will assume that agents make their membership decisions taking into account the possibility of future membership changes by other agents - that is, we will assume that agents are farsighted and are concerned with the long run consequences of their immediate actions in choosing their club memberships. We begin by formalizing a notion of farsighted dominance. Then, using this farsighted dominance relation over club networks, we will identify club networks (i.e., club structures) that are farsightedly stable.

3.1 Farsighted Dominance

Throughout let $S$ denote a nonempty subset of $N$.

Definition 2 (Feasible Change and Improvement) Let $g_0$ and $g_1$ be two club networks in $K$ ($g_0 \neq g_1$).

(1) (Feasible Change) We say that agents $i \in S$ can feasibly change club network $g_0$ to club network $g_1$, denoted

$$ g_0 \xrightarrow{S} g_1, $$

if the move from network $g_0$ to network $g_1$ only involves a change in club memberships by agents in $S$, leaving unchanged the memberships of agents outside group $S$, that is, if

$$ g_0(i) = g_1(i) \text{ for all agents } i \in N \setminus S \text{ (i.e., } i \text{ not contained in } S). $$

(2) (Improvement) We say that club network $g_1$ is an improvement over club network $g_0$ for agents $i \in S$, denoted

$$ g_1 \triangleright_S g_0, $$

if $u(\lvert g_1^2(i) \rvert) > u(\lvert g_0^2(i) \rvert)$ for agents $i \in S$.

(3) (Feasible Improvement) We say that club network $g_1$ is a feasible improvement over club network $g_0$ for agents $i \in S$, denoted

$$ g_1 \triangleright_S g_0, $$

freely and unilaterally in switching their memberships, our model does not address the question of how agents $i_1$ and $i_5$ come to simultaneously switch their memberships, whether by communication and collusion or by serendipity. In order to formally address this question additional structure would have to be added to the current model. Page, Wooders, and Kamat (2005) make a start on addressing this question via the introduction of the supernetwork (i.e., a network of networks) in which the arcs represent coalitional moves and coalitional preferences (see also Page and Wooders (2004)).
if $g_0 \rightarrow_S g_1$ and $g_1 \succ_S g_0$.

(4) (Farsightedly Feasible Improvement) We say that club network $g^* \in \mathbb{K}$ is a farsightedly feasible improvement over club network $g \in \mathbb{K}$ (or equivalently, we say that club network $g^*$ farsightedly dominates club network $g$), denoted $g^* \triangleright g$, if there exists a finite sequence of club networks, $g_0, \ldots, g_n$, with $g := g_0$ and $g^* := g_n$, and a corresponding sequence of sets of agents, $S_1, \ldots, S_n$, such that for $k = 1, 2, \ldots, n$,

$$g_{k-1} \rightarrow_S g_k \text{ and } g_n \succ_S g_{k-1}.$$ 

Thus, club network $g^*$ is a farsighted feasible improvement over club network $g$ if (i) there is a finite sequence of feasible changes in club networks starting with network $g$ and ending with network $g^*$, and if (ii) payoffs

$$(u(g^2(i)))_{i \in N}$$

in ending club network $g^*$ are such that for each $k$ and for the agents in each coalition $S_k$, payoffs in the ending club network $g^*$ are greater than the payoffs agents in $S_k$ would have received in club network $g_{k-1}$ (i.e., in the club network that agents in $S_k$ changed) - that is, for each $k$

$$u(g^2(i)) := u(g^2(i)) > u(g_{k-1}^2(i)) \text{ for } i \in S_k.$$ 

The definition of farsighted feasible improvement above is a network rendition of Chwe’s (1994) definition.

**Example 3** Suppose that there are seven agents and two clubs and that the optimal club size is three. Figure 3 depicts three feasible club networks, $g_0$, $g_1$, and $g_2$. Club network $g_2$ farsightedly dominates club network $g_0$.

Figure 3: Three Possible Club Structures
To see this, consider the following sequence of moves. First, agents $i_6$ and $i_7$ switch their memberships from club $c_2$ to club $c_1$. This feasible move by agents $i_6$ and $i_7$ changes club network $g_0$ to club network $g_1$ and is denoted by

$$g_0 \xrightarrow{\{i_6,i_7\}} g_1.$$ 

Second, agents $i_1$ and $i_2$ switch their memberships from club $c_1$ to club $c_2$. This feasible move by agents $i_1$ and $i_2$ changes club network $g_1$ to club network $g_2$ and is denoted by

$$g_1 \xrightarrow{\{i_1,i_2\}} g_2.$$ 

Given an optimal club size of 3 and given the assumption of single-peaked payoffs, the initial moves by agents $i_6$ and $i_7$ makes them worse off. In particular, agents $i_6$ and $i_7$ start out in club $c_2$ in network $g_0$ with 4 members $\{i_4,i_5,i_6,i_7\}$ and payoffs given by

