Team Contests with Multiple Pairwise Battles*

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Abstract

We study a contest between two teams. The teams include an equal number of players. Each player of a team participates in a component battle, competing against a matched rival from the other team. A team wins if and only if its players prevail in a sufficiently large number of component battles. The winning team is awarded a prize, which is a public good for all its players. In addition, a player may receive a private reward by winning his own battle, regardless of the win or loss of his team. We demonstrate in an all-pay-auction setting that the “strategic-momentum effect” or “discouragement effect” identified in individual multi-battle contests does not appear in team contests. The outcomes of past battles do not distort the stochastic outcomes of future plays. In addition, neither the bidding efficiency nor the stochastic outcome of the contest depends on the temporal structure of the contest, i.e., the sequencing or clustering of component battles, or the prevailing information disclosure and feedback policy. These results yield rich implications for contest design and policy experimentation. The main results and their logic remain valid in a wide array of alternative settings.

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Key Words: Team Contests, Multi-Battle, All-Pay Auction, Collective Action, History-Independence, Sequence-Independence, Temporal-Structure-Independence

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1 Introduction

Many competitive events can be generically viewed as contests. In a contest, economic agents expend scarce resources in order to win a limited number of prizes, while they forfeit the resources regardless of win or loss. Such competitive activities appear in a diverse array of environments including political campaigns, sports, R&D races, warfare, and even internal labor markets inside firms.

This widespread phenomenon has spawned numerous formal models dedicated to unveiling the various strategic concerns involved in these activities. The evolving literature has increasingly recognized that a contest often consists of more than a single static “battle” (see Konrad, 2009). Instead, it may require rivaling parties to confront each other on multiple fronts. One’s success cannot be accomplished in a single stroke of effort, but rather depends on its overall performance in a series of shots. Consider, for instance, an R&D race for the innovation of a final product (see Harris and Vickers, 1987). The development effort is often an accumulative process that proceeds in multiple stages. In each stage, firms carry out a specific task—i.e., the development of a component technology. A firm attains its ultimate success only if it achieves a sufficient number of advances on component technologies ahead of its rival. Alternatively, to stand out as a U.S. presidential candidate, in the primaries a politician has to defeat his opponents in the majority of state elections (see Klumpp and Polborn, 2006).

Existing studies conventionally assume that the grand prize of a multi-battle contest is competed for by individual contenders, and that each of them participates in all component battles. Many contests, however, involve competitions between teams. Each team includes a set of affiliated but independent players. Each member of a team is matched to his counterpart from the rival team on an individual battlefield. A team succeeds only if its members secure sufficient victories in component battles. For instance, major pharmaceutical and biotechnology companies, such as Roche and L’Oréal, have evolved into network organizations that rely heavily on extensive webs of strategic alliances in R&D. In a race for a new drug, component tasks of the development project are often carried out by affiliated but independent entities within each firm’s network. Partisan elections, such as general elections in most democracies, also resemble team contests. A political party is allowed to form a government only if its candidates prevail in the majority of constituencies, while the success or failure of an individual candidate, to a large extent, depends on his own effort in rais-

\[^1\] Cox and Magar (1999) and Hartog and Monroe (2008) established that majority status of a political party in a national legislative body is a valuable asset worth considerable collective effort to attain. The competition for the majority status between political parties thus resembles a team contest where a team’s victory benefits individual players.
ing fund and campaigning. Alternatively, a large-scale military operation usually includes a series of separate battles between matched individual units, e.g., in World War II. The outcome on an individual battlefield depends on the maneuvers and commitments of the participating units. Such a contest can be even more saliently exemplified by many sports events with team titles, in which the competitions consist of a series of individual matches, such as the Ryder Cup in men’s golf, the Davis Cup in men’s tennis, the Thomas Cup in men’s badminton, and the Swaythling Cup in men’s table tennis, to name a few.

In a team contest, a player’s payoff depends on the outcome of not only his own battle, but also those of others. He benefits from the team’s win in the overall contest—and therefore the efforts sunk by his fellow teammates—while retaining independent control over his own contributions. Players’ strategic behavior naturally involves coordination and collective action; it gives rise to a drastically different strategic mindset than that in a contest between individuals. The literature, however, has provided little in formal modeling to shed light on this widespread phenomenon. The present paper provides a formal analysis of a multi-battle team contest to fill this gap. A snapshot of the game follows.

Two teams, consisting of an equal number of players, compete for a common object. Each player on a team competes against his matched opponent from the rival team in one component battle. A team wins the contest if and only if its players secure more victories in component battles than its rival team. The winning team receives a trophy, which is a public good that accrues to the benefit of all its players. A player may also secure a private trophy for winning his own battle. To provide an analogy, a newly synthesized chemical not only contributes to P&G’s innovation on a grooming product, but also creates a private benefit for the affiliated lab that successfully develops this ingredient, i.e., royalty revenues from future alternative uses of the chemical. Alternatively, Novak Djokovic may attach substantial importance to his personal victory over Rafael Nadal even when he represents the Serbian team in the Davis Cup. A politician who runs for a seat in the legislature benefits from his own success in his political career. Players can be heterogeneous in terms of their abilities, with their marginal effort costs being drawn, potentially, from different distributions. In our baseline setting, component battles are carried out successively. Players observe the outcomes of previous battles (i.e., the current state of the contest), and strategically place their bids on their battlefields to maximize their own expected payoffs.

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2Political party committees’ contributed funds directly to candidates are subject to the contribution limits (currently $5k), while they may make additional "coordinated expenditures," subject to limits, to help their nominees in general elections. National party committees may however make unlimited "independent expenditures" to support or oppose federal candidates. Snyder (1989, p656) pointed out that political parties supply only a small fraction of the total campaign resource of their candidates. In the US 2010 Congressional races, the sources of campaign contributions are available from wikipedia (http://en.wikipedia.org/wiki/Campaign_finance_in_the_United_States).
Our results for team contests are in stark contrast to those for multi-battle contests between individuals, which reveals the fundamentally different incentives offered by team competitions. Conventional wisdom in the existing literature holds that a dynamic multi-battle contest typically demonstrates a feature of “history-dependence.” One’s (perhaps purely accidental) victories in early battles generate *strategic momentum* for him, while *discouraging* the other. The more motivated leader thus attains easy wins in subsequent confrontations with the laggard; the outcomes of earlier battles distort subsequent competitions, and determine the ultimate winner. This dynamic provides a rationale for the well observed “New Hampshire effect” in the history of U.S. presidential primaries (see Polborn and Klumpp, 2006). Due to the strategic importance of early battles, Polborn and Klumpp (2006) predict that candidates invest heavily on campaigns in constituencies with early election dates. Aldrich (1980) and Strumpf (2002) stress that the particular sequence of a given series of heterogeneous battles can distort the ultimate outcome in a sequential contest: The winner who stands out in a sequential election, e.g., U.S. presidential primaries, could have been the loser if the elections in different constituencies had been ordered differently.

The strategic-moment effect or discouragement effect, however, does not arise in a team contest in which matched players engage in head-to-head competition in a series of component battles. Three main neutrality results from the present analysis can be highlighted, as follows:

1. **History-Independence** The outcomes of early battles do not distort the balance of future confrontations. Throughout the contest, a player’s probability of winning his battle does not depend on the state of the contest, i.e., by how much one’s team is leading or lagging behind the rival team, and how many additional wins each team needs to reach the finish line. The stochastic outcome of each battle, i.e., the ex ante likelihood of each player’s win, is purely determined by participating players’ innate abilities.

2. **Sequence-Independence** Let us fix the matching of players, and let each battle be contested between a fixed pair of players. The battle would yield a fixed amount of ex ante expected overall effort, which is independent of the order of the battle in the entire sequence. The overall expected effort of the contest, as well as its stochastic outcome, is invariable if the sequence of battles is reshuffled. In addition, each individual player’s ex ante expected payoff is also independent of the sequencing arrangement.

3. **Temporal-Structure-Independence** (or Disclosure-Independence) The ex ante expected effort of a battle between a fixed pair of players would not vary, even when the battles of the contest—or a portion of these battles—are carried out simultaneously. 

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3 Readers are referred to Harris and Vickers (1987), Klumpp and Polborn (2006), Konrad and Kovenock (2009a), and Malueg and Yates (2010).
its stochastic outcome. To interpret this alternatively, it also predicts that the stochastic outcome and expected effort of the contest are independent of the prevailing information-disclosure or feedback policy. Neither the expected effort nor the stochastic outcome of each battle would vary, regardless of whether players observe the history of past battles or how much they observe.

These results yield several useful implications for contest design and practical policy experimentation. For instance, our analysis sheds light on the strategic trade-offs and efficient design of a general electoral institution, in which the candidate in each constituency plays an active role on behalf of his political party. For brevity, we discuss these only in Section 6, following our formal analysis.

In Section 2, we briefly review the relevant literature. In Section 3, we present a simple and stylized example. The fundamental logic will be illustrated using this example. It highlights the unique strategic trade-offs involved in the current setting, which distinguish a team contest from its counterpart between individuals. Section 4 sets up a general model of a multi-battle contest between teams. Section 5 conducts the main analysis and presents a complete set of results. In Section 6, we discuss the main implications of our results and several possible extensions. Section 7 concludes the paper.

2 Link to Literature

This paper primarily belongs to the literature on multi-battle contests, which dates back to Harris and Vickers (1987). They analyze a dynamic R&D race in which competing parties meet in a sequence of component contests successively: One wins a prize if and only if it has won a sufficient number of them. Harris and Vickers were among the first to identify the discouragement effect in a race. Snyder (1989), in contrast, studies a simultaneous multi-battle contest, which models the competition for majority status between two political parties in the parallel elections of multiple constituencies. Klumpp and Polborn (2006), as aforementioned, focus on U.S. presidential primaries and compare the efficiency of sequential campaigning to that of its simultaneous counterpart, which provides a rationale for the New Hampshire Effect observed in presidential primaries, and demonstrates that a sequential arrangement helps reduce campaign expenditures. Malheg and Yates (2010) demonstrate theoretically the ex post asymmetry caused by one’s winning early battle in a best-of-three contest between two symmetric players. They further provide empirical evidence from professional tennis matches. All these studies are essentially carried out while Tullock contest is adopted for each component battle.

Konrad and Kovenock (2009a) were the first to examine multi-battle contests in which the component contests are all-pay auctions with complete information. In contrast to
lottery contest where uncertainty in contest outcome is determined by exogenous factors, the all-pay auction captures the notion of endogenous uncertainty generated by the use of nondegenerate mixed strategies in equilibrium. They provide a complete characterization of the unique subgame perfect equilibrium of sequential multi-battle contests between two players, while allowing the players to be asymmetric in various aspects, e.g., their bidding costs, the minimum numbers of battles to win for victory in the contest, etc. Their analysis shows that individual and aggregate effort, as well as individual winning probabilities, can be non-monotonic in the closeness of the race when bidders are asymmetric. With positive intermediate prizes for each battle incorporated in their analysis, even a large lead by one player does not fully discourage the other. In a further study where the two players’ bidding costs are ex ante uncertain, Konrad and Kovenock (2010) demonstrate that the aforementioned discouragement effect can also be attenuated by the uncertainties in the players’ bidding costs.

These studies typically assume that players compete with each other in disjoint battlefields. Kovenock, Sarangi, and Wiser (2011) introduce interdependence between component battles by allowing for “complementarity”: A contestant wins the contest if he wins a certain combination of component battles. Kovenock and Roberson (2009) study a two-stage contest. In each stage, contestants meet each other in multiple parallel battles, and one’s win on one battlefield at the first stage grants him a head start when they meet again on the same front.

Existing studies on multi-battle contests typically assume that two players meet and compete against each other in all component battles. Our paper differs from them by exploring team contests, in which each component battle is fought between a particular pair of matched players from rival teams. The main questions raised in this paper are analogous to those of Klumpp and Polborn (2006), i.e., (1) How do the outcomes of early battles distort those of later ones? and (2) How do differing temporal structures affect contestants’ behavior? Our study, however, focuses on a different setup from that of Klumpp and Polborn, i.e., contests between teams. The results thus have a different scope of application, and our paper complements that of Klumpp and Polborn in this regard. In a war-of-attrition model, Strumpf (2002) focuses on the distortionary effect of a particular sequence of heterogeneous battles on the outcome of a dynamic contest between two individuals. He highlights a new "strategic effect" stemming from strategic behavior of forward-looking candidates under costly participation: Typically a player who is favored in later battles is more likely to prevail than one who is favored in earlier ones. As Klumpp and Polborn, Strumpf focuses on contests between individuals, and also interprets his analysis in the context of presidential primaries. Our setting leads to the Sequence-Independence result, which contends that the sequencing of heterogeneous battles does not distort the stochastic outcome of the overall contest. This
prediction is unique in a team contest setting.

Our paper is naturally linked to the extensive literature on group contests or contests between allied players, which includes Skaperdas (1998), Nitzan (1991), Esteban and Ray (2001, 2008), Nitzan and Ueta (2009), Münster (2007, 2009), Konrad and Leininger (2007), and Konrad and Kovenock (2009b), among many others. These studies typically assume that contestants in each coalition join forces on a single front and compete against each other by a collectively produced composite output. In contrast, our setting requires each player to carry out his part on a single front, and teams compete against each other in a set of disjoint battles. A few notable exceptions can be identified. Kovenock and Roberson (2012) and Rietzke and Roberson (2012) consider rivalry between two allied players and one independent player. Each of the former competes against the latter in a series of simultaneous disjoint battles in a Colonel Blotto game. The two studies investigate the incentives for allied players to exchange their resources and the subsequent strategic plays in the contests. Our paper differs fundamentally from these papers in several respects. First, a player in those settings fights a series of parallel battles (by the nature of Colonel Blotto games), while one in ours fights only one. Second, a player in their settings secures rent by winning each individual battle; he maximizes the sum of his private payoffs from all his battles, while remaining independent of spillover from his ally’s wins or losses. In contrast, a player in our setting derives utility from his team’s winning the majority of battles, but may not receive any private reward from winning his own battle. Third, players in the other contexts form alliances to exchange resources, while players in our setting act collectively to compete for a team trophy.

Our analysis concludes that the expected effort of the team contest does not depend on whether players learn the outcomes of previous battles, or how much they have learned. The paper can thus be linked to the small but growing literature on communication and feedback in dynamic contests. This strand of literature, e.g., Gershkov and Perry (2009), Aoyagi (2010), Ederer (2010), Gürtler and Harbring (2010), and Goltsman and Mukherjee (2011), typically focuses on whether a contest designer should reveal intermediate rankings to the contestants in a dynamic contest. In contrast to the present paper, players in those studies compete on single tasks and supply continuing efforts that add to their bids. The winner is determined by players’ accumulated output in carrying out a single task, instead of by the accumulated number of wins in disjoint battles.

