The Structure and Performance of The World Market in a Cobb-Douglas Example

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ABSTRACT: In an international trading economy where countries set tariffs strategically, modeled using a Cobb-Douglas example, this paper studies the relationship between the structure and the performance of the world market. Using new results from monotone comparative statics in a Shapley-Shubik market game, replication of such an international trading economy is studied. It is shown that, as the economy is replicated, the equilibrium converges monotonically towards the equilibrium of a competitive equilibrium model of international trade. The distributional implications of replication are also evaluated.

KEYWORDS. Efficiency, free trade, market performance, market structure, Nash tariffs, Shapley-Shubik market game, tariff war.

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1. Introduction

It used to be said that international trade theory was a showcase for the theory of general competitive equilibrium. A central assumption in trade theory at that time was that countries are price takers on world markets. Research in that framework gave rise to some of the fundamental results of international trade, not least the efficiency of free trade. However, in recent years the idea that countries have power on world markets, and hence that they interact strategically, has become central to the economic theory of the world trading system (see Bagwell and Staiger 2002 for a synthesis of the literature on the economics of the world trading system). This shift in perspective has lead to greater scope for the analysis of situations where international trade in equilibrium may not be efficient.

The purpose of this paper is to investigate in a strategic setting the relationship between the structure of the world market and its performance in terms of world economic efficiency and equity. Of course, the existing literature on the economics of the world trading system does analyze the implications for efficiency of opening up to international trade. But the previous literature has largely ignored the issue of how a progressive change in the structure of the world market would affect the performance of the market in terms of efficiency and equity. The reason for this is almost certainly because the analytical tools for carrying out this type of analysis did not exist until recently.

The present paper introduces new tools in monotone comparative statics (Amir and Lambson 2000, Amir 2002, Amir and Bloch 2004) to the field of international trade. These tools have been developed specifically to address questions of how a change in market structure affects market performance. The tools have been developed in the study of Cournot competition and Shapley-Shubik market games. This present paper introduces the tools to an international trade setting and shows how they can be used to yield new insights about the relationship between the structure of the world market and its performance. As far as I know, this present paper represents the first application in which a Shapley-Shubik market game is embedded in a policy game (here a tariff game) and the insights of strategic behavior in a market game used to understand the incentives that drive international tariff policy-making.
The model that is developed in this paper blends Johnson’s classic (1953-54) characterization of a tariff war with a model of bilateral oligopolies. Although it was not fully appreciated at the time of his writing, Johnson’s characterization of a tariff war is in fact a static Nash equilibrium of a tariff game, where the tariff game can be thought of as a terms-of-trade driven Prisoner’s Dilemma. The model of bilateral oligopolies has its origins in Bloch and Ghosal (1997) and Gabszewicz and Michel (1997), which build in turn on the market game originated by Shapley and Shubik (1977). A feature of the bilateral oligopoly model is used in the present paper to extend Johnson’s original tariff war model in a new direction. Johnson investigates a tariff war between two countries trading two goods. While Johnson models just two countries trading two goods, the bilateral oligopoly model has a large number of traders trading two goods. The model of this present paper brings together these two features; a number of countries each have a comparative advantage in the production of one of the two goods. Thus, a number of countries compete in the export of a single good in equilibrium. Another way to put this is that an international market is characterized as having two ‘sides’. There is more than one country on each side of the market. In Johnson’s original model there is only one country on each side of the market.

The analysis of a tariff game in the present paper follows Amir and Bloch’s analysis of a Shapley Shubik market game. Using their approach, the comparative statics can be carried out on two levels. The effects of a change in the number of countries on one side of the market can be analyzed. It is also possible to analyze the effects of a simultaneous increase in the number of countries on both sides of the market that holds them in the same proportion. This we will refer to as a ‘replication’ of the international trading economy.

There are some parallels in the set-up of the present model to Bagwell and Staiger’s (1997) ‘competing exporters’ model. And some of the key features of the model turn out to be the same in that tariffs between countries that compete in the export of a particular good are strategic complements. But in fact the set-up of the respective models are quite different, particularly with regard to the endowment structure. Also, Bagwell and Staiger use their model to look at the formation of bilateral and multilateral trade agreements, while the present paper does not focus on trade agreements. And the model of this present paper yields insights that would not be available from Bagwell and Staiger (1997).