$$u(|g_0^2(i_6)|) = u(|g_0^2(i_7)|) = u(|\{i_4,i_5,i_6,i_7\}|) = u(4),$$

and move to club $c_1$ creating a new club network $g_1$ in which club $c_1$ has 5 members $\{i_1,i_2,i_3,i_6,i_7\}$. As a result, agents $i_6$ and $i_7$ are made worse off with payoffs given by

$$u(|g_1^2(i_6)|) = u(|g_1^2(i_7)|) = u(|\{i_1,i_2,i_3,i_6,i_7\}|) = u(5).$$

However, due to the second round of moves by agents $i_1$ and $i_2$, agents $i_6$ and $i_7$ end up in a smaller club $c_1$ in club network $g_2$, and thus end up better off. In particular, in the second round of moves, agents $i_1$ and $i_2$ leave club $c_1$ and move to club $c_2$ - changing club network $g_1$ to club network $g_2$. This move makes agents $i_1$ and $i_2$ better off, but also makes agents $i_6$ and $i_7$ better off. In particular, agents $i_1$ and $i_2$ move from club $c_1$ in network $g_1$ with 5 members $\{i_1,i_2,i_3,i_6,i_7\}$ and payoffs given by

$$u(|g_1^2(i_1)|) = u(|g_1^2(i_2)|) = u(|\{i_1,i_2,i_3,i_6,i_7\}|) = u(5),$$

to club $c_2$ in network $g_2$ with 4 members $\{i_1,i_2,i_4,i_5\}$ and payoffs given by

$$u(|g_2^2(i_1)|) = u(|g_2^2(i_2)|) = u(|\{i_1,i_2,i_4,i_5\}|) = u(4).$$

These second round moves by agents $i_1$ and $i_2$ leave agents $i_6$ and $i_7$ in a smaller club $c_1$ and thus make agents $i_6$ and $i_7$ better off. Thus, agents $i_6$ and $i_7$ who started out in club $c_2$ in network $g_0$ with 4 members $\{i_4,i_5,i_6,i_7\}$ and payoffs given by

$$u(|g_0^2(i_6)|) = u(|g_0^2(i_7)|) = u(|\{i_4,i_5,i_6,i_7\}|) = u(4),$$

end up in club $c_1$ in network $g_2$ with 3 members, $\{i_3,i_6,i_7\}$ and payoffs given by

$$u(|g_2^2(i_6)|) = u(|g_2^2(i_7)|) = u(|\{i_3,i_6,i_7\}|) = u(3).$$

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*Allowing coalitions to initially be made worse off but then eventually better off, as in this example, differentiates farsighted dominance from other dominance relations.*
3.2 Path Dominance

We say that a sequence of club networks \( \{ g_k \}_k \) in \( K \) forms a farsighted domination path (i.e., a \(<\<\)-path) if for any two consecutive networks \( g_{k-1} \) and \( g_k \),

\[
g_k \text{ farsightedly dominates } g_{k-1},
\]

that is,

\[
g_{k-1} \text{ <\< } g_k.
\]

Using the terminology of graph theory, we can think of the farsighted dominance relation \( g_{k-1} \text{ <\< } g_k \) between club networks \( g_k \) and \( g_{k-1} \) as defining a \(<\<\)-arc from network \( g_{k-1} \) to network \( g_k \). The length of \(<\<\)-path \( \{ g_k \}_k \) is defined to be the number of \(<\<\)-arcs in the path. We say that network \( g_1 \in K \) is \(<\<\)-reachable from network \( g_0 \in K \) if there exists a finite \(<\<\)-path in \( K \) from \( g_0 \) to \( g_1 \).

We can use the notion of \(<\<\)-reachability to define a new relation on the feasible set of club networks \( K \). In particular, for any two networks \( g_0 \) and \( g_1 \) in \( K \) define

\[
g_1 \succeq_K g_0 \text{ if and only if } \begin{cases} g_1 \text{ is } <\<\text{-reachable from } g_0 \text{ through } K, \text{ or} \\ g_1 = g_0 \end{cases}
\]

(1)

The relation \( \succeq_K \) is a weak ordering on \( K \). In particular, \( \succeq_K \) is reflexive (\( g \succeq_K g \)) and \( \succeq_K \) is transitive (\( g_2 \succeq_K g_1 \) and \( g_1 \succeq_K g_0 \) implies that \( g_2 \succeq_K g_0 \)). We shall refer to the relation \( \succeq_K \) as the farsighted domination path (FDP) relation.\(^7\)