Our study yields immediate implications for the design of election schemes. While the majority of these studies are concerned with the behavior of voters, for a few notable exceptions,

4For instance, Dekel and Piccione (2000) focus on sophisticated voters’ strategic information-gathering activities. Bikhchandani, Hirshleifer, and Welch (1992) study the information cascades in primaries; and their model depicts the herding behavior of voters in states with later elections.
including Klumpp and Polborn (2006) and Strumpf (2002), focus on the impact of electoral institutions on candidates’ behavior. As pointed out above, while the previous work studies competitions between two individual candidates, we are interested in competitions among players in coalitions, e.g., political parties. The different context of interests leads to different implications for candidates’ behavior, and, in this regard, our study complements this literature.

3 Example: Symmetric “Best-of-Three” Contest

For the sake of expositional efficiency, we adopt a simple “best-of-three” setting with complete information to illustrate the distinct features of multi-battle team contests. For this purpose, we first consider a benchmark contest between two symmetric individual players. They meet in three successive battles, and one wins the contest by winning two of them. We then consider a contest between two teams. Each team includes three players, with each player confronting his matched player from the rival team in one battle. Similar to the individual contest, three battles are carried out successively. In each component battle, players simultaneously commit to their effort outlays. A team wins the contest if and only if its players secure two victories in component battles.

In both settings, we assume that one’s effort incurs a unity marginal cost, which is commonly known. For the moment, we do not need to fix a particular form of contest technology. Instead, we only impose mild restrictions on bidding behavior. First, we require that the contest technology better incentivize the player who has a higher valuation of winning the current battle. The player with a higher valuation thus tends to place a higher bid, and therefore is more likely to win in equilibrium. Second, a symmetric bidding equilibrium exists when two players in a battle equally value the win. Evenly motivated players adopt the same bidding strategy and win the battle with equal probability.

We then compare the observations obtained from the two settings (individual contest vs. team contests). The main insights and logic that underpin our main results can be revealed by the comparison.

3.1 Benchmark: Individual Contest

The benchmark contest between two individuals resembles the setup of Malueg and Yates (2010). Two individual players, indexed by $i = 1, 2$, compete for a prize of value $1$. They

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5 Our general setup, which is presented in Section 4, allows each team to consist of $2n + 1$ players. Players’ effort costs are allowed to be heterogeneous and privately known.

6 A standard Tullock contest or an all-pay auction clearly satisfies these requirements and thus serves our purpose.
confront each other in three successive battles, which are indexed by $t = 1, 2, 3$. The single prize will be awarded to the player who wins at least two battles in the series.

The game can be analyzed by backward induction. Suppose that one player has won the first two battles. Battle 3 becomes irrelevant and elicits zero effort. Next, suppose that each has won one battle. The outcome of the subsequent battle determines the ultimate winner. In a symmetric equilibrium, each wins with a probability of $\frac{1}{2}$. Let $E(x_3)$ be one’s expected effort in the equilibrium. Each player thus expects an expected payoff of $v_3 = \frac{1}{2} - E(x_3)$ from participating in this battle.

We then consider players’ incentives in the second battle. When battle 2 is to be carried out, one player has won one (i.e., the first) battle. Without loss of generality, we assume that player 1 is the leader. For player 1, winning battle 2 terminates the contest and allows him to collect a prize of value 1. Losing the battle would force him to participate in the deciding battle, from which he expects a gain of $v_3$. Hence, player 1 responds to an “effective prize spread” of $1 - v_3 = \frac{1}{2} + E(x_3)$ when placing his bid. In contrast, player 2 has an effective prize spread of exactly $v_3 = \frac{1}{2} - E(x_3)$. The contest continues and proceeds to a deciding battle if player 2 wins, while it ends if he fails to even the score. Hence, he has an effective prize spread of $v_3 - 0 = v_3$.

Player 1 is thus better motivated in the second battle simply because of his earlier victory. Therefore, the winner of the first battle is expected to be more likely to win the second battle. The entire contest is more likely to end with a “clean sweep.” The ex ante symmetric contest is diverted into an asymmetric path after one obtains a lead. The outcome of battle 1 amplifies the incentive of the leader to win the subsequent battle, while it attenuates that of the laggard. This simple example illustrates the source of the strategic-momentum or discouragement effect identified in the literature on multi-battle contests between individuals.

### 3.2 Team Contest

We now demonstrate that the ex post asymmetry does not appear in a team contest with successive pairwise battles. Two teams, each consisting of three players, compete for a trophy. The trophy is a public good and has a value of 1 to all players. A player is indexed by $i(t)$ if he is affiliated with a team $i$, and is assigned to the $t$–th battle.

Suppose that one team has won the first two battles. Battle 3 becomes irrelevant. Next, suppose that two teams score evenly in the first two battles. Battle 3 is a symmetric match. In a symmetric equilibrium, each player wins the deciding battle with a probability of $\frac{1}{2}$. We then consider the immediately preceding battle. Assume without loss of generality that player 1(1) has won battle 1 on behalf of team 1.

\footnote{For the moment, players are assumed not to receive a private reward for winning a component battle.}
If player 1(2) wins the battle, the contest ends and he receives an immediate reward of 1. If he loses, the contest proceeds to battle 3. Player 1(2) expects a payoff of $\frac{1}{2}$ from this event, because his teammate, i.e., player 1(3), may win the battle and secure the trophy for his team with probability $\frac{1}{2}$. Hence, player 1(2)’s effective prize spread is $1 - \frac{1}{2} = \frac{1}{2}$.

If player 2(2) wins the battle, the contest proceeds to the deciding battle, which is to be contested by players 1(3) and 2(3). Player 2(2) would stand a chance of winning the contest with probability $\frac{1}{2}$ (if player 2(3) wins battle 3). He expects a payoff of $\frac{1}{2}$ from winning his battle. If he loses, the contest ends. Hence, player 2(2)’s effective prize spread is $\frac{1}{2} - 0 = \frac{1}{2}$.

We observe that player 2(2) values his win as much as his rival 1(2) does. Battle 2 remains symmetric in spite of team 1’s lead. Thus, the players involved would win the second battle with equal probability. One’s initial lead does not distort subsequent competitions. The strategic-momentum or discouragement effect, which is identified in the multi-battle individual contest, no longer prevails in the context of team competitions.

### 3.3 Intuition

As demonstrated by the above example, the predictions from a team-contest setting contrast sharply with those obtained from its individual-contest counterpart. To elucidate the logic that underpins this contrast, we first elaborate on the nature of the strategic-momentum or discouragement effect, which typically looms large in sequential contests between individuals.

For a laggard in a “best-of-three” individual contest, the benefit of winning the second battle is to even the score; no actual reward would accrue unless he wins the third battle. He is disincentivized by two factors. First, the uncertainty of the third battle discounts the incentive provided by the prize purse. Second, he has to continue to sink costly effort into the third battle if he wins the current one. The additional outlay dissipates his future rent, thereby attenuating his incentive to remain in contest. As revealed in Section 3.1, his incentive to win battle 2 is given by a prize spread $\frac{1}{2} - E(x_3)$.

By contrast, the incentive of his leading opponent to win is magnified by two factors. First, he would reap the actual benefit immediately if he wins. Second, the contest would be prolonged if he loses, which forces him to endure another uncertain and costly battle in the future while continuing to supply effort. The additional outlay burns the rent, and therefore aggravates the pain from the current loss. As evidenced by our example, he receives 1 if he wins, but only $\frac{1}{2} - E(x_3)$ if he loses. In summary, the leader has more to win (or more to lose), which makes him a favorite, despite the players’ ex ante symmetry.

The same, however, cannot be said of team contests. We first consider player 1(2) of the leading team (i.e., team 1), who is up for battle 2. Analogous to individual contests, a win allows him to secure the prize immediately. The team’s lead, however, allows him to slack off: His team would not lose the contest immediately because of his loss, and he would still
receive the prize as long as his teammate prevails. He suffers less from a prolonged struggle than his counterpart in individual contests, because a subsequent battle does not require his own contribution. The value of winning this battle is thus suppressed by the opportunity of free-riding. As revealed by our previous discussion, a loss renders an expected payoff of \( \frac{1}{2} \): The third battle is an effortless fair draw for player 1(2). He would be punished less for losing the battle than his counterpart in individual contests, as he is allowed to (partially) free-ride on another’s effort.

We then consider the player of the lagging team who is up for battle 2. Analogous to his counterpart in individual contests, his valuation of the win is discounted by the future uncertainty caused by the lag, because he receives the prize only if his team also wins battle 3. However, as he does not have to bear the cost of future battle, the rent from the win will not be dissipated by the subsequent struggle when the contest continues. He expects a value of \( \frac{1}{2} \) from his own win, which pays off more to him than to his counterpart in individual contests. Furthermore, the disadvantage, due to his team’s lag, is a double-edged sword that could also incentivize him. In contrast to his leading rival, the player on a lagging team is prevented from free-riding on his teammate because the loss would end the contest immediately. These effects thus amplify the effective prize spread in his battle.

Opposite forces are exercised on players in leading and lagging teams. They reinforce each other and eliminate the impact of the discouragement effect on the lagging team, thereby maintaining an even battlefield, regardless of the outcome of past plays. In subsequent sections, we provide a complete analysis of team contests in a general setup, which allows for a general number of battles, private trophies, and various asymmetries between teams and players. We demonstrate that the fundamentals we observe in the stylized example remain valid in general.

4 General Setup

Two teams, indexed by \( i = 1, 2 \), compete in a contest for an indivisible object. Each team consists of \( 2n + 1 \) risk-neutral players.

Each player on one team is matched to a player on the rival team.\(^8\) Each pair of matched players is assigned to a disjoint battlefield, and they compete head-to-head in a component battle. In each of the \( 2n + 1 \) component battles, the players involved simultaneously place their bids. A team is awarded the object if and only if it accumulates at least \( n + 1 \) victories from the \( 2n + 1 \) pairwise component battles.

\(^8\) As will be reemphasized later, throughout the paper we assume that the two players in a particular battle remain unchanged once the matching is complete. However, the sequence of battles is allowed to vary in Section 5.3.
4.1 Temporal Structure

We begin with a setting in which the $2n+1$ component battles are carried out successively. Battles are indexed by $t = 1, 2, ..., 2n + 1$ by their orders in the given unfolding sequence. A player on team $i$ is indexed by $i(t)$ if he is assigned to the $t$-th battle.

Before an arbitrary battle $t$ is fought, the history of past battles, or the state of the contest, is observed by players $1(t)$ and $2(t)$. The state of the contest is summarized by a tuple $(k_1, k_2)$, where $k_i$ is the number of wins secured by team $i$, with $t = k_1 + k_2 + 1$.

The contest reaches a terminal state $(k_1, k_2)$ once a team has accumulated exactly $n + 1$ victories, i.e., $\max(k_1, k_2) = n+1$.

In this case, subsequent battles become irrelevant to the outcome of the team contest, as the winning team has been determined. A component battle $t$ is called trivial if the contest is carried out in a state $(k_1, k_2)$ if $\max(k_1, k_2) \geq n+1$. However, players in a trivial battle may still put forth positive efforts if they derive private benefit from winning their own battles. More detail is provided when we lay out the fundamentals of the payoff structure.

In our setting, the contest would not be terminated unless all $2n+1$ component battles have been carried out. The contest continues after a terminal state has been reached. Konrad and Kovenock (2009a) adopt an alternative termination rule, assuming that the contest ends immediately once it reaches a terminal state. These alternative settings depict differing contexts. Our main results do not depend on the prevailing termination rule, which is discussed in Section 6 in more detail.

4.2 Payoffs

The rewards to each player arise from two sources: team prize and battle prize. First, one benefits from his team’s win. The trophy of the winning team is a public good for its players. All players equally value the prize awarded to the team, and we normalize the common valuation to 1. Second, a player may reap private benefit from winning his own battle, irrespective of his team’s success or failure. Specifically, a player $i(t)$ receives an intermediate private reward $\pi(t) \geq 0$ if he wins his own battle. The value of the private reward $\pi(t)$ is common to the fixed pair of matched players $1(t)$ and $2(t)$. It is specifically determined by the characteristics of the particular battlefield and allowed to vary across battlefields.

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9. In Section 5, we will present formal analysis of an alternative setting in which component battles are contested (partially or completely) simultaneously.

10. The other team must have not accumulated more than $n$ victories, given that a total of $2n+1$ battles are to be carried out.

11. For the sake of expositional efficiency, we collapse the asymmetry among players into a setting of heterogeneously distributed effort costs. Our model can be readily adapted for a setting in which players value team victory differently; the main insights do not depend on the current modelling approach.
For instance, U.S. Senate membership could value more to Californian politicians than to their North Dakota counterparts.

A trivial battle $t$ is no different from a standard static contest in which two players compete for a single prize $\pi(t)$. To the extent that $\pi(t) = 0$, a trivial battle elicits zero effort.

### 4.3 Component Battles

In each component battle $t$, matched players simultaneously exert their efforts $x_{i(t)}$. Following Konrad and Kovenock (2009a, 2010), we model each component battle as an all-pay auction. One wins the battle with certainty if he contributes a higher effort than the rival player. When both players exert the same amount of effort, the winner is picked randomly.

Players can be heterogeneous in terms of their innate abilities. The ability differential is reflected by the possibly different marginal effort costs. Each player $i(t)$’s effort entry $x_{i(t)}$ incurs a constant marginal cost $c_i(t) > 0$. The marginal cost $c_i(t)$ is continuously and independently distributed over an interval $[\underline{c}_{i(t)}, \overline{c}_{i(t)}]$, with a cumulative distribution function $F_i(t)(\cdot)$ and a continuously differentiable density function $f_i(t)(\cdot) > 0$. The distribution of $c_i(t)$ is commonly known. The main analysis focuses on the case in which the realization of $c_i(t)$ is privately known only to player $i(t)$. The pairwise battle is thus modeled as an incomplete-information all-pay auction.\(^\text{12}\)

It should be noted that a variety of alternative information structures can be entertained. To the extent that the realizations of both $c_1(t)$ and $c_2(t)$ are commonly known, the battle $t$ is a standard complete-information all-pay auction.\(^\text{13}\) To the extent that the realization of $c_i(t)$ is commonly known, while that of $c_j(t)$ is only known by player $j(t), i \neq j, \forall i, j \in \{1, 2\}$, the setting evolves into an all-pay auction with one-sided asymmetric information.\(^\text{14}\) All of our results extend to these alternative settings. Details will be discussed in Section 6.