The idea behind increasing the number of countries is to understand how the structure of the market affects its performance. It may be that the present framework could be extended to consider how trade between WTO members is affected by the introduction of new members, or indeed how inter-block membership affects the efficiency and equity properties of a trade block. While these are potentially interesting ideas, these extensions are beyond the scope of the present paper.
The analysis of comparative statics in this extended model of Johnson’s tariff war contributes to our understanding of the effects of international market structure on an international trading equilibrium in three different ways. First, the analysis highlights conditions under which an increase in the number of countries affects the equilibrium terms of trade and volume of trade in a predictable manner. Extending Amir and Bloch, a key result provides conditions under which the trading equilibrium of the tariff game converges monotonically to the perfectly competitive equilibrium of the economy (the efficient outcome of free trade) as the number of countries is replicated. This provides a nice interface between the early international trade literature based on perfect competition and the newer literature based on strategic interaction between countries.

Second, it is possible to test the robustness of predictions made about strategic trade policy obtained in a partial equilibrium environment to the context of general equilibrium. While Brander and Spencer’s (1984) model of strategic tariff setting has become a workhorse model for the analysis of strategic trade policy, it is criticized because it only allows analysis of strategic tariff setting in one sector, while trade in the numeraire good goes unfettered. The model of this present paper allows the extension of such strategic tariff policy analysis to a general equilibrium setting in which both sectors are subject to trade intervention simultaneously.

Finally, the model can be used to provide a new perspective on the equity implications of international trade. In particular, the more scarce is the good that a country exports in equilibrium, the worse off a country becomes when the economy is replicated. This surprising result occurs in the model that we will develop because although a country gains from the increase in competition to supply it with exports from the other side of the market, at the same time its own export becomes less scarce on world markets and the overall effect of replication is to undermine its power on world markets.

In terms of policy implications for the real world of this paper, the mechanics of the third (final) effect seem to be of greatest interest. Attention has focused recently on developing country complaints that they are badly served under the dispute settlement system of the World Trade Organization (see Bagwell, Mavroidis and Staiger 2005). According to conventional wisdom this is because developing countries are small, or the import elasticity of demand for their exports is assumed to be high because they export similar or
homogeneous goods (agricultural products, for example). Consequently developing coun-
try power to retaliate using tariffs is weak, since their optimal tariff is low, following a
development from a trade agreement by a large developed country.\textsuperscript{4} The model of the present
paper suggests an alternative reason for why developing country tariff retaliation is weak
(all countries in the model are the same size). It is that there are many such develop-
ing countries relative to developed countries and so they compete more aggressively with
one another to put goods on their side of the market, hence undermining their collective
terms-of-trade.

The paper proceeds as follows. In the next section, we will set up the basic model
of the international market. This defines the countries that constitute the players in
the tariff game, the tariffs set by the countries, which constitute their strategies, and
the payoffs to each country from its own tariff choice and the tariff choices of all other
countries. Section 3 then sets up the tariff game which is played using the model set
up in Section 2, and Section 4 defines equilibrium of that game. Section 5 introduces a
Cobb-Douglas example, which is then used to simplify the characterization of equilibrium
that is undertaken in Section 6. Section 7 presents possibilities for extending the results
to a more general setting. Conclusions are drawn in Section 8.

2. The Model of an International Trading Economy

Let us first set out the model of an international trading economy. This will form the
basis for the tariff game that we will analyze.

There are two sets of countries: manufacturers $A = \{1, 2, ..., a, ..., m\}$; and primary
product producers $B = \{m + 1, m + 2, ..., b, ..., n + m\}$.\textsuperscript{5} There are $m$ countries in the set
$A$ and there are $n$ countries in the set $B$. Following notation from the matching literature,
a will be used to denote the representative member of $A$ and $b$ will be used to denote
the representative member of $B$. Where convenient, we will use $i$ to refer to a country in
either set $A$ or in set $B$.

\textsuperscript{4}Syropoulos (2002) undertakes a comprehensive analysis of the effect of relative country size on optimal
tariffs in a tariff war.