Note that if club network \( g_1 \) is a feasible improvement over club network \( g_0 \) for agents \( i \in S \), then \( g_1 \) also dominates \( g_0 \) with respect to the farsighted domination path (FDP) relation, \( \succeq_K \). Thus,

\[
\text{if } g_1 \triangleright_S g_0 \text{ for some coalition } S, \text{ then } g_1 \succeq_K g_0.
\]

This applies even if the \( S \) consists of a single agent, that is, even if \( S = \{ i \} \) for some agent \( i \in N \). Thus,

\[
\text{if } g_1 \triangleright_{\{i\}} g_0 \text{ for some agent } i \in N, \text{ then } g_1 \succeq_K g_0.
\]

**Remark 1** If network \( g_0 \in K \) is \(<\<\)-reachable from network \( g_0 \), then we say that \( K \) contains a \(<\<\)-circuit. Thus, a \(<\<\)-circuit in \( K \) starting at club network \( g_0 \in K \) is a finite \(<\<\)-path from \( g_0 \) to \( g_0 \). A \(<\<\)-circuit of length 1 is called a \(<\<\)-loop. Note that because the relation \(<\<\) is irreflexive (i.e., because it is not possible to have \( g <\< g \)) \(<\<\)-loops are in fact ruled out. However, because the farsighted dominance relation, \(<\<\), is not transitive, it is possible to have \(<\<\)-circuits of length greater than 1.

\(^7\)The relation \( \succeq_K \) is sometimes referred to as the transitive closure in \( K \) of the farsighted dominance relation, \(<\<\), on \( K \).
4 Club Formation Games and the Farsighted Core

A club formation game with farsighted agents is a pair \((K, \preceq_K)\), where \(K\) is the feasible set of club networks and \(\preceq_K\) is the farsighted domination path (FDP) relation on \(K\).

One of the most fundamental stability notions in game theory is the core. Here we define the notion of core for club formation games with respect to farsighted path dominance. We call this notion of the core the farsighted core.

**Definition 3 (The Farsighted Core)**

Let \((K, \preceq_K)\) be a farsighted club formation game. A subset \(C\) of club networks in \(K\) is said to be the farsighted core of \((K, \preceq_K)\) if for each club network \(g \in C\) there does not exist a club network \(g' \in K\), \(g' \neq g\), such that \(g' \preceq_K g\).

Note that any club network \(g\) contained in the farsighted core \(C\) is a Nash club network - and in fact is a strong Nash club network.\(^8\) Letting \(\text{NE}\) denote the set of Nash club networks in \(K\) and letting \(\text{SNE}\) denote the set of strong Nash club networks in \(K\), we can conclude from our definition of the farsighted core that

\[
C \subseteq \text{SNE} \subseteq \text{NE}.
\]

Example 3 is particularly interesting as it demonstrates that farsighted behavior may generate quite different outcomes than myopic behavior and strong Nash equilibria (or Nash club equilibria). In Example 3, the number of clubs is not sufficiently large to permit all players to be in clubs of optimal size (i.e., \(|C| < \frac{|N|}{s^*}\) for \(|C| = 2, |N| = 7\), and \(s^* = 3\)). As shown in Arnold and Wooders (2002), in this case, it is a strong Nash equilibrium for the agents to be divided into clubs that are as close as possible to the same size – in this example, into clubs of sizes 3 and 4. No group of agents (nor any single agent) can improve upon his own payoff - but, nevertheless, the farsighted core is empty. This is because, as the example illustrates, farsighted agents, unlike myopic agents, will switch their club memberships to an already overcrowded club, temporarily making themselves worse off, if in the end switching induces an out migration that makes them better off.

When the number of clubs is unconstraining, the situation is quite different. Our next results give necessary and sufficient conditions for the farsighted core of a club formation game to be nonempty when there is an ample number of clubs, that is, when the number of clubs is unconstraining.

**Theorem 1 (Necessary and sufficient conditions for nonemptiness of the farsighted core)**

Consider a farsighted club formation game \((K, \preceq_K)\) with \(N\) agents, \(C\) clubs, and optimal club size \(s^*\), \(1 \leq s^* < |N|\). Suppose that assumptions (A-1)-(A-4) hold. In addition, assume that

\(^8\)A club network \(g \in K\), is a Nash club network if there does not exist another club network \(g' \in K\) such that \(g' \succ_i g\) for some agent \(i \in N\).

A club network \(g \in K\), is a strong Nash club network if there does not exist another club network \(g' \in K\) such that \(g' \succ_S g\) for some coalition \(S\).
Proof. Suppose that

(a) \(|C| \geq \frac{|N|}{s^*}\), and

(b) \(|N| = rs^* + l\) for nonnegative integers \(r\) and \(l\), \(l < s^*\).