### 5 Analysis

Before we proceed to the formal analysis, we introduce the following additional terminologies. Let $v_i(k_1, k_2)$ be the continuation value to team $i$ when the contest is in a state $(k_1, k_2)$. The continuation value denotes the gross payoff players of team $i$ expect from winning the team trophy when it is rationally assessed in a given state. As the value of the grand trophy is

\[^{12}\text{See Amann and Lenninger (1996), Moldovanu and Sela (2001, 2006), and Moldovanu, Sela, and Shi (2007).}\]

\[^{13}\text{See Hillman and Riley (1989), Baye, Kovenock, and de Vries (1996), and Konrad and Kovenock (2009a, 2010).}\]

\[^{14}\text{A similar contest is studied by Morath and Münster (2010). Our results do not depend on the prevailing information structure.}\]
normalized as 1, the \textit{continuation value} equals the expected winning probability of the team. Apparently, we must have $v_1(k_1, k_2) = 1$ if $k_1 \geq n + 1$, and $v_1(k_1, k_2) = 0$ if $k_2 \geq n + 1$. In the former case, team 1 has won the contest, and all its players receive a gross benefit of value 1 from the team trophy. In the latter case, team 1 has lost the contest and the trophy is awarded to players of the rival team. Similarly, $v_2(k_1, k_2) = 1$ if $k_2 \geq n + 1$, and $v_2(k_1, k_2) = 0$ if $k_1 \geq n + 1$.

We then consider an arbitrary battle $t$. Suppose that the contest is in a state $(k_1, k_2)$. Note that $t = k_1 + k_2 + 1$ must hold. For player 1($t$), the contest reaches a state $(k_1 + 1, k_2)$ if he wins battle $t$. He receives his battle-specific award $\pi(t)$. In addition, his expected award from his team’s eventual win is given by the continuation value $v_1(k_1 + 1, k_2)$. If he loses, the contest reaches a state $(k_1, k_2 + 1)$. The gross payoff he expects from a loss is simply the continuation value $v_1(k_1, k_2 + 1)$ in the corresponding state. Hence, player 1($t$) has an effective prize spread $\pi(t) + [v_1(k_1 + 1, k_2) - v_1(k_1, k_2 + 1)]$. Similarly, the prize spread for player 2($t$) is $\pi(t) + [v_2(k_1, k_2 + 1) - v_2(k_1 + 1, k_2)]$.

We claim that winning a battle is equally valued by the matched players, i.e., $v_1(k_1 + 1, k_2) - v_1(k_1, k_2 + 1) = v_2(k_1, k_2 + 1) - v_2(k_1 + 1, k_2)$, regardless of the current state of the contest or the history of past battles. The following simple but critical fact resolves the entire puzzle.

\textbf{Lemma 1} $v_1(k_1, k_2) + v_2(k_1, k_2) = 1$, $\forall k_1, k_2 \in \{1, 2, \ldots, 2n + 1\}$, with $k_1 + k_2 \leq 2n + 1$.

The simple fact of Lemma 1 does not require a formal proof. The continuation value $v_i(k_1, k_2)$ reflects a team $i$’s prospect of winning the trophy of value 1. The outcome of a battle shifts the state of the contest and changes teams’ prospects for their eventual success. However, because the rent has to be won by one and only one team in the end, the shift of state only affects the probabilistic division of the rent between the two teams, while it does not change the overall size of the rent.

By Lemma 1, player 2($t$)’s stake in the battle can be reformulated as

$$\pi(t) + [v_2(k_1, k_2 + 1) - v_2(k_1 + 1, k_2)]$$

$$= \pi(t) + [(1 - v_1(k_1, k_2 + 1)) - (1 - v_1(k_1 + 1, k_2))]$$

$$= \pi(t) + [v_1(k_1 + 1, k_2) - v_1(k_1, k_2 + 1)],$$

which is exactly the same as the prize spread faced by player 1($t$).

We use $\Delta v(k_1, k_2)$ to denote $v_1(k_1 + 1, k_2) - v_1(k_1, k_2 + 1)$ or equivalently $v_2(k_1, k_2 + 1) - v_2(k_1 + 1, k_2)$. The following obtains.

\textbf{Theorem 1} Every battle $t$, with $t \in \{1, 2, \ldots, 2n + 1\}$, is a common-valued all-pay auction in which the two matched players, 1($t$) and 2($t$), symmetrically value the win, regardless
of the state of the contest. In a battle \( t \), players have a common effective prize spread of \( V(t)(k_1, k_2) = \pi(t) + \Delta v(k_1, k_2) \). The component \( \Delta v(k_1, k_2) \) is determined by the specific state \((k_1, k_2)\) of the contest.

Theorem 1 generalizes the insights we obtain from the best-of-three example. To a fixed pair of matched players, the outcomes of preceding battles vary their valuations of winning the battle. The impact, however, is constantly symmetric. As a result, matched players are symmetrically motivated over all possible paths as the contest goes on.

With these preliminaries, we are ready to present our main results.

5.1 History Independence

In our model, matched players are allowed to be ex ante heterogeneous, with their marginal effort costs to be drawn from different distributions as stated in Section 4. We first have the following lemma, which characterizes the bidding equilibrium.\(^{15}\)

**Lemma 2** Consider a battle \( t \) with a symmetric prize spread \( V(t) > 0 \). Assume that \( c_{i(t)}, i = 1, 2, \) is distributed over an interval \([\xi_{i(t)}, \xi_{i(t)}] \) where \( \xi_{i(t)} > 0 \), with a continuous cumulative distribution function \( F_{i(t)}(\cdot) \) and a continuously differentiable density function \( f_{i(t)}(\cdot) > 0 \). There exists a unique pure strategy equilibrium \( b_{i(t)}(\cdot; V(t)), i = 1, 2, \) with \( b_{i(t)}(\cdot; V(t)) = \xi_{i(t)}(\cdot)V(t) \). The function \( \xi_{i(t)}(\cdot) \) depends solely on the two players’ effort cost distributions.

**Proof.** See Appendix. The proof for the existence and uniqueness of the equilibrium roughly follows Amann and Leininger (1996). \( \blacksquare \)

In a battle with prize \( V(t) = 0 \) (e.g., a trivial battle with \( \pi(t) = 0 \)), matched players simply bid zero. Such a trivial battle is irrelevant to the outcome of the whole contest, regardless of the prevailing tie-breaking rule. Lemma 2 leads to the following.

**Proposition 1** Consider a battle \( t \) with a symmetric prize spread \( V(t) > 0 \).

(i) A player \( i(t) \)’s expected winning probability \( \mu_{i(t)} \) in battle \( t \) is determined solely by the two players’ effort cost distributions, while it does not depend on the common prize spread \( V(t) > 0 \): 

(ii) A player \( i(t) \)’s expected effort \( E(x_{i(t)}) \) in battle \( t \) can be written as \( E(x_{i(t)}) = \rho_{i(t)} \cdot V(t) \). The constant \( \rho_{i(t)} \) depends solely on the two players’ effort cost distributions.

Theorem 1 and Proposition 1 lead to the following result.

\(^{15}\)As in Amann and Leininger (1996), the equilibrium requires a tie-breaking rule that favors the stronger player when both players bid zero. The event, however, happens with zero probability.
Theorem 2 (History Independence) Consider an arbitrary nontrivial battle $t$ in state $(k_1, k_2)$ with $k_1, k_2 \leq n$, and $t = k_1 + k_2 + 1$. The equilibrium winning likelihoods $(\mu_{1(t)}, \mu_{2(t)})$, with $\mu_{1(t)} + \mu_{2(t)} \equiv 1$, do not depend on the state of the contest $(k_1, k_2)$.

By Theorem 2, the contest can be viewed as a series of independent random draws. Each nontrivial battle $t$ is resolved by the winning likelihoods $(\mu_{1(t)}, \mu_{2(t)})$, which are predetermined by the profiles of contenders’ bidding competence. This stochastic outcome remains invariant regardless of the actual path in which the contest evolves, so long as either the winning team has not been decided, or the battle awards a positive private prize $\pi(t) > 0$. The history of earlier battles does not distort the outcomes of subsequent competitions, even though players’ equilibrium effort outlays must be contingent on the exact state of the contest, which determines the prize spread $V(t)$.

These results are remarkably robust, and the main logic extends to broader settings. First, Theorem 1, which verifies common prize spread between matched players, is entirely independent of the prevailing contest technologies or the simplifying assumption of equal finish lines between teams. Second, the result of history independence remains robust even when players are allowed to value the team trophy unequally if there is no battle-specific award associated with the battle.\(^{16}\)

5.2 Continuation Value and Effective Prize Spread

The history-independence property allows us to compute explicitly each team’s continuation value $v_i(k_1, k_2)$ at all possible states and the effective prize spread of each battle. In this part, we derive a few formulae that will be critical in subsequent analysis. The analysis in this part also yields useful implications that elucidate the strategic nature of the team contest.

Consider an arbitrary nontrivial battle $t$, in which case $V(t) > 0$ regardless of $\pi(t)$. Recall by Proposition 1(i) that a player $i(t)$ prevails with a probability $\mu_{i(t)}$ irrespective of the state of the contest. For expository efficiency, we introduce the following terminology. We define $\theta_i(h|m)|^{t+m-1}_{t}$, with $t \in \{1, 2, \ldots, 2n + 1\}$, $m \in \{0, 1, 2, \ldots, (2n + 1) - (t - 1)\}$ and $h \in \{0, 1, \ldots, m\}$, to be the probability of the event that players $i(t)$ to $i(t+m-1)$ will win exactly $h$ of $m$ battles (battles $t$ to $t + m - 1$), conditional on all these $m$ battles’ being nontrivial.

The probability $\theta_i(h|m)|^{t+m-1}_{t}$ can be computed explicitly with the set of probabilities $\mu = \{(\mu_{1(t)}, \mu_{2(t)}), t = 1, 2, \ldots, 2n + 1\}$, which are ultimately determined by the cost distributions of matched players. Consider, for example, the special case of symmetric team contests,\(^{16}\) Detailed analysis is omitted in the current paper for brevity, but is available from the authors on request. Lemma 1 and Theorem 1 would have to be formulated alternatively, but an identical logic leads to the same prediction.

\(^{16}\) Detailed analysis is omitted in the current paper for brevity, but is available from the authors on request. Lemma 1 and Theorem 1 would have to be formulated alternatively, but an identical logic leads to the same prediction.
in which the marginal effort costs of matched players are identically distributed. The contest renders a series of fair draws, \( \mu_1(t) = \mu_2(t) = \frac{1}{2} \), for all \( t \in \{1, \ldots, 2n + 1\} \). We then have
\[
\theta_i(h/m)|_{t+m-1}^t = C^h_{m}(\frac{1}{2})^m.
\]

We are now ready to compute each team’s continuation value \( v_i(k_1, k_2) \) in an arbitrary given state. Recall that the team trophy has a value of 1. The continuation value is in fact no different from the team’s expected winning odds assessed in the state. When the contest is in a state \( (k_1, k_2) \), with \( k_1, k_2 < n + 1 \), team \( i \) wins the contest once its players secure exactly another \( (n + 1) - k_i \) wins in subsequent (nontrivial) battles. Beginning at a state \( (k_1, k_2) \), the winning team \( i \) can close the contest by securing its last nontrivial victory, i.e., its \( (n+1) \)th win, in any battle \( (k_1 + k_2 + m) \), with \( m \in \{(n + 1) - k_i, \ldots, (2n + 1) - (k_1 + k_2)\} \). The following can be obtained.

**Proposition 2** Suppose that the contest is in a state \( (k_1, k_2) \). The continuation value of team \( i \) at this state can be expressed as follows:

\[
v_i(k_1, k_2) = \begin{cases} 
\sum_{m=(n+1)-k_i}^{(2n+1)-(k_1+k_2)} \left[ \mu_i((k_1+k_2)+m) \cdot \theta_i(n-k_i|m-1) \right]_{(k_1+k_2)+1}^{(k_1+k_2)+(m-1)} & \text{if } k_i, k_j \leq n; \\
1 & \text{if } k_i \geq n + 1; \\
0 & \text{if } k_j \geq n + 1.
\end{cases}
\]

(1)

The formula of \( \sum_{m=(n+1)-k_i}^{(2n+1)-(k_1+k_2)} \left[ \mu_i((k_1+k_2)+m) \cdot \theta_i(n-k_i|m-1) \right]_{(k_1+k_2)+1}^{(k_1+k_2)+(m-1)} \) is obtained by summing up the probabilities of a team \( i \)'s securing its \( (n+1) \)th win in all possible battles \( (k_1 + k_2 + m) \), with \( m \in \{(n + 1) - k_i, \ldots, (2n + 1) - (k_1 + k_2)\} \).

Consider again the aforementioned special case of a symmetric contest. The general formula of Proposition 2 allows us to obtain the following.

**Corollary 1** In a symmetric team contest, the continuation value of team \( i \) when the contest is in a state \( (k_1, k_2) \) is given as follows:

\( i \) \( v_i(k_i, k_j) = \sum_{l=0}^{n-k_i} \left[ (\frac{1}{2})^{n+1-k_i+l} \times \frac{(n-k_i+l)!}{(n-k_i)!l!} \right] \), if \( k_i, k_j \leq n; \)

\( ii \) \( v_i(k_i, k_j) = 1 \) if \( k_i \geq n + 1; \)

\( iii \) \( v_i(k_i, k_j) = 0 \) if \( k_j \geq n + 1. \)

Recall that matched players in a battle \( t \) respond to a common prize spread \( V(t)(k_1, k_2) = \pi(t) + [v_1(k_1+1, k_2) - v_1(k_1, k_2+1)] \) when the contest is in a state \( (k_1, k_2) \), with \( k_1 + k_2 + 1 \equiv t \), and \( k_1 + k_2 \leq 2n \). Proposition 2 allows us to compute explicitly the size of the players’ stake in the battle in every particular contingency.

**Proposition 3** Consider an arbitrary battle \( t \) when the contest is at a state \( (k_1, k_2) \) with \( t \equiv k_1 + k_2 + 1 \). The effective prize spread in battle \( t \), i.e., \( V(t)(k_1, k_2) = \pi(t) + \Delta v(k_1, k_2) \), is given by:
(i) $\pi(t) + \theta_i(n - k_i|2n - k_1 - k_2)|_{t+1}^{2n+1}$, or, equivalently, $\pi(t) + \theta_j(n - k_j|2n - k_1 - k_2)|_{t+1}^{2n+1}$, for all $i, j \in \{1, 2\}$, if $k_1, k_2 \leq n$ and $\min(k_1, k_2) < n$;
(ii) $\pi(t) + 1$ if $k_1 = k_2 = n$;
(iii) $\pi(t)$ if $\max(k_1, k_2) \geq n + 1$.

Proof. See Appendix.