\textsuperscript{5}The intersection of sets $A$ and $B$ is empty.
Each country in A has an endowment (normalized to unity) of a homogeneous manufactured good, referred to as Good 1, and each country in B has an endowment (also normalized to unity) of a homogeneous agricultural product, referred to as good 2. Goods 1 and 2 are the only two goods available.\footnote{Equivalently, assume a (trivial) production technology in which each country is endowed with a factor of production that can produce 1 unit of good 1 or 1 unit of good 2. The framework and results could be generalized to non-trivial production technologies but the analysis would quickly become intractable and this would not add insight.}

Each country has an identical mass of atomistic consumers who are (domestic) price takers. Each consumer behaves non-strategically, and solves a standard consumption problem. The government of each country behaves as a benevolent dictator, manipulating prices by setting tariffs in order to maximize the welfare of the representative citizen.\footnote{The assumption that the government is a ‘benevolent dictator’ has a long history in economics, being used by Johnson (1953-54) in his seminal specification of a tariff game. It simply says that while consumers behave competitively, and cannot ‘see through’ the government’s budget constraint, the government ‘adopts the preferences of its citizens’ and sets tariffs strategically on their behalf to maximize their welfare, taking as given the tariffs set by other governments. Full specification of the tariff game is the subject of Section 3.}

Denote by \( x_{ij} \) the consumption of good \( j \in \{1, 2\} \) in country \( i \). The preferences of the representative consumer in country \( a \) over \( (x_{aj}) \) are specified by the utility function,

\[
u_a = u^A(x_{a1}, x_{a2}).
\]

Similarly, for the representative consumer in country \( b \), preferences are defined over \( (x_{bj}) \) by the utility function,

\[
u_b = u^B(x_{b2}, x_{b1}).
\]

Following the literature on market games, utility functions are assumed to satisfy the following general properties:\footnote{Since the work of Grossman and Helpman (1994, 1995) research on international trade policy has accounted for distributional considerations in strategic tariff setting, including those brought about by lobbying and other forms of special interest. Such an extension to the present framework could well yield interesting insights but it will not be taken up here. The present model is kept as simple as possible in order to highlight the links to the literature on market games as clearly as possible. Even without allowing for special interest groups, the model yields some interesting insights about the distributional implications of international trade, as has already been mentioned and as we shall see in depth below.}

\[
\frac{\partial u^A(x_{a1}, x_{a2})}{\partial x_{a1}} = u^A_{x1}, \quad \frac{\partial u^A(x_{a1}, x_{a2})}{\partial x_{a2}} = u^A_{x2}, \quad \frac{\partial^2 u^A(x_{a1}, x_{a2})}{\partial x_{a1} \partial x_{a2}} = u^A_{xy}
\]

and so on. Symmetrically for \( u^B(x_{a2}, x_{a1}) \), \( \frac{\partial u^B(x_{a2}, x_{a1})}{\partial x_{a2}} \) will appear as \( u^B_{x2} \) and so on.
1. The utility functions are twice continuously differentiable, strictly increasing and strictly concave.\(^9\)

2. The utility functions satisfy the boundary conditions \( \lim_{x_{ij} \to 0} u_i = +\infty, \ i \in \{x, y\}, I \in \{A, B\} \).

3. The utility functions satisfy the symmetry assumption \( u^A(x, y) = u^B(y, x) \).

The consumer in country \( i \) faces a budget constraint

\[
\sum_{j=1}^{2} p_j (1 + \tau_i) x_{ij} = p_i + R_i, \tag{2.1}
\]

where \( p_j, \tau_i, R_i = p_j \tau_i x_{ij} \) are, respectively, the world price of good \( j \), the tariff set by country \( i \) on good \( j \), and tariff revenue in country \( i \), which as usual is returned to the consumer in a lump-sum. Without loss of generality, countries in \( A \) impose a zero tariff on good 1 and countries in \( B \) impose a zero tariff on good 2. For all other tariffs, let \( \tau_i \in \mathbb{R}_+; \) tariffs are non-negative.