The following statements are true.

1. The farsighted core of \((\mathbb{K}, \succeq_\mathbb{K})\) is nonempty if and only if \(u(l) \geq u(s^* + 1)\).

2. Club network \(g_*\) is contained in the farsighted core if and only if \(g_*\) has \(r\) clubs of size \(s^*\) and one club of size 1.

Consider a club network \(g_*\) with \(r\) clubs of size \(s^*\) and one club of size \(l\) \((l < s^*)\). Let \(I\) be the group of agents such that each agent \(i\) in \(I\) is a member of as \(s^*\) club (i.e., a club of size \(s^*\)) and let \(E\) be the group of agents in the club of size \(l\). Because

\[ u(|g^2_*(i)|) \geq u(|g^2(i)|) \text{ for all } g \in \mathbb{K} \text{ and all } i \in I, \]

no coalition requiring the participation of agents from \(I\) will be able to initiate a change in club network \(g_*\), which leads to another club network making the participants from \(I\) better off. Moreover, because

\[ u(l) \geq u(s^* + 1) \text{ and payoffs are single peaked,} \]

no coalition of agents from \(E\) alone will be able to initiate a change in club network \(g_*\), which leads to another club network making the agents from \(E\) better off. Thus, for any club network \(g_*\) with \(r\) clubs of size \(s^*\) and one club of size \(l\), there does not exist a club network \(g \in \mathbb{K}\), \(g \neq g_*\), such that \(g \succeq_\mathbb{K} g_*\). Therefore, if \(|C| \geq \frac{|N|}{s^*}\) and \(u(l) \geq u(s^* + 1)\), then any club network \(g_*\) with \(r\) clubs of size \(s^*\) and one club of size \(l\) is in the farsighted core.

Suppose now that \(|C| \geq \frac{|N|}{s^*}\) but that \(u(l) < u(s^* + 1)\). Let \(g \in \mathbb{K}\) and given \(g\) define the following club subcollections:

\[ C^+_g := \{ c \in C : |g(c)| > s^* \}, \]

\[ C^*_g := \{ c \in C : |g(c)| = s^* \}, \]

and

\[ C^-_g := \{ c \in C : |g(c)| < s^* \}. \]

Given that \(|C| \geq \frac{|N|}{s^*}\), \(C^-_g \neq \emptyset\) for all \(g \in \mathbb{K}\).

Let \(g \in \mathbb{K}\) and suppose that \(C^+_g \neq \emptyset\). Consider clubs \(c_1 \in C^+_g\) and \(c_2 \in C^-_g\) and let \(S_1\) be a coalition of agents from club \(c_1\) of size \(s^* - |g(c_2)|\). Observe that if agents in coalition \(S_1 \subseteq g(c_1)\) switch their memberships to club \(c_2\), then the new larger club \(c_2\) will be of optimal size \(s^*\) and all members of coalition \(S_1\) will be made better off.
by making the switch. Let \( g' \in \mathbb{K} \) be the club network which results from this switch. Then we have
\[
g' \succ_{S_1} g \quad \text{and thus} \quad g' \succeq_{\mathbb{K}} g.
\]
Let \( g \in \mathbb{K} \) and suppose that \( C_g^+ = \emptyset \). If \( |C_g^*| = r \), then there is an agent \( i \) in some club \( c_1 \in C_g^- \) who can switch his membership to some club \( c_2 \in C_g^* \) and be made better off because \( u(l) < u(s^* + 1) \). Letting \( g' \in \mathbb{K} \) be the club network resulting from this switch we have
\[
g' \succ_{\{i\}} g \quad \text{and thus} \quad g' \succeq_{\mathbb{K}} g.
\]
If \( |C_g^*| < r \) (maintaining the assumption that \( C_g^+ = \emptyset \)) then sufficiently many agents from clubs in \( C_g^- \) can switch their memberships to some club \( c' \in C_g^- \) resulting in a new, larger club \( c' \) of optimal size \( s^* \). Moreover, all agents making this membership switch will be better off. Letting \( S' \) denote the coalition of agents making the switch and letting \( g' \in \mathbb{K} \) be the resulting club network we have
\[
g' \succ_{S'} g \quad \text{and thus} \quad g' \succeq_{\mathbb{K}} g.
\]

5 Conclusions

An aspect of our work which we find particularly interesting is relationships between the outcomes of the dynamic process in Arnold and Wooders (2002) and the outcomes of farsighted strategic behavior. Research in progress addresses these questions.

References