In all three cases stated in Proposition 3, $V(t)(k_1, k_2)$ can be written uniformly as $\pi(t) + \theta_i(n - k_i|2n - k_1 - k_2)|_{t+1}^{2n+1}$. By Proposition 3, in addition to the private prize $\pi(t)$, winning the current battle in a state $(k_1, k_2)$ provides a marginal benefit of $\nu_i(k_1 + 1, k_2) - \nu_i(k_1, k_2 + 1) = \theta_i(n - k_i|2n - k_1 - k_2)|_{t+1}^{2n+1}$ (or, equivalently, $\theta_j(n - k_j|2n - k_1 - k_2)|_{t+1}^{2n+1}$) to each player $i(t)$. The expression $\theta_i(n - k_i|2n - k_1 - k_2)|_{t+1}^{2n+1}$ is simply the ex ante probability of the event that player $i(t)$’s teammates will win exactly $n - k_i$ out of the remaining $2n - k_1 - k_2$ battles after battle $t$ has been contested, conditional on all these battles’ being nontrivial. In this event, battle $t$ turns out to be pivotal to the entire contest: Its outcome ex post breaks the tie and determines the winning team. It should be noted that the event of team $i$'s winning $n - k_i$ battles is equivalent to that of team $j$'s winning $n - k_j$ of them. Hence, $\theta_i(n - k_i|2n - k_1 - k_2)|_{t+1}^{2n+1} = \theta_j(n - k_j|2n - k_1 - k_2)|_{t+1}^{2n+1}$ must hold.

This finding yields interesting implications and sheds light on the nature of a team contest. Winning the current battle allows player $i(t)$ to derive additional benefit from the team trophy if and only if this battle turns out to be a tie-breaker or it is ex post pivotal: His win does not ex post contribute to the team’s ultimate victory, to the extent that his teammates turn out to win more than $n - k_i$ of them, while it does not pay off (from the team trophy) either, to the extent that they turn out to secure less than $n - k_i$ of them. In either event, his win or loss ex post does not matter to the outcome of the contest. This observation further explains the logic behind Theorem 1. A team’s lead may disincentivize its players and encourage them to free-ride, as their battles are less likely to be pivotal, and their own loss can be made up for by their teammates’ efforts. By contrast, a team’s lags may, paradoxically, incentivize its players, as their battles can be more likely to contribute to the team’s win, while they are less able to rely on the efforts of their teammates to remedy their own losses. These opposite forces reinforce each other and eliminate the impact of the usual discouragement effect, thereby leading to evenly motivated battles.

Again, the general formula provided by Proposition 3 allows us to obtain the following result for the special case of a symmetric team contest.

Corollary 2 Consider a symmetric team contest, in which matched players in each battle are homogeneous. For a battle $t$ in a state $(k_1, k_2)$, the prize spread for battle $t$, i.e., $V(t)(k_1, k_2) = \pi(t) + \Delta v(k_1, k_2)$, $t \equiv k_1 + k_2 + 1$, is given by:
(i) $\pi(t) + (\frac{1}{2})^{2n-k_1-k_2} \cdot \frac{(2n-k_1-k_2)!}{(n-k_1)!(n-k_2)!}$ if $k_1, k_2 \leq n$ and $\min(k_1, k_2) < n$;
Proposition 3 and Corollary 2 allow us to derive the equilibrium efforts of each battle \( t \) in every possible state. By Proposition 1, player \( i(t) \)'s expected effort, when the contest is in a state \((k_1, k_2)\), is given by

\[
E(x_{i(t)} | (k_1, k_2)) = \rho_{i(t)} V_{t}(k_1, k_2),
\]

where the constant \( \rho_{i(t)} \) is a function of \((F_{1(t)}(\cdot), F_{2(t)}(\cdot))\), his and his rival’s marginal effort cost distributions.

### 5.3 Sequence Independence

We now conduct the following thought experiment. Fix the pairwise matching between players of the two teams, and let each pair be assigned to a fixed battlefield. We reschedule the sequence of these (heterogeneous) battles. The \( t \)-th battle in a sequence, which is fought between a given pair of players on a fixed battlefield, can be carried out in a different order \( \tilde{t} \) under a rescheduled sequence. Each given pair of players, however, continues to fight on the same front and to compete for a given private trophy.

We investigate the following issues. First, we explore how the sequencing arrangement of heterogeneous battles affects the outcome of each battle, as well as that of the entire contest. Second, we study how the amount of equilibrium effort supply in the contest depends on the particular sequence of battles. Third, we investigate how the players’ expected payoff depends on the sequence of battles.

To formally investigate these issues, we begin by deriving the ex ante expected effort of a battle \( t \) in a given sequence. As implied by Lemma 2, the equilibrium expected effort of a given battle depends on the particular state of the contest when the battle is fought. By Propositions 1 and 3, a player \( i(t) \) exerts in equilibrium an expected effort \( E(x_{i(t)} | (k_1, k_2)) = \rho_{i(t)} V_{t}(k_1, k_2) \), where the constant \( \rho_{i(t)} \) is a function of \((F_{1(t)}(\cdot), F_{2(t)}(\cdot))\), his and his rival’s marginal effort cost distributions. Let \( \rho_{(t)} = \rho_{1(t)} + \rho_{2(t)} \). Further define \( x(t) = x_{1(t)} + x_{2(t)} \), which is the overall effort of battle \( t \). Let \( \Pr((k_1, k_2) | t) \) denote the ex ante probability that battle \( t \) will take place in a state \((k_1, k_2)\). Apparently, \( E(x(t) | (k_1, k_2)) = E(x_{1(t)} | (k_1, k_2)) + E(x_{2(t)} | (k_1, k_2)) \).

Summing up \( E(x(t) | (k_1, k_2)) \) over all possible states in which the battle could take place,
we obtain the ex ante expected overall effort of this battle:

\[ E(x_t) = \sum_{k_1+k_2=t-1} \Pr((k_1,k_2)|t) E(x(t)|k_1,k_2) \]

\[ = \sum_{k_1+k_2=t-1} \Pr((k_1,k_2)|t) \rho(t) V(t)(k_1,k_2) \]

\[ = \rho(t) \sum_{k_1+k_2=t-1} \Pr((k_1,k_2)|t) \cdot V(t)(k_1,k_2) \]

\[ = \rho(t) E(V(t)), \]  

(2)

where \( E(V(t)) = \sum_{k_1+k_2=t-1} \Pr((k_1,k_2)|t) \cdot V(t)(k_1,k_2) \) is the ex ante expected prize spread for battle \( t \). The expected total effort of the entire contest amounts to

\[ E(x) = \sum_{t=1}^{2n+1} \rho(t) E(V(t)). \]  

(3)

Let us define \( \theta(n|2n)|_{-t} \equiv \theta_i(n|2n)|_{-t}; \forall i \in \{1,2\} \), which is the probability that each team will win exactly \( n \) out of the \( 2n \) battles other than \( t \), provided that all these battles are nontrivial. Note that we must have \( \theta_1(n|2n)|_{-t} = \theta_2(n|2n)|_{-t} \). When one team wins \( n \) out of the \( 2n \) battles, the rival team must prevail in exactly \( n \) battles, i.e., all the other battles, as well. The probability can be computed based on the set of fixed winning odds \( \{(\mu_1(t),\mu_2(t))\} \). A solution of \( E(V(t)) \) is obtained.

**Proposition 4** Consider an arbitrary battle \( t \) under a given sequence of battles. The ex ante expected prize spread in this battle is \( E(V(t)) = \pi(t) + \theta(n|2n)|_{-t} \).

**Proof.** See Appendix. \( \blacksquare \)

The prize spread \( V(t)(k_1,k_2) \) in a battle \( t \) is contingent on the exact state \((k_1,k_2)\) of the contest when the battle is fought. Proposition 4, however, shows that the ex ante expected prize spread of a particular battle faced by a fixed pair of players, when aggregating over all possible ex post contingencies, depends only on the odds of each team’s winning \( n \) of the other \( 2n \) battles. As shown by Proposition 4, the incentive provided by the team trophy is ex ante discounted by the probability \( \theta(n|2n)|_{-t} \), i.e., the ex ante probability that the battle will be ex post pivotal. The same intuition that underpins Proposition 3 extends in this context.

We are now ready to formally address the thought experiment we proposed in the beginning of this subsection. Fix the pairwise matching between players and their assignment to specific battlefields. For expositional clarity, we index by \( g \in \{1,\ldots,2n+1\} \) the (fixed) pairs of matched players, and denote by \( i(g) \) the player from team \( i \) in pair \( g \). The purpose is to distinguish the index \((g)\) of (fixed) pairwise matching of players between teams from the (variable) temporal ordering \((t)\) of these battles when the sequence is allowed to be reshuffled. The (fixed) private trophy contested by players \( i(g) \) is thus denoted by \( \tau(g) \) accordingly, which is specific to the battlefield on which players \( 1(g) \) and \( 2(g) \) compete.

Proposition 1, Theorem 2, and Proposition 4 can be readily adapted to conclude the following.
Theorem 3 (Sequence Independence) (i) The likelihoods of a fixed pair \((g)\) of players’ winning their battle, \((\mu_1(g), \mu_2(g))\), are independent of the sequence of battles, provided that the battle is carried out before a terminal state is reached, or there is a positive private trophy, i.e., \(\pi(g) > 0\).

(ii) The ex ante expected likelihood of each team’s winning the contest is independent of the specific sequence of battles.

(iii) Neither the ex ante expected effort of each battle between a fixed pair \((g)\) of players nor the expected total effort of the contest would change if the sequence of battles were reshuffled. In particular, a battle between a fixed pair \((g)\) of players yields an ex ante expected overall effort

\[
E(x(g)) = \rho(g)[\pi(g) + \theta(n|2n)|_{-g}],
\]

where \(\theta(n|2n)|_{-g}\) is the probability that player \(i(g)\)’s teammates win exactly \(n\) out of the remaining \(2n\) (nontrivial) battles.

(iv) For a fixed pair \((g)\) of players, their ex ante expected payoffs are independent of the sequence of battles in the contest.

It is unnecessary to formally present a technical proof; rather, we illustrate these results casually by the following arguments. The impact on the outcome of an individual battle is relatively straightforward. Recall the fact that matched players respond to a common (positive) prize spread regardless of the state of the contest, which leads to the history-independence result. It continues to hold when battles are alternatively ordered. A battle between a fixed pair \((g)\) of players is resolved with fixed winning likelihoods \((\mu_1(g) ; \mu_2(g))\), which depend solely on their cost distributions \(F_1(g)(\cdot) ; F_2(g)(\cdot)\) but not on the prevailing sequencing arrangement, provided that the contest has yet to reach a terminal state, or that a positive private award is available in the battle. We thus have Theorem 3(i).

It must be noted, however, that additional complication arises when a battle does not award a private trophy, i.e., \(\pi(g) = 0\): Once the winning team has been decided in the contest, the battle becomes trivial and elicits no effort. The winner is picked according to the prevailing tie-breaking rule, while the winning likelihoods \((\mu_1(g) ; \mu_2(g))\) may no longer apply. Reshuffling the sequence in the contest could change the probability of a given battle’s being (non)trivial when it is being carried out. For instance, a battle that is ordered at \(t\)-th position under a sequence, with \(t \leq n + 1\), would never be a trivial one. However, when it is ordered at \(\tilde{t}\)-th position under an alternative sequence, with \(\tilde{t} > n + 1\), it ends up as trivial with positive probability.

Despite this marginal impact of sequencing arrangement on the outcome of an individual battle (when it becomes trivial and does not carry a private reward), it does not distort the stochastic outcome of the overall contest, i.e., the ex ante probability of each team’s winning the contest. Some clarifying remarks are in order to corroborate this claim, i.e., Theorem...
3(ii). A team $i$’s ex ante winning probability is simply the likelihood of its securing no less than $n+1$ wins in the $2n+1$ component battles. To the extent that all battles award positive private trophies, the entire contest is a series of independent lotteries, with each battle $g$ being resolved stochastically by the probabilities $\{\mu_{1(g)}, \mu_{2(g)}\}$. Reshuffling apparently would not make a difference in the stochastic outcome of the entire contest.

To the extent that a battle $g$ is trivial when it is being carried out, and it awards no private trophy, players exert zero effort and the winning likelihoods $\{\mu_{1(g)}, \mu_{2(g)}\}$ may no longer apply. However, this nuance does not affect our claim. The ex ante likelihood of a team $i$’s winning the contest under a given sequence is in fact given by $v_i(0, 0)$. To win the contest, a team $i$ could secure its $(n + 1)$th win in any battle $m$, with $m \in \{n + 1, \ldots, 2n + 1\}$. The value of $v_i(0, 0)$ can be computed by aggregating the probabilities of all these possibilities. By Proposition 2, it is written as

$$v_i(0, 0) = \sum_{m=n+1}^{2n+1} [\mu_i(m) \cdot \theta_i(n | m - 1)]^{(m-1)}_1,$$

with each element in the sum, $\mu_i(m) \cdot \theta_i(n | m - 1)]^{(m-1)}_1$, being the probability of a team $i$’s securing its $(n + 1)$th win in battle $m$. In each possible event (i.e., team $i$ closes the contest in exactly battle $m$), all the $m$ battles, which are considered in the calculation of $\theta_i(n | m - 1)]^{(m-1)}_1$, are nontrivial. By contrast, the probabilities of a player’s winning subsequent trivial battles, i.e., battles $t > m$, are irrelevant, and they do not affect the value of $v_i(0, 0)$ under the given sequence of battles. Hence, the computed ex ante winning odds of each team would remain the same irrespective of those in subsequent trivial battles. As a result, $v_i(0, 0)$ would not vary if the winning likelihoods $\{\mu_{1(g)}, \mu_{2(g)}\}$ are assumed to apply when a battle $g$ turns out to be trivial and awards no private trophy. By the same arguments laid out in the previous paragraph, we conclude that $v_i(0, 0)$, i.e., a team $i$’ ex ante winning odds, would remain invariant to reshuffled sequences.

Theorem 3(iii) is directly adapted from the observation highlighted in Proposition 3. The proposition states the ex ante expected effort that a battle $t$ would elicit under a given sequence, which is given by $E(x(t)) = \rho(t)E(x(t)) = \rho(t)[\pi(t) + \theta(n | 2n)]$. Apparently, $E(x(t))$ depends entirely on players’ cost distributions and the characteristics of the particular battle to which they are assigned. Hence, it would not vary when battles are ordered alternatively.

Theorem 3(iv) is then straightforward. A player $i(g)$’s ex ante expected payoff can be written as

$$U_{i(g)} = v_i(0, 0) + \mu_{i(g)} \pi(g) - \rho_{i(g)}[\pi(g) + \theta(n | 2n)]_g.$$

He has an ex ante expected payoff $v_i(0, 0)$ from the prospect of his team’s success; he wins a private trophy $\pi(g)$ with a probability $\mu_{i(g)}$ if $\pi(g) > 0$, while he gains no private reward if $\pi(g) = 0$; and he is ex ante expected to contribute an effort $\rho_{i(g)}[\pi(g) + \theta(n | 2n)]_g$. As aforementioned, none of the three elements depends on the particular sequence of battles.
The general formula for $E(x_{(g)})$ reduces to the following expression for a symmetric contest, in which each battle is contested by ex ante equally competent players.