Denote the vector of tariffs set by all countries in \( A \) as \( \tau_a = \{\tau_1, \ldots, \tau_a, \ldots, \tau_m\} \) and the vector of tariffs set by all countries in \( B \) as \( \tau_b = \{\tau_{m+1}, \ldots, \tau_b, \ldots, \tau_{n+m}\} \). It will also be convenient to have notation for the tariffs of all countries in \( A \) except country \( a \); \( \tau_a = \{\tau_1, \ldots, \tau_a-1, \tau_a+1, \ldots, \tau_m\} \). The tariff vector \( \tau_{-b} \) is defined analogously. Finally, for the total tariff vector we have \( \tau = \{\tau_a, \tau_b\} = \{\tau_1, \ldots, \tau_m, \tau_{m+1} \ldots, \tau_{n+m}\} \).

3. The Tariff Game

The extensive form of the game is as follows. First, each country \( i \) simultaneously chooses an import tariff \( \tau_i \). Then, given world prices \( p = \{p_1, p_2\} \), and the tariff \( \tau_i \), the consumer in country \( i \) chooses \( x_{i1} \) and \( x_{i2} \) to maximize \( u_i \) subject to the budget constraint. This yields the usual excess demands and the indirect utility function. For country \( a \),

\[
u_a = u^A(p, \tau_a) = u^A(x_{a1}(p, \tau_a), x_{a2}(p, \tau_a)).
\]

Then, conditional on \( \tau \), markets clear and

\(^9\)The assumption here of strict concavity is slightly stronger than the usual market games assumption of strict quasi-concavity. It is made here to ensure that a well defined and smoothly varying solution exists to the consumer problem based on tariffs.
world prices $p$ are determined.\footnote{As this is a general equilibrium model, prices are determined only up to a scalar, and so some normalization (e.g. choice of numeraire) must be made. This technical detail and others are dealt with below.} These world prices will of course depend on tariffs i.e. $p = p(\tau)$. If equilibrium prices are unique, given tariffs, then the mapping $p(.)$ is one-to-one. Then the indirect utility function can be written as a function only of tariffs.\footnote{Transfers are not allowed between countries. This is deemed to be a reasonable assumption in the trade literature, since although transfers do happen they are often constrained by extraneous factors. Thus, an equilibrium without transfers is often deemed to be a reasonable characterization of observed outcomes.}

Finally, as this is an endowment economy, exports of Good 1 by $a$ can be written as $e_a (p, \tau_a) = 1 - x_{a1}(p, \tau_a)$. Similarly, exports of Good 2 by $b$ can be written as $e_b (p, \tau_b) = 1 - x_{b2}(p, \tau_b)$. Then for total exports we have $E_A (p, \tau_a) = \sum_{a \in A} e_a (p, \tau_a)$ and $E_B (p, \tau_b) = \sum_{b \in B} e_b (p, \tau_b)$. Also, it will be convenient to have notation for exports by all countries in $A$ except country $a$; $E_{A-a} (p, \tau_{-a}) = E_A (p, \tau_a) - e_a (p, \tau_a)$. Symmetrically, $E_{B-b} (p, \tau_{-b}) = E_B (p, \tau_b) - e_b (p, \tau_b)$.

We can now derive an expression for world prices strictly in terms of tariffs. By world market clearing,

$$p_1 E_A (p, \tau_a) = p_2 E_B (p, \tau_b).$$

Without loss of generality, let $p_1 = p$ and let $p_2 = 1$. Rearranging, we have

$$p = \frac{E_B (p, \tau_b)}{E_A (p, \tau_a)}. \quad (3.1)$$

If each function $e_a (p, \tau_a)$ and $e_b (p, \tau_b)$ is continuous, the world market clearing condition implicitly defines a mapping from the vector of tariffs $\{\tau_a, \tau_b\}$ to $p$:

$$p = p (\tau).$$

The above set-up reveals a simple relationship between a market game and a tariff game. In a market game, a player is able to choose directly the quantity that he puts on the market. To make the analogy to the present setting, a market game would model a situation in which each government were running a command economy and able to dictate the quantity exported by its country; the government of $a$ could dictate $e_a$ directly. In the present model, instead, each government can only set tariffs, influencing the quantity that a country exports by manipulating consumers’ decisions through tariff setting. In
all other respects, the underlying economic exchange that is taking place is the same in a tariff game as in a market game.