**Corollary 3** With symmetric players in each battle, players in a fixed pair $(g)$ exert an ex ante expected effort regardless of the sequence of battles

$$E(x_{(g)}) = \rho(g)\pi(g) + \rho(g)\frac{(2n)!}{n!n!}\left(\frac{1}{2}\right)^{2n},$$

where $\rho(g) = \rho_{1(g)} + \rho_{2(g)}$.

### 5.4 Temporal-Structure Independence

In this part, we relax our assumption of sequential contest and explore the ramifications of alternative temporal structures. We allow component battles to be carried out (partially) simultaneously.

More specifically, the set of $2n + 1$ component battles is partitioned into $Z \leq 2n + 1$ clusters. Battles included in the same cluster are carried out simultaneously. Players in a battle do not observe the outcomes of parallel battles within the same cluster, while they observe the outcomes of battles included in previous clusters. This setup accommodates various possible temporal structures. With $Z = 2n + 1$, the sequential setting we consider in the baseline setting is restored. With $Z = 1$, a perfectly simultaneous setting obtains in which all battles are carried out at the same time.

We then explore how alternative temporal structures affect the outcome and efficiency of the contest, and obtain the following.

**Theorem 4** (Temporal-Structure Independence) (i) Each battle $t$ has a stochastic outcome $(\mu_1(t), \mu_2(t))$, provided that it is nontrivial or there is a positive private trophy $(\pi(t) > 0)$. The winning likelihoods $(\mu_1(t), \mu_2(t))$ are predetermined by matched players’ marginal effort cost distributions, but are independent of the history of prior battles or the temporal structure of the contest.

(ii) Each team’s ex ante probability of winning is independent of the temporal structure.

(iii) Each battle $t$ yields an ex ante expected overall effort $E(x_{(t)}) = \rho(t)\pi(t) + \rho(t)\theta(n|2n)|_t$, which is independent of the temporal structure.

(iv) The ex ante expected payoffs for the players in each battle $t$ are independent of the temporal structure.

**Proof.** See Appendix. ■

Theorem 4 is in fact a direct extension and generalization of Theorems 2 and 3. Neither $(\mu_1(t), \mu_2(t))$, nor $\rho(t)$, nor $\theta(n|2n)|_t$ depends on the prevailing temporal structure of the contest. Regardless of the temporal structure, the winner of each nontrivial battle is an independent draw, with the probability of one’s winning to be predetermined by players’
cost distributions. By the same arguments laid out in the previous subsection, the prevailing
temporal structure does not affect each team’s ex ante winning odds and players’ expected
payoffs for each battle.

By Theorem 4(iii), the team prize lures a fixed pair of players in a battle $t$ to contribute an
ex ante expected effort $\rho(t) \theta(n|2n)|_{-t}$, where $\rho(t)$ is determined purely by the characteristics
of the players. The team prize is further discounted by $\theta(n|2n)|_{-t}$, which, as previously
stated, is the probability that each team will win exactly $n$ of $2n$ nontrivial battles other
than $t$. In other words, this is the probability that battle $t$ will be ex post pivotal.

In summary, we conclude that the ex ante expected outcome, the ex ante expected effort
of the contest, and players’ expected payoffs are entirely independent of contest sequence,
the temporal structure of the contest, or the information available to the players about the
outcomes of precedent battles.

6 Discussions and Extensions

Our results yield rich implications for the design of various competitive mechanisms. In
this section, we briefly discuss the implications and applications of our results, and further
discuss a few extensions. We demonstrate that our results and the main logic continue to
hold (qualitatively) in these alternative settings.

6.1 Implications and Applications

One immediate application of our results is the design of sports tournaments with team
titles, e.g., the Thomas Cup (men’s badminton). In these events, athletes are typically
sorted and matched by their professional rankings. Given the fixed matching of players
between the two teams, which is determined by players’ professional performance records, it
leaves open the question of how to sequence the matches between different (fixed) pairs of
players. For instance, should the match between the top players of the teams be scheduled
for an early round or a later round? Our Sequence-Independence result indicates that the
ex ante expected effort of a match between a given pair of players does not depend on its
order in the sequence; neither does the stochastic outcome of the contest. Furthermore, the
temporal-structure-independence result demonstrates that neither the expected effort nor
the stochastic outcome of these tournaments would vary even if matches were to be carried
out (partially) simultaneously.

More importantly, the analysis allows us to address concerns about the design of political
systems. For instance, how does the temporal arrangement of a general election affect its
outcome, and the strategic behaviors of candidates? Consider, for instance, a policy designer
who attempts to economize on electoral-campaign expenditures in political races. One natural instrument is to fine-tune the timing of the elections in different constituencies. Klumpp and Polborn (2006) explore the efficiency implications of temporal arrangement in individual multi-battle contests, i.e., U.S. presidential primaries. They show that a sequential multi-battle contest, e.g., the format for U.S. Primaries, leads to lesser campaign expenditures than a “national election day” (a simultaneous contest), due to the aforementioned discouragement effect. They show that players dump more resources on early elections, while they give up in later ones as lags accumulate. As a result, a New Hampshire effect arises.

Klumpp and Polborn (2006) focus on the context of electoral competitions between individual candidates, e.g., U.S. presidential primaries. By contrast, national general elections in most democracies resemble a simultaneous multi-battle team contest. Regional candidates, who represent rival political parties in each constituency, compete head-to-head for seats in a national legislative body. Votes are cast on election day across all constituencies. The party that achieves majority status in the legislative body forms the government. In the U.S., the election of house representatives resembles a general election in other democracies, and also has a simultaneous temporal arrangement. The election for the Senate, however, has a partially sequential structure. One third of the seats in the U.S. Senate are up for election every other year, while all elections for these seats are carried out simultaneously. In these contexts, political parties, as coalitions of affiliated politicians, compete against each other.

Our analysis of team contests provides a framework to address policy concerns in these contexts. A number of questions have yet to be formally explored in the existing literature. Would campaign expenditures be reduced if elections in different constituencies were carried out successively? Under a (partially) sequential race, e.g., election for the Senate in the U.S., would a party’s lead distort the outcomes of subsequent elections? How would the sequence of elections in differing constituencies affect the electoral outcome, if these elections were carried out successively? Our analysis immediately sheds light on the ramifications of the differing temporal arrangements and suggests that the temporal structure of elections in different constituencies does not cause distortion to either their outcomes or the expected rent-dissipation rate. Our study focuses on a different context (e.g., an electoral competition between parties) than that of Klumpp and Polborn (2006). It complements the existing literature in this regard.

Finally, our paper also sheds light on a classical question in the literature on contest design: Does it pay to provide intermediate feedback to contestants in a dynamic contest? It should be noted that a (partially) simultaneous temporal structure can be equivalently

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17Empirical studies, such as Cox and Magar (1999) and Hartog and Monroe (2008), among others, have shown that a party’s majority status (in the Senate) generates nontrivial values to individual politicians or businesses.
interpreted as one without or with limited intermediate feedback. Under such a feedback system, players cannot perfectly observe the history of past plays. Our analysis in Section 5.4 shows that the feedback or disclosure policy does not impact the performance or the outcome of the contest.

6.2 Extensions

Our baseline analysis is executed in a fairly general setting, and the main insights rely little on the particulars of the setup. A number of possible extensions can be entertained. We subsequently discuss these situations, which will demonstrate the generality and versatility of the main logic of our analysis. The discussion also allows us to ascertain and test the limits of our results.

6.2.1 Alternative Information Structures

We assume that each player’s marginal effort cost is randomly distributed and privately known. As aforementioned, these results continue to hold when alternative information structures are in place.

One direct variation is to assume that the marginal effort costs of matched players are commonly known. The battle is a standard asymmetric all-pay auction with complete information. Existing analyses on complete-information all-pay auctions, e.g., Baye, Kovenock, and de Vries (1996), imply that similar results of Lemma 2 would emerge, except that both players would play a mixed strategy at equilibrium. In particular, players’ winning probabilities are functions of \( \frac{c_i(t)}{c_j(t)} \), the ratio between their marginal effort costs, irrespective of the prevailing state of the contest.

Another variation is to allow one player in a battle to possess superior information: He knows the realization of his rival’s marginal effort cost, while his own cost is only privately known. The battle is an all-pay auction with one-sided asymmetric information. Some formal analysis would show that the results of Lemma 2 still hold in this noncanonical setting, except

\[ ^{18} \text{For instance, our setup can be readily adapted for a setting in which players in a battle value their teams’ win asymmetrically. The current analysis extends to cases in which the battle-specific awards can be ignored.} \]

\[ ^{19} \text{We follow Konrad and Kovenock (2009a, 2010) to model each component battle as an all-pay auction. Klumpp and Polborn (2006) assume that the winner of each component battle, i.e., state primary elections, is selected through a Tullock contest success function. The latter setting allows noisy factors to come into play in determining the winner. Our results remain robust under alternative contest technologies in complete-information settings. It should be noted that the key fact, as stated by Lemma 1 and Theorem 1—which underpins the analysis—is completely independent of the prevailing winner-selection rule. Further, an observation analogous to that of Lemma 2 can be obtained in a Tullock setting as well, as implied by the numerous studies in contest literature (see Konrad, 2009).} \]
that the player with public bidding cost plays a mixed strategy at equilibrium while the other still plays a pure strategy.\textsuperscript{20}

Therefore, our main results are robust to the above variations in information structure, as information structure plays no role in the rest of the analysis.

\subsection*{6.2.2 Alternative Contest Termination Rule}

We have assumed that the contest continues after a terminal state has been reached, until all $2n + 1$ battles are fought. Konrad and Kovenock (2009a) adopt an alternative termination rule, assuming that the contest ends immediately once it reaches a terminal state and the winner is determined. The prediction of history independence is not sensitive to the alternative termination rule. It does not affect the ex ante winning probabilities of each team, because all trivial battles are irrelevant for the outcome of the contest. However, additional qualifications are required for the claims of sequence independence and temporal-structure independence. Neither the sequence of battles nor the temporal structure of the contest would affect the stochastic outcome of each battle—provided that it is carried out—or that of the entire contest. It might, however, affect the ex ante expected effort in the contest when battle-specific awards are available to players.

Recall that a battle $t$ in our main setting elicits an ex ante expected effort $E(x(t)) = \rho(t)\pi(t) + \rho(t)\theta(n|2n)|_{-t}$. The private trophy $\pi(t)$ always elicits effort from players $i(t)$ under the termination rule assumed in our setting. Under the alternative termination rule, it could elicit effort only if battle $t$ takes place before a terminal state is reached. Hence, the incentive provided by $\pi(t)$ must be ex ante discounted by the probability of battle $t$’s being nontrivial. As a result, an alternative sequence would vary the ex ante expected overall effort of the contest, when the private trophy $\pi(t)$ and matched players are heterogeneous across battles. Temporal structure would also make a difference in this context. With positive private trophy $\pi(t)$, a completely simultaneous contest must elicit the maximum amount of effort, even though it does not distort the stochastic outcome of the contest.

It should be noted, however, that when the contest awards no private trophy, i.e., $\pi(t) = 0$, $\forall t$, all our results remain under the alternative termination rule.

\subsection*{6.2.3 Unequal “Finish Line”}

Another extension is to allow teams to race toward different finish lines, such that one can win the contest by accumulating a smaller number of victories in component battles.

\textsuperscript{20}For brevity, the proof is omitted here. More details are available from the authors on request. This noncanonical setting has rarely been studied in the literature, with the exception of Morath and Münster (2010). They study an all-pay auction with one-sided asymmetric information regarding players’ valuations of the object.
There are altogether \( T = \tilde{k}_1 + \tilde{k}_2 \) battles. Team 1 wins the first-past-the-post contest by securing \( \tilde{k}_1 \) victories before team 2 wins \( \tilde{k}_2 \), with \( \tilde{k}_1 \neq \tilde{k}_2 \). For instance, in an R&D race toward a final product, one firm may have secured patents on a few critical component technologies, to which the other has no access.

The setup in our baseline setting can be readily adapted for this setting. This \( T \)-battle race can be viewed as a subcontest of a “best-of-(2n + 1)” contest, with \( T < 2n + 1 \). Specifically, the first battle in this race is equivalent to the \((2n + 2 - T)\)th battle in the “best-of-(2n + 1)” contest in a state \((n + 1 - \tilde{k}_1, n + 1 - \tilde{k}_2)\). Similarly, an arbitrary battle \( k \) in this race is equivalent to the \((2n + 1 - T + k)\)th battle in the “best-of-(2n + 1)” contest.

Our prediction of history independence can be exercised immediately: It allows us to apply existing analysis to last \( T \) battles. All main predictions obtained in our baseline setting naturally extend based on similar arguments.

### 6.2.4 Unequally Weighed Battles

Another direct extension is to allow component battles to carry different weights for teams’ wins. To be more specific, let each contest include \( T \geq 2 \) component battles, which are carried out successively. A team is awarded a score \( s_i(t) \) if its player \( i(t) \) wins battle \( t \). Each team can maximally score 1 if its players prevail in all battles, i.e., \( \sum_{t=1}^{T} s_i(t) = 1 \). A team wins the entire contest if and only if it obtains a higher accumulated score \( \tilde{s} \) than the rival after all \( T \) battles have been contested. The score awarded for winning a battle is allowed to vary across battles for a given team, i.e., \( s_i(t) \neq s_i(t') \) for \( t \neq t' \). It is also allowed to vary across teams for a given battle, i.e., \( s_1(t) \neq s_2(t) \). The score \( s_i(t) \) indicates a battle \( t \)'s importance to team \( i \)'s success. A team, by winning a given battle, can take a bigger step toward eventual success than by winning another. It can also take a bigger step than its rival team toward success by winning a given battle. Our baseline model becomes a special case of this extended setting, with \( s_1(t) = s_2(t') = \frac{1}{2n+1}, \forall t, t' \in \{1, \ldots, 2n + 1\} \).

The contest would reach a terminal state in a battle \( \tilde{t} \) if and only if the accumulated score of the team that wins battle \( \tilde{t} \) exceeds the maximum score the other team can earn, i.e., the score the other team can earn if it wins all remaining battles.

We now demonstrate that the fundamentals of our analysis remain valid in this extended setting. Let the state of the contest be summarized by the tuple \((\tilde{s}^1, \tilde{s}^2)\), which indicates each team’s accumulated score before the next battle is fought. Let a battle \( t \) be fought in an arbitrary state \((\tilde{s}^1, \tilde{s}^2)\). Assume for simplicity that players do not receive private benefit from winning the battle. Again, let \( v_i(\tilde{s}^1, \tilde{s}^2) \) be the continuation value of team \( i \) assessed when the contest is in this state.

In this battle, player 1\( (t) \) has an effective prize spread \([v_1(\tilde{s}^1 + s_1(t), \tilde{s}^2) - v_1(\tilde{s}^1, \tilde{s}^2 + s_2(t))]\), while player 2\( (t) \) has \([v_2(\tilde{s}^1, \tilde{s}^2 + s_2(t)) - v_2(\tilde{s}^1 + s_1(t), \tilde{s}^2)]\). We still observe that matched players
in each battle equally value the win, i.e., \( v_1(\bar{s}^1 + s^1_t, \bar{s}^2) - v_1(\bar{s}^1, \bar{s}^2 + s^2_t) = v_2(\bar{s}^1, \bar{s}^2 + s^2_t) - v_2(\bar{s}^1 + s^1_t, \bar{s}^2) \). Note that the simple fact of Lemma 1 remains valid in the extended setting:

\[
v_1(\bar{s}^1, \bar{s}^2 + s^2_t) + v_2(\bar{s}^1, \bar{s}^2 + s^2_t) = 1,
\]

which corroborates our claim. The shift of state in the contest changes the probabilistic division of the rent but does not change the sum of the rent. Analogous to the baseline setting, each battle would be a common-valued static contest, and its winner would be picked in an independent draw, irrespective of the current state in which the battle takes place.

In spite of the alternative scoring rule and heterogeneous weights, a “reciprocity” can be stated intuitively. Although winning a battle moves one team toward the eventual victory by a different margin than it does the other, the increase in one team’s winning odds implies a decrease of the other team’s winning odds to the same degree.

In conclusion, this extension does not change our main predictions on the stochastic outcome of each battle or that of the entire contest. The same trade-offs and strategic mindset persist in the alternative setting, although the exact formula for effective prize spread and expected effort must be computed quantitatively in a different fashion. For example, Proposition 4 shows that the team trophy is ex ante discounted by the probability of battle \( t \)'s being ex post pivotal in the contest, i.e., \( \theta(n|2n)[-] \). The same insight holds in the generalized setting. The formula of this probability, however, has to be redefined to accommodate the alternative scoring rule.

### 7 Concluding Remarks

In this paper, we study a multi-battle-contest between two teams. The contest consists of a series of pairwise battles fought between matched players from rival teams. We demonstrate in an all-pay-auction setting that the well-studied strategic-momentum or discouragement effect in individual multi-battle contests no longer prevails. The history of past plays in the sequential contest does not cause distortion to the stochastic outcomes of future battles. In addition, the expected effort of the contest is independent of the temporal structure of the contest, i.e., the sequencing or clustering of component battles. These results yield rich implications for policy experimentation and contest design. As we have demonstrated, the main logic of our baseline analysis extends beyond the current setting, and remains robust in many variations of the basic model.

Our paper contributes to the burgeoning literature on multi-battle contests by introducing a setting that involves collective actions. Team competitions, as stated in introduction, are pervasive in the real world, yet they have rarely been studied in way of formal modeling. Our
analysis unveils the fundamental difference in players’ strategic mindsets between individual contests and team contests.

Our analysis is one of the earliest steps toward a more comprehensive understanding of this intriguing phenomenon. In addition to the extensions discussed in Section 6, the framework still leaves a tremendous room for many extensions. Our results, given the generality and versatility of our approach, would be useful in future analysis of team contests in alternative settings.

In our baseline analysis, we assume that players’ effort costs are independently distributed. One interesting but technically challenging extension is to allow players’ cost distributions to involve team-specific characteristics, i.e., the costs of players from the same team have affiliated distributions. For instance, the members of a team often attend training camps together before major tournaments. Joint training programs affect the performance of all involved athletes. In that case, the team contest would involve strategic information updating, as players who turn up in later rounds would infer the competence of their rivals by observing their teammates’ past plays. Alternatively, a player’s performance and incentives can be perturbed by various behavioral factors. For instance, a player may use his teammates’ performance as a reference to set his own expectations: He may feel pressured when teammates perform well, or relieved of pressure otherwise. Team contests provide an interesting context in which we can introduce such peer effects into formal modeling. These extensions, among many others, would give rise to substantially more extensive strategic interaction and broaden the scope of this framework. They deserve to be studied seriously, and will be attempted by the authors in future. In addition, our study generates a number of testable hypotheses. The theoretical predictions have yet to be tested, either in the field or in the laboratory, to identify their limits and to ascertain other factors that could be involved in the strategic trade-offs of players. An experimental study is in progress that will shed light on economic agents’ behavior in multi-battle team contest. These variations would introduce a wide array of new elements that could enrich the strategic interactions in the game and uncover various other forces involved in the trade-offs players face in team competitions. They deserve to be explored more comprehensively in future research.

Appendix

Proof of Lemma 2

Proof. For simplicity, we drop the subscript \((t)\) throughout this proof. Let bidder \(i\) have a marginal bidding cost \(c_i\), which follows a distribution \(F_i(c_i)\) on the support \([c_i, \bar{c}_i]\), with \(c_i > 0\). The density is denoted by \(f_i(c_i) \in [f, \bar{f}]\), with \(f > 0\). The common prize is denoted
Consider the following differential equation system

\begin{align}
\zeta_1(b) &= -\frac{\zeta_2(b)}{V f_1(\zeta_1(b))}, \quad (4) \\
\zeta_2(b) &= -\frac{\zeta_1(0)}{V f_2(\zeta_2(b))}, \quad (5)
\end{align}

with boundary conditions \( \zeta_i(0) = \bar{c}_i, i = 1, 2 \). The function \( \zeta_i(b), i = 1, 2 \) can be pinned down starting from \( b_0 = 0 \). As the slopes are known from (4) and (5) at \( b_0 = 0 \), \( \zeta_i(b) \) can be pinned down at a higher \( b_1 \). Then the slopes at \( b_1 \) can be pinned down, and we can move to a higher \( b_2 \). Clearly, \( \zeta_i(\cdot), i = 1, 2 \), which are determined by (4) and (5), strictly decrease with its argument.

Suppose that there exists a common \( \bar{b} > 0 \) such that \( \zeta_i(\bar{b}) = \zeta_j, i = 1, 2 \). We define \( c^*_i = \bar{c}_i, i = 1, 2 \). Otherwise suppose that \( \zeta_i(\cdot) \) first reaches \( \bar{c}_i \) at some \( \bar{b} > 0 \), while \( \zeta_j(\bar{b}) > \zeta_j \). Here and hereafter, we use \( i \) and \( j \) to denote the two different bidders. Let \( c^*_j = \bar{c}_j \), and define \( j \) as the weaker bidder, and \( i \) as the stronger bidder. We show that there exists a \( c^*_j \in (\bar{c}_j, \bar{c}_j) \) and a common \( \bar{b} > 0 \) such that \( \zeta_i(\bar{b}) = \zeta_i, i = 1, 2 \), where \( \zeta_i(\cdot), i = 1, 2 \) are the solutions of (4) and (5) with boundary conditions \( \zeta_i(0) = c^*_i, i = 1, 2 \).

Extend the support of \( f_j(c_j) \) to allow \( c_j < c_j \), and define \( f_j(c_j) = f_j(c_j), \forall c_j < c_j \). For an arbitrary \( \bar{c}_j \in [\bar{c}_j, \bar{c}_j] \), consider \( \zeta_i(\cdot), i = 1, 2 \) determined by (4) and (5) with boundary conditions \( \zeta_i(0) = c^*_i = \bar{c}_i \), and \( \zeta_j(0) = \bar{c}_j \). Define \( \bar{b}(\bar{c}_j) \) such that \( \zeta_i(\bar{b}) = \zeta_i \) and \( \xi(c_j) = \zeta_j(\bar{b}(\bar{c}_j)) \). Apparently, \( \bar{b}(\bar{c}_j) \) is continuous, and \( \xi(\bar{c}_j) \) is also continuous. In addition, when \( \bar{c}_j = \bar{c}_j \), we have \( \xi(\bar{c}_j) > \zeta_j \) by assumption. When \( \bar{c}_j = \bar{c}_j \), \( \xi(c_j) < c_j \) must hold due to the monotonicity of \( \zeta_1(\cdot) \) and \( \zeta_2(\cdot) \). Hence, there must exist a \( c^*_j \in (\bar{c}_j, \bar{c}_j) \) such that \( \xi(c^*_j) = \zeta_j(\bar{b}(c^*_j)) = \zeta_j \). Let \( c^*_j = \bar{c}_j \) and \( \bar{b} = \bar{b}(c^*_j) \). By their definitions, (4) and (5), with boundary conditions \( \zeta_i(0) = c^*_i, i = 1, 2 \), admit solution of decreasing functions \( \zeta_i(\cdot), i = 1, 2 \) such that \( \zeta_i(\bar{b}^*) = \zeta_j, i = 1, 2 \).

We are now ready to characterize a pure strategy bidding equilibrium. As a usual practice in the contest literature (e.g., Amann and Leininger, 1996), we assume a tie-breaking rule that favors the stronger bidder \( i \) when both players bid zero. Note that this event occurs with zero probability in the equilibrium. We search for (weakly) monotonic pure-strategy bidding equilibrium. The equilibrium bidding strategy is denoted by \( b_i(c_i), i = 1, 2 \), which is continuous above zero and can have mass point at zero only. Apparently, \( b_i(c_i), i = 1, 2 \) must share a common upper bound \( \bar{b} \). Assume \( b_i(c_i) = 0, i = 1, 2 \) when \( c_i \geq c^*_i \in [\bar{c}_i, \bar{c}_i] \), and that \( b_i(c_i), i = 1, 2 \) strictly decreases when \( c_i \leq c^*_i \), with \( c^*_i \) being defined as above.

For stronger bidder \( i \), define \( \varphi_i(b) = b_i^{-1}(b) \in [c_i, c^*_i] = [\bar{c}_i, \bar{c}_i] \), where \( b \in [0, \bar{b}] \). His expected payoff, when bidding \( b_i \), is as follows:

\[
\pi_i(b_i, c_i) = V \cdot \Pr(b_j(c_j) \leq b_i) - c_i \cdot b_i, \forall c_i \in [\bar{c}_i, \bar{c}_i], \forall b_i \geq 0. \quad (6)
\]
Stranger bidder $i$ of type $c_i \in [c_i, \bar{c}_i]$ chooses $b_i$ to maximize his expected payoff:

$$\max_{b_i \geq 0} \pi_i(b_i, c_i) = V \Pr(b_j(c_j) \leq b_i) - c_i \cdot b_i$$

$$= V \Pr(c_j \geq b_j^{-1}(b_i)) - c_i \cdot b_i$$

$$= V[1 - F_j(b_j^{-1}(b_i))] - c_i \cdot b_i.$$  

The first order condition is given by:

$$-V \cdot f_j(b_j^{-1}(b_i)) \frac{db_j^{-1}(b_i)}{db_i} - c_i = 0. \quad (7)$$

For weaker bidder $j$, define $\varphi_j(b) \equiv b_j^{-1}(b) \in [c_j, c_j^*]$, with $b \in [0, \bar{b}]$. Weaker bidder $j$ of type $c_j \in [c_j, c_j^*]$ chooses $b_j \geq 0$ to maximize his expected payoff:

$$\max_{b_j \geq 0} \pi_j(b_j, c_j) = V \Pr(b_i(c_i) < b_j) - c_j \cdot b_j$$

$$= V \Pr(c_i > b_i^{-1}(b_j)) - c_j \cdot b_j$$

$$= V[1 - F_i(b_i^{-1}(b_j))] - c_j \cdot b_j.$$  

The first order condition is given by:

$$-V \cdot f_i(b_i^{-1}(b_j)) \frac{db_i^{-1}(b_j)}{db_j} - c_j = 0. \quad (8)$$

Solving $b_i(c_i)$, $i = 1, 2$ is equivalent to solving for $\varphi_i(b)$, $i = 1, 2$. Equations (7) and (8) can be rewritten as

$$\varphi_i'(b) = -\frac{\varphi_j(b)}{V f_i(\varphi_i(b))}, \quad i, j = 1, 2, i \neq j, \quad (9)$$

with boundary conditions $\varphi_i(0) = c_i^*$, $i = 1, 2$. Clearly, $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$, which are determined by (4) and (5) with boundary conditions $\zeta_i(0) = c_i^* = \bar{c}_i$ and $\zeta_j(0) = c_j^*$, must satisfy (9). In addition, we have $\zeta_i(\bar{b}^*) = \xi_i$, $i = 1, 2$ by construction of $c_i^*, c_j^*$ and $\bar{b}^*$.

Let $b_i(c_i) \in [0, \bar{b}^*]$, $c_i \in [c_i, c_i^*]$, $i = 1, 2$ denote the inverse of $\xi_i(\cdot)$. For the weaker bidder $j$, we define $b_j(c_j) = 0$, $\forall c_j \in [c_j^*, \bar{c}_j]$. For weaker bidder $j$ of type $c_j^*$, the optimal bid is zero, which renders a zero payoff. Hence, the optimal bid must be zero for all types with $c_j > c_j^*$: if type $c_j > c_j^*$ has an optimal bid that differs from zero and renders non-negative expected payoff, then type $c_j^*$ can adopt the same bid and earn a positive expected payoff.

Let $\xi_i(c_i) = b_i(c_i; V = 1)$, $i = 1, 2$. We now establish that $b_i(c_i; V) = \xi_i(c_i) \cdot V$, $i = 1, 2$ constitutes a bidding equilibrium for an arbitrary common prize $V > 0$. For this purpose, it suffices to show that $\tilde{b}_i(c_i) = \beta \cdot b_i(c_i)$, $i = 1, 2$ constitutes the bidding equilibrium for prize $V = \beta \cdot V, \forall \beta > 0$, if $b_i(c_i)$, $i = 1, 2$ constitutes the equilibrium for prize $V$.  