There is an idea here that will appear unfamiliar in the context of a tariff game and it should be clarified. The idea is that a country manipulates tariffs strategically in order to determine the quantity exported. From the past literature, going back at least as far as Johnson (1953-54), our thinking is that in a tariff game a country manipulates the quantity imported by setting tariffs, and in doing so manipulates the terms of trade to its advantage. But we know by the Lerner symmetry theorem that an export tax has an equivalent import tariff. The present paper uses the principle of the Lerner symmetry theorem to make a novel switch in focus from imports to exports, in order to highlight the link to the literature on market games. While the Lerner symmetry theorem is well known, it has not been exploited in a strategic setting such as this before. But it should be emphasized that the game is essentially equivalent to a standard tariff game, in which the focus has traditionally been on the effect of tariffs on imports.

4. Equilibrium and Efficiency

In a trading equilibrium,

\[ E_A(p(\tau),\tau_a) > 0 \text{ and } E_B(p(\tau),\tau_b) > 0. \]

For any vector of tariffs \( \{\tau_a, \tau_b\} \) in a trading equilibrium, the final allocation obtained by country \( a \) is given by

\[ (x_{a1}(p(\tau),\tau_a), x_{a2}(p(\tau),\tau_a)) = (1 - e_a(p(\tau),\tau_a), e_a(p(\tau),\tau_a) p(\tau)) \]

and the final allocation obtained by country \( b \) is given by

\[ (x_{b2}(p(\tau),\tau_b), x_{b1}(p(\tau),\tau_b)) = \left(1 - e_b(p(\tau),\tau_b), \frac{e_b(p(\tau),\tau_b)}{p(\tau)} \right). \]

Otherwise there is autarky.

This way of writing the final allocation that each country obtains in a trading equilibrium emphasizes the effect of tariff policy on exports.

We can now define an equilibrium in tariffs as a vector of tariffs \( (\hat{\tau}_1, ..., \hat{\tau}_m, \hat{\tau}_{m+1}, ..., \hat{\tau}_{n+m}) \) such that
(i) for any country $a \in A$, $\hat{\tau}_a$ maximizes
\[ u^A \left(1 - e_a \left(p(\tau_a, \hat{\tau}_a, \hat{\tau}_a), \tau_a \right), e_a \left(p(\tau_a, \hat{\tau}_a, \hat{\tau}_a), \tau_a \right) \right). \]

(ii) for any country $b \in B$, $\hat{\tau}_b$ maximizes
\[ u^B \left(1 - e_b \left(p(\hat{\tau}_a, \tau_b, \hat{\tau}_b), \tau_b \right), e_b \left(p(\hat{\tau}_a, \tau_b, \hat{\tau}_b), \tau_b \right) \right). \]

Standard arguments can be used to prove existence of an equilibrium, which is just a Nash equilibrium of the tariff game.

Following standard definitions, world welfare is the sum of all national welfares, and world efficiency is given by
\[ \max_{\tau} \sum_{a \in A} u^A (p(\tau), \tau) + \sum_{b \in B} u^B (p(\tau), \tau). \]

5. The Cobb-Douglas Example

One of the appealing features of the analysis carried out by Amir and Bloch is that it imposes minimal restrictions on consumer preferences which have a natural economic interpretation. Only two restrictions are imposed. The first is the familiar notion of normality of goods. The second is the property that goods are either gross complements or gross substitutes. In this present paper we will characterize and analyze equilibrium within a more restrictive framework based on Cobb-Douglas preferences. The reason for doing this is to emphasize the links between market games and a tariff game. Since a major purpose of the present paper is to highlight these links and to build intuition around these ideas, it seems worth the compromise of generality to focus on this simpler framework. Section 7 provides formal indications that the results established for Cobb-Douglas should extend to a more general setting.