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Define inverse functions \( \tilde{\varphi}_i(\tilde{b}) \equiv \tilde{b}_i^{-1}(\tilde{b}) = \varphi_i(\frac{\tilde{b}}{\tilde{b}_{i+}}) \in [c_i, c_i^*] \), \( i = 1, 2 \) on \([0, \beta \tilde{b}]\) where \( \tilde{b} = b_i(\tilde{c}_i) \), \( i = 1, 2 \). We only need to show that \( \tilde{\varphi}_i(\tilde{b}) \), \( i = 1, 2 \) satisfy

\[
\tilde{\varphi}_i'(\tilde{b}) = \frac{-\varphi_j(\tilde{b})}{V \cdot f_i(\tilde{\varphi}_i(\tilde{b}))}, \quad i, j = 1, 2, i \neq j.
\]

That is,

\[
\varphi_j'(\frac{\tilde{b}}{\beta}) \frac{1}{\beta} = \frac{-\varphi_j'(\tilde{b})}{\beta V \cdot f_i(\varphi_i'(\frac{\tilde{b}}{\beta}))}, \quad i, j = 1, 2, i \neq j,
\]

which further reduces to

\[
\varphi_j'(\frac{\tilde{b}}{\beta}) = \frac{-\varphi_j'(\tilde{b})}{V \cdot f_i(\varphi_i'(\frac{\tilde{b}}{\beta}))}, \quad i, j = 1, 2, i \neq j,
\]

which coincides with (9), i.e., \( \varphi_j'(b) = \frac{-\varphi_j'(b)}{V f_i(\varphi_i'(b))}, \quad i, j = 1, 2, i \neq j \).

We thus have \( b_i(c_i; \beta V) = \beta \cdot b_i(c_i; V) \) for any \( V \). Setting \( V = 1 \), we have \( b_i(c_i; \beta) = \beta \cdot b_i(c_i; 1) = \xi_i(c_i) \cdot \beta \). This verifies that \( b_i(c_i; \beta) \) is linear in \( \beta \).

We next establish the uniqueness of the pure-strategy equilibrium. Suppose that \( b_i(c_i), \quad i = 1, 2 \) constitutes a bidding equilibrium and \( b_i(c_i) \) is strictly decreasing on support \([c_i, c_i^*]\). Note that at least one of \( c_i^*, \quad i = 1, 2 \) is the upper bound \( \tilde{c}_i \). Following Amann and Leininger (1996), we define function \( k : [c_1, c_1^*] \rightarrow [c_2, c_2^*], \quad k(c_1) = b_2^{-1}(b_1(c_1)) \). We have

\[
k'(c_1) = (b_2^{-1})'(b_1(c_1)) \cdot b_1'(c_1).
\]

According to the FOCs (9) for bidding equilibrium \( b_i(c_i), \quad i = 1, 2 \), we have

\[
(b_2^{-1})'(b_1(c_1)) = -\frac{c_1}{V \cdot f_2(b_2^{-1}(b_1(c_1)))} = -\frac{c_1}{V \cdot f_2(k(c_1))}.
\]

In addition, note \( b_1'(c_1) = [(b_1^{-1})'(b_1(c_1))]^{-1} \), and

\[
(b_1^{-1})'(b_1(c_1)) = -\frac{k(c_1)}{V \cdot f_1(c_1)},
\]

from FOCs (9). Hence,

\[
k'(c_1) = -\frac{c_1}{V \cdot f_2(k(c_1))} \cdot \left[-\frac{k(c_1)}{V \cdot f_1(c_1)}\right]^{-1} = \frac{c_1 \cdot f_1(c_1)}{f_2(k(c_1))} \cdot k(c_1).
\]

Together with boundary condition \( k(c_1) = c_2 \), the above ordinary first order differential equation fully characterizes the unique trace of increasing function \( k(\cdot) \). There are three possible cases. In case I, \( k(\bar{c}_1) = \bar{c}_2 \), then we must have \( c_1^* = \bar{c}_1 \) and \( c_2^* = \bar{c}_2 \); in case II, \( k(\bar{c}_1) < \bar{c}_2 \), then we must have \( c_1^* = \bar{c}_1 \) and \( c_2^* < \bar{c}_2 \); in case III, \( k(\bar{c}_1) > \bar{c}_2 \), then we must have \( c_1^* < \bar{c}_1 \) and \( c_2^* = \bar{c} \). In any case, \( c_1^* \) and \( c_2^* \) are uniquely pinned down. It follows that FOCs (9) deliver unique solutions for equilibrium by standard arguments.
Proof of Proposition 3

Proof. Cases (ii) and (iii) are straightforward. We now focus on case (i). In case (i), we have \(k_1, k_2 \leq n\) and \(\min(k_1, k_2) < n\). We first consider the case \(k_2 < n\).

The following fact must hold:

\[
v_1(k_1, k_2) = \mu_1(k_1+k_2+1) v_1(k_1 + 1, k_2) + (1 - \mu_1(k_1+k_2+1)) v_1(k_1, k_2 + 1),
\]

which leads to

\[
v_1(k_1, k_2) - v_1(k_1, k_2 + 1) = \mu_1(k_1+k_2+1) [v_1(k_1 + 1, k_2) - v_1(k_1, k_2 + 1)] \iff
\]

\[
v_1(k_1+1, k_2) - v_1(k_1, k_2 + 1) = \frac{v_1(k_1, k_2) - v_1(k_1, k_2 + 1)}{\mu_1(k_1+k_2+1)}.
\]

Proposition 2 gives

\[
v_1(k_1, k_2) = \sum_{m=(n+1)-k_1}^{(2n+1)-(k_1+k_2)} \mu_1((k_1+k_2+m) \cdot \theta_1(n - k_1 \mid m - 1))_{n \mid (k_1+k_2)+1},
\]

and

\[
v_1(k_1, k_2 + 1) = \sum_{m=(n+1)-k_1}^{(2n+1)-(k_1+k_2)} \mu_1((k_1+k_2+m) \cdot \theta_1(n - k_1 \mid m - 1))_{n \mid (k_1+k_2)+1}.
\]

Hence,

\[
v_1(k_1, k_2) - v_1(k_1, k_2 + 1) = \left\{ \begin{array}{l}
\sum_{m=(n+1)-k_1}^{(2n+1)-(k_1+k_2)} \mu_1((k_1+k_2+m) \cdot \theta_1(n - k_1 \mid m - 1))_{n \mid (k_1+k_2)+1} \\
- \sum_{m=(n+1)-k_1}^{(2n+1)-(k_1+k_2)} \mu_1((k_1+k_2+m) \cdot \theta_1(n - k_1 \mid m - 1))_{n \mid (k_1+k_2)+1} \\
\end{array} \right.
\]

\[
= \mu_1(k_2+n+1) \cdot \theta_1(n - k_1 \mid n - k_1)_{n \mid (k_1+k_2)+1} + \mu_1(k_2+n+2) \cdot \theta_1(n - k_1 \mid n - k_1)_{n \mid (k_1+k_2)+1} + \sum_{m=(n+2)-k_1}^{(2n+1)-(k_1+k_2)} \mu_1((k_1+k_2+m) \cdot \theta_1(n - k_1 \mid m - 1))_{n \mid (k_1+k_2)+1} \\
- \theta_1(n - k_1 \mid n - 2)_{n \mid (k_1+k_2)+1}.
\]

Define

\[
\Delta(m) = \mu_1((k_1+k_2+m) \cdot \theta_1(n - k_1 \mid m - 1))_{n \mid (k_1+k_2)+1} - \theta_1(n - k_1 \mid m - 2)_{n \mid (k_1+k_2)+1}.
\]

We claim

\[
\Delta(m) = \mu_1((k_1+k_2+m) \cdot \theta_1(n - k_1 \mid m - 1))_{n \mid (k_1+k_2)+2} - \theta_1(n - k_1 \mid m - 2)_{n \mid (k_1+k_2)+2}.
\]
For this purpose, we introduce the following two facts:

**Fact 1** \( \theta_1(n - k_1 | m - 1)^{(k_1 + k_2) + m} \) \( (k_1 + k_2) + 1) \cdot \theta_1(n - k_1 - 1 | m - 2)^{(k_1 + k_2) + m - 1} \) 

\(+ (1 - \mu_1((k_1 + k_2) + m)) \cdot \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + m - 1} .

**Fact 2** \( \theta_1(n - k_1 | m - 1)^{(k_1 + k_2) + (m - 1)} \) \( (k_1 + k_2) + 1) \cdot \theta_1(n - k_1 - 1 | m - 2)^{(k_1 + k_2) + (m - 1)} \) 

\(+ (1 - \mu_1((k_1 + k_2) + 1)) \cdot \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + (m - 1)} .

**Fact 1** leads to

\[
\mu_1((k_1 + k_2) + m) \cdot \theta_1(n - k_1 - 1 | m - 2)^{(k_1 + k_2) + m - 1} \\
= \theta_1(n - k_1 | m - 1)^{(k_1 + k_2) + m} - (1 - \mu_1((k_1 + k_2) + m)) \cdot \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + m - 1} .
\]

By **Fact 2**, for \( m \geq n + 2 - k_1 \),

\[
\Delta(m) = \mu_1((k_1 + k_2) + m) \left[ \theta_1(n - k_1 | m - 1)^{(k_1 + k_2) + (m - 1)} - \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + (m - 2)} \right] \\
= \mu_1((k_1 + k_2) + m) \cdot \left[ \mu_1((k_1 + k_2) + 1) \cdot \theta_1(n - k_1 - 1 | m - 2)^{(k_1 + k_2) + (m - 1)} \\
+ (1 - \mu_1((k_1 + k_2) + 1)) \cdot \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + (m - 1)} \\
- \mu_1((k_1 + k_2) + m) \cdot \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + (m - 2)} \right] \\
= \mu_1((k_1 + k_2) + 1) \cdot \left[ \mu_1((k_1 + k_2) + m) \cdot \theta_1(n - k_1 - 1 | m - 2)^{(k_1 + k_2) + (m - 1)} \\
- \mu_1((k_1 + k_2) + m) \cdot \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + (m - 1)} \right] .
\]

Further, by **Fact 1**, the last expression is equal to

\[
\mu_1((k_1 + k_2) + 1) \left[ \theta_1(n - k_1 | m - 1)^{(k_1 + k_2) + m} - \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + m - 1} \right] \\
= \mu_1((k_1 + k_2) + 1) \left[ \theta_1(n - k_1 | m - 1)^{(k_1 + k_2) + m} - \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + m - 1} \right] .
\]

Hence,

\[
v_1(k_1, k_2) - v_1(k_1, k_2 + 1) \\
= \mu_1(k_2 + n + 1) \cdot \theta_1(n - k_1 | n - k_1)^{k_2 + n} \\
+ \mu_1((k_1 + k_2) + 1) \sum_{m = (n + 2) - k_1}^{(2n + 1) - (k_1 + k_2)} \left[ \theta_1(n - k_1 | m - 1)^{(k_1 + k_2) + m} - \theta_1(n - k_1 | m - 2)^{(k_1 + k_2) + m - 1} \right] .
\]

Note \( \mu_1(k_2 + n + 1) \cdot \theta_1(n - k_1 | n - k_1)^{k_2 + n} = \mu_1((k_1 + k_2) + 1) \cdot \theta_1(n - k_1 | n - k_1)^{k_2 + n + 1} \).

Hence, we must have

\[
v_1(k_1, k_2) - v_1(k_1, k_2 + 1) \\
= \mu_1((k_1 + k_2) + 1) \cdot \theta_1(n - k_1 | 2n - k_1 - k_2)^{2n + 1} .
\]
We then conclude
\[ v_1(k_1 + 1, k_2) - v_1(k_1, k_2 + 1) = \frac{v_1(k_1, k_2) - v_1(k_1, k_2 + 1)}{\mu_1((k_1 + k_2)+1)} = \theta_1(n - k_1|2n - k_1 - k_2)|^{2n+1}_{(k_1 + k_2)+2}, \] (16)
which completes the proof for the case where \( k_2 < n \).

When \( k_1 < n \), the same procedure leads to
\[ v_2(k_1, k_2 + 1) - v_2(k_1 + 1, k_2) = \frac{v_2(k_1, k_2) - v_2(k_1 + 1, k_2)}{\mu_2((k_1 + k_2)+1)}, \]
and
\[ v_2(k_1, k_2) - v_2(k_1 + 1, k_2) = \mu_2((k_1 + k_2)+1) \theta_2(n - k_2|2n - k_1 - k_2)|^{2n+1}_{(k_1 + k_2)+2}. \]

Hence, we obtain
\[ v_2(k_1, k_2 + 1) - v_2(k_1 + 1, k_2) = \theta_2(n - k_2|2n - k_1 - k_2)|^{2n+1}_{(k_1 + k_2)+2} = \theta_1(n - k_1|2n - k_1 - k_2)|^{2n+1}_{(k_1 + k_2)+2}, \] (17)
which completes the proof. ■

**Proof of Proposition 4**

**Proof.** Recall
\[ E(V(t)) = \sum_{k_1 + k_2 = t-1} \Pr((k_1, k_2)|t)V(t)(k_1, k_2). \]
It should be noted that \( \sum_{k_1 + k_2 = t-1} \Pr((k_1, k_2)|t) = 1 \) regardless of \( t \).

We first look at the case where \( t \leq n + 1 \). In this case, we must have \( \max\{k_1, k_2\} \leq n \) and \( \min(k_1, k_2) < n \). By Proposition 3, we have
\[ E(V(t)) = \sum_{k_1 + k_2 = t-1} \Pr((k_1, k_2)|t)V(t)(k_1, k_2) = \pi(t) + \sum_{k_1 + k_2 = t-1} \theta_1(k_1|k_1 + k_2)|^{t-1}_1 \times \theta_1(n - k_1|2n - k_1 - k_2)|^{2n+1}_{t+1} = \pi(t) + \theta_1(n|2n)|^{-t} - \pi(t) + \theta_2(n|2n)|^{-t}, \]
where \( \theta_1(n|2n)|^{-t} \) is the overall probability of team \( i \)'s winning exactly \( n \) out of the \( 2n \) battles (nontrivial) other than battle \( t \).

We now turn to the case where \( t \geq n + 2 \). This case can be divided into the following subcases. (i) When \( \max\{k_1, k_2\} \geq n + 1 \), \( V(t) = \pi(t) \); (ii) When \( \max\{k_1, k_2\} \leq n - 1 \), \( V(t) = \pi(t) + \theta_1(n - k_1|2n - k_1 - k_2)|^{2n+1}_{t+1} \); (iii) When \( k_1 = k_2 = n \), \( V(t) = \pi(t) + \theta_1(n - k_1|2n - k_1 - k_2)|^{2n+1}_{t+1} \).
Proof of Theorem 4

Proof. Part (i), (ii) and (iv) of Theorem 4 can be shown following the same arguments for their counterparts in Theorem 3. We now focus on part (iii).

In this extended setting, we continue to denote by \((k_1, k_2)\) the state of the contest. The tuple indicates the number of victories each team has secured before battles in a cluster \(z\) are carried out.

We first illustrate that for any battle \(t\) within any arbitrary cluster, the prize spreads of the two teams must be symmetric regardless of the prevailing state \((k_1, k_2)\). The case of \(\max(k_1, k_2) \geq n + 1\) is trivial. In this case, one team has won. The prize spread is simply \(\pi(t)\), which is symmetric.

We now focus on the case of \(k_1, k_2 < n + 1\). Let \(n(z)\) be the number of battles included in a cluster \(z\). Let \(T_z\) denote the set of battles in a cluster \(z\). In this case, a player \(1(t)\) receives \(\pi(t)\) if he wins, and the contest may enter any state \((k_1 + 1 + l, k_2 + n(z) - l - 1)\), with \(l \in \{0, 1, \ldots, n(z) - 1\}\), after all the \(n(z)\) battles in \(z\) are fought. Note that \((k_1 + l, k_2 + n(z) - 1 - l)\) can be used to denote a stochastic contest state facing battle \(t\). If he loses, then the contest may enter any state \((k_1 + l, k_2 + n(z) - l)\), with \(l \in \{0, \ldots, n(z) - 1\}\), after all the battles in \(z\) are fought.

Suppose that after \(z\) the contest is in state \((\tilde{k}_1, \tilde{k}_2)\). Let \(\tilde{\nu}_i(\tilde{k}_1, \tilde{k}_2)\) denote team \(i\)'s conditional winning probability. Clearly, \(\tilde{\nu}_1(\tilde{k}_1, \tilde{k}_2) = 1\) and \(\tilde{\nu}_2(\tilde{k}_1, \tilde{k}_2) = 0\) when \(\tilde{k}_1 \geq n + 1\).

As a result, player \(1(t)\)'s effective spread amounts to

\[
V_{1(t)}(k_1, k_2) = \pi(t) + \Delta v_{1(t)}^1(k_1, k_2), \forall t \in T_z. 
\]  

Here, \(\Delta v_{1(t)}^1(k_1, k_2) = \sum_{l=0}^{n(z)-1} \left\{ \tilde{\nu}_1(k_1 + (1 + l), k_2 + n(z) - (1 + l)) - \tilde{\nu}_1(k_1 + l, k_2 + n(z) - l) \cdot \tilde{\theta}_1(l|n(z) - l) \right\}_{T_z \setminus \{t\}} \right\},
\]

where \(T_z\) denotes the set of all battles in cluster \(z\), and \(\tilde{\theta}_1(l|n(z) - l)\) is the probability that team 1 wins \(l\) out of the \(n(z) - 1\) simultaneous battles in cluster \(z\), excluding battle \(t\). Here and hereafter, we use \(t\) to index a battle in the cluster with state \((k_1, k_2)\).
Similarly, the effective prize spread of player 2(t) is

\[ V^2_{(t)}(k_1, k_2) = \pi(t) + \Delta v^2_{(t)}(k_1, k_2), \forall t \in T_z. \quad (20) \]

Here, \[ \Delta v^2_{(t)}(k_1, k_2) = \sum_{l=0}^{n(z)-1} \left\{ \begin{array}{c}
\bar{v}_2(k_1 + l, k_2 + n(z) - l) \\
-\bar{v}_2(k_1 + (1 + l), k_2 + n(z) - (1 + l)) \\
\cdot \bar{\gamma}_1(l|n(z) - 1)|_{T_z \setminus \{i\}}
\end{array} \right\}, \]

Note that Lemma 1 still holds for every possible contest state. We thus obtain \[ V^1_{(t)}(k_1, k_2) = V^2_{(t)}(k_1, k_2). \] Hence, each battle in \( z \) is symmetrically valued, and each nontrivial battle \( t \) in any cluster \( z \) has a stochastic outcome \((\mu_1(t), \mu_2(t))\), which is solely determined by matched players’ cost distributions by Proposition 1.

We then consider \( V^1_{(t)}(k_1, k_2) \) to further pin down the symmetric prize spread \( V(t)(k_1, k_2) \) for a battle \( t \) in any cluster. The case of \( \max\{k_1, k_2\} \geq n + 1 \) is trivial. One team has won, thus the prize spread is simply \( \pi(t) \). We now focus on the case of \( k_1, k_2 \leq n \).

For the unclustered battles after the last clustered battles, the results of Proposition 3 apparently hold. We now consider the last cluster that contains more than one battle, which is denoted by \( z_1 \). Without loss of generality, assume that there are unclustered battles following this cluster. \(^{21}\) Propositions 2 and 3 apply to all (unclustered) battles that follow cluster \( z_1 \). Recall that \( \Delta v(k_1, k_2) \) is defined in Section 5 (just before Theorem 1) as \( v_1(k_1 + 1, k_2) - v_1(k_1, k_2 + 1) \) or equivalently \( v_2(k_1, k_2 + 1) - v_2(k_1 + 1, k_2) \). Note that in all three cases of Proposition 3, \( \Delta v(k_1, k_2) \) can be conveniently written as \( \theta_i((n - k_i)|2n - k_1 - k_2)|_{t+1}^{2n+1} \).

Hence, at a stochastic state \((k_1 + l, k_2 + n(z_1) - 1 - l)\), we must have

\[
\Delta v^1_{(t)}(k_1 + l, k_2 + n(z_1) - 1 - l) \\
= \bar{\gamma}_1(k_1 + (1 + l), k_2 + n(z_1) - (1 + l)) - \bar{\gamma}_1(k_1 + l, k_2 + n(z_1) - l) \\
= v_1(k_1 + (1 + l), k_2 + n(z_1) - (1 + l)) - v_1(k_1 + l, k_2 + n(z_1) - l) \\
= \Delta v(k_1 + l, k_2 + n(z_1) - 1 - l) \\
= \theta_i((n + 1) - (k_1 + 1 + l)|2n + 1 - (k_1 + k_2 + n(z_1)))|_{k_1+k_2+n(z_1)+1}^{2n+1},
\]

if \( k_1 + l < n + 1 \), and \( k_2 + n(z_1) - 1 - l < n + 1 \) (cases (i) and (ii) in Proposition 3); and it

\(^{21}\) The case in which no more battles follow the cluster \( z \) is simpler. The same result holds.
boils down to zero otherwise (cases (iii) in Proposition 3). Hence,

\[
\Delta v^1_t(k_1, k_2) \\
= \sum_{l=0}^{n(z_1) - 1} \Delta v^1_t(k_1 + l, k_2 + n(z_1) - 1 - l) \cdot \tilde{\theta}_1(l \mid n(z_1) - 1) \bigg|_{T_{z_1} \setminus \{t\}} \\
= \sum_{l=\max\{0, k_2 + n(z_1) - 1 - n\}}^{\min\{n(z_1) - 1, n - k_1\}} \Delta v^1_t(k_1 + l, k_2 + n(z_1) - 1 - l) \cdot \tilde{\theta}_1(l \mid n(z_1) - 1) \bigg|_{T_{z_1} \setminus \{t\}} \\
= \sum_{l=\max\{0, k_2 + n(z_1) - 1 - n\}}^{\min\{n(z_1) - 1, n - k_1\}} \left[ \theta_1((n + 1) - (k_1 + 1 + l))2n + 1 - (k_1 + 2 + n(z_1)) \right]_{k_1 + k_2 + n(z_1) + 1}^{2n + 1} \\
= \theta_1(n - k_1 \mid 2n - (k_1 + k_2)) \bigg|_{\{\tilde{i} \geq k_1 + k_2 + 1, \tilde{i} \neq t\}} , \forall t \in T_{z_1}.
\]

It follows that \( V_t(k_1, k_2) = V^1_t(k_1, k_2) , t \in T_{z_1} \) can be rewritten as

\[
\pi_t + \Delta v^1_t(k_1, k_2) \\
= \pi_t + \sum_{l=\max\{0, k_2 + n(z_1) - 1 - n\}}^{\min\{n(z_1) - 1, n - k_1\}} \left[ \theta_1((n + 1) - (k_1 + 1 + l))2n + 1 - (k_1 + 2 + n(z_1)) \right]_{k_1 + k_2 + n(z_1) + 1}^{2n + 1} \\
= \pi_t + \theta_1(n - k_1 \mid 2n - (k_1 + k_2)) \bigg|_{\{\tilde{i} \geq k_1 + k_2 + 1, \tilde{i} \neq t\}} ,
\]

where \( \theta_1(n - k_1 \mid 2n - (k_1 + k_2)) \bigg|_{\{\tilde{i} \geq k_1 + k_2 + 1, \tilde{i} \neq t\}} \) is the probability of team 1’s winning exactly \( n - k_1 \) out of the \( 2n - k_1 - k_2 \) (nontrivial) battles (excluding battle \( t \)) when the contest enters the state \( (k_1, k_2) \) after the cluster that precedes the last multi-battle cluster \( z_1 \) is contested. The above formula for \( V_t(k_1, k_2) \) still applies for the case \( \max\{k_1, k_2\} \geq n + 1 \). In this case, clearly \( \theta_1(n - k_1 \mid 2n - (k_1 + k_2)) \bigg|_{\{\tilde{i} \geq k_1 + k_2 + 1, \tilde{i} \neq t\}} = 0 \). Because \( k_1 + l < n + 1 \), and \( k_2 + n(z_1) - 1 - l < n + 1 \), the range of \( l \) (i.e. from \( \max\{0, k_2 + n(z_1) - 1 - n\} \) to \( \min\{n(z_1) - 1, n - k_1\} \)) excludes all (and only) the events that the result of battle \( t \) does not make a difference in determining the winning probability of the whole contest. Note that when no more battles follow \( z_1 \), we have \( n(z_1) = (2n + 1) - (k_1 + k_2) \). It can be verified that \( \max\{0, k_2 + n(z_1) - 1 - n\} = \min\{n(z_1) - 1, n - k_1\} = n - k_1 \) in the case of \( k_1, k_2 \leq n \).

We then consider the cluster that immediately precedes \( z_1 \), which is denoted by \( z_2 \). Suppose that it faces an arbitrary state \( (k_1, k_2) \). Note that we allow it to include only one battle. We have

\[
\Delta v^1_t(k_1, k_2) \\
= \sum_{l_2=0}^{n(z_2) - 1} \left\{ [\tilde{v}_1(k_1 + (1 + l_2), k_2 + n(z_2) - (1 + l_2)) - \tilde{v}_1(k_1 + l_2, k_2 + n(z_2) - l_2)] \cdot \tilde{\theta}_1(l_2 \mid n(z_2) - 1) \bigg|_{T_{z_2} \setminus \{t\}} \right\}.
\]
Suppose that after \( z_2 \) the contest is in a state \((\tilde{k}_1, \tilde{k}_2)\). Recall that \( \tilde{v}_i(\tilde{k}_1, \tilde{k}_2) = 1 \) and \( \tilde{v}_j(\tilde{k}_1, \tilde{k}_2) = 0 \) when \( \tilde{k}_i \geq n + 1 \). Note that we have shown that in cluster \( z_1 \), every nontrivial battle \( t \) has winning probabilities \((\mu_1(t), \mu_2(t))\). We now calculate \( \tilde{v}_i(\tilde{k}_1, \tilde{k}_2) \) for \( \tilde{k}_1, \tilde{k}_2 \leq n \). Note in this case, every battle in cluster \( z_1 \) is nontrivial. Thus

\[
\tilde{v}_1(\tilde{k}_1, \tilde{k}_2) = \sum_{l_1=0}^{n(z_1)} [v_1(\tilde{k}_1 + l_1, \tilde{k}_2 + n(z_1) - l_1) \cdot \tilde{\theta}_1(l_1 \mid n(z_1))]_{T_{z_1}}, \tilde{k}_1 + \tilde{k}_2 = k_1 + k_2 + n(z_2).
\]

Hence,

\[
\tilde{v}_1(k_1 + (1 + l_2), k_2 + n(z_2) - (1 + l_2)) - \tilde{v}_1(k_1 + l_2, k_2 + n(z_2) - l_2) \\
= \sum_{l_1=0}^{n(z_1)} [v_1(k_1 + (1 + l_2) + l_1, k_2 + n(z_2) - (1 + l_2) + n(z_1) - l_1) \\
- v_1(k_1 + l_2 + l_1, k_2 + n(z_2) - l_2 + n(z_1) - l_1)] \cdot \tilde{\theta}_1(l_1 \mid n(z_1))]_{T_{z_1}} \\
= \sum_{l_1=0}^{n(z_1)} \Delta v(k_1 + l_2 + l_1, k_2 + n(z_2) - (1 + l_2) + n(z_1) - l_1) \cdot \tilde{\theta}_1(l_1 \mid n(z_1))]_{T_{z_1}} \\
= \sum_{l_1=0}^{n(z_1)} \{ \theta_1(n - (k_1 + l_2 + l_1)|2n + 1 - (k_1 + k_2 + n(z_2) + n(z_1)))^{2n+1}_{k_1+k_2+n(z_2)+n(z_1)+1} } \\
\cdot \tilde{\theta}_1(l_1 \mid n(z_1))]_{T_{z_1}} \\
= \theta_1(n - (k_1 + l_2)|2n + 1 - (k_1 + k_2 + n(z_2)))^{2n+1}_{k_1+k_2+n(z_2)+1} \cdot \forall l_2 \in \{0, 1, ..., n(z_2) - 1\}. \quad (21)
\]

Therefore,

\[
\Delta v^1_{(t)}(k_1, k_2) \\
= \sum_{l_2=0}^{n(z_2)-1} \{ \theta_1(n - (k_1 + l_2)|2n + 1 - (k_1 + k_2 + n(z_2)))^{2n+1}_{k_1+k_2+n(z_2)+1} \\
\cdot \tilde{\theta}_1(l_2 \mid n(z_2) - 1)\}_{T_{z_2} \setminus \{t\}} \\
= \theta_1(n - k_1|2n - (k_1 + k_2))|\{\tilde{i} \geq k_1+k_2+1, \tilde{i} \neq t\}, \forall t \in T_{z_2}. \quad (22)
\]

In view of (21), we can also obtain \( \Delta v^1_{(t)}(k_1, k_2), \forall t \in T_{z_3} \) by considering clusters \( z_3 \) and \( z_2 \) when applying the procedure of deriving (22) by considering clusters \( z_2 \) and \( z_1 \). The following general formula can then be obtained: for any battle \( t \) in any cluster \( z_k \),

\[
\Delta v^1_{(t)}(k_1, k_2) = \theta_1(n - k_1|2n - (k_1 + k_2))|\{\tilde{i} \geq k_1+k_2+1, \tilde{i} \neq t\}, \forall t \in T_{z_k}, \quad (23)
\]

where \((k_1, k_2)\) is the contest state before cluster \( z_k \) is carried out.
By repeating the exercise in the proof of Proposition 4, we can conclude that each battle \( t \) has an ex ante expected prize spread of \( \theta_t(n|2n)|_{-t} \), and thus it elicits an ex ante expected total effort \( \rho_t \pi(t) + \rho_t \theta_t(n|2n)|_{-t} \), which does not depend on how these battles are clustered or sequenced. We then complete the proof.

References