The preferences of the representative consumer in country $a$ over $(x_{aj})$ are given by the following Cobb-Douglas utility function:
\[ u_a = u^A (x_{a1}, x_{a2}) = (x_{a1})^{1-\alpha} (x_{a2})^\alpha. \]
The preferences of the representative consumer in country $b$ over $(x_{bj})$ are given by the utility function,

$$u_b = u^B (x_{b2}, x_{b1}) = (x_{b2})^{1-\alpha} (x_{b1})^{\alpha}.$$  \hfill (5.2)

It is easily checked that (5.1) and (5.2) satisfy properties 1-3 introduced above. We can also verify that both goods are normal under Cobb-Douglas preferences. Following Amir and Bloch (2004), Good 1 is a normal good for country $a$ if and only if $\Delta_{a1} \equiv u^A_y u^A_{xx} - u^A_x u^A_{xy} > 0$. Good 2 is a normal good for country $a$ if and only if $\Delta_{a2} \equiv u^A_x u^A_{xy} - u^A_y u^A_{xx} > 0$. By the symmetry of preferences, Good 1 is a normal good for country $b$ if and only if $\Delta_{b1} \equiv u^B_y u^B_{xx} - u^B_x u^B_{xy} > 0$, and Good 2 is a normal good for country $b$ if and only if $\Delta_{b2} \equiv u^B_x u^B_{xy} - u^B_y u^B_{xx} > 0$. As Amir and Bloch point out, this definition of normality requires the property that demand for the good is increasing in income for all prices.

Before undertaking a characterization of equilibrium, let us solve the model for the Cobb-Douglas preferences specified above under the assumption that the vectors of tariffs $\tau_a$ and $\tau_b$ are given. By doing this, we will be able to highlight a key property of the model under Cobb-Douglas that will help to highlight the link between a market game and a tariff game. The property is that $e_a$ depends only on $\tau_a$. In general, exports are given by $e_a (p (\tau), \tau_a)$. Under Cobb-Douglas preferences, the model has the special feature that $e_a$ can be written only as a function of $\tau_a$; $e_a (\tau_a)$. This is because the terms in $p (\tau)$ cancel as we shall see below. Thus, while in the present tariff game setting the government cannot choose $e_a$ directly as in a market game, under Cobb-Douglas preferences we can analyze in a tractable way how the government can set $\tau_a$ in order to choose $e_a$ indirectly.

We will work out the problem of country $a$, the preferences of which are given by (5.1). The problem of country $b$ is analogous. The consumer optimization problem gives demands for the two goods:

$$x_{a1} = (1-\alpha) \left( \frac{p_1 + R_a}{p_1} \right); \hfill (5.3)$$

$$x_{a2} = \alpha \left( \frac{p_1 + R_a}{p_2 (1+\tau_a)} \right). \hfill (5.4)$$

Using the fact that $R_a = p_2 \tau_a x_{a2}$, we have

$$x_{a1} = \frac{(1-\alpha) (1 + \tau_a)}{(1-\alpha) \tau_a + 1}.$$
Rewriting in terms of exports,
\[ e_a (\tau_a) = 1 - \frac{(1 - \alpha) (1 + \tau_a)}{(1 - \alpha) \tau_a + 1} = \frac{\alpha}{1 + (1 - \alpha) \tau_a}. \]  
(5.5)

Here we see how \( e_a \) depends only on \( \tau_a \). We can use \( e_a (\tau_a) \) to calculate the response of exports to a change in tariffs:
\[ \frac{de_a}{d\tau_a} = -\frac{(1 - \alpha) \alpha}{(1 + (1 - \alpha) \tau_a)^2} < 0. \]

From this we see that \( e_a (\tau_a) \) is everywhere decreasing in \( \tau_a \).

Choosing \( p_1 = p \) and letting \( p_2 = 1 \), we can now derive an expression for the indirect utility function in terms of exports:
\[ u^A (1 - e_a (\tau_a), e_a (\tau_a) p) = \left( \frac{(1 - \alpha) (1 + \tau_a)}{(1 - \alpha) \tau_a + 1} \right)^{1-\alpha} \left( p \frac{\alpha}{((1 - \alpha) \tau_a + 1)} \right)^\alpha \]

So far we have been writing \( p \) as if it were parametric. We can now introduce the fact that, by (3.1), \( p \) depends on \( \tau = (\tau_1, ..., \tau_m, \tau_{m+1}, ..., \tau_{n+m}) \);
\[ p(\tau_1, ..., \tau_m, \tau_{m+1}, ..., \tau_{n+m}) = \frac{E_B (\tau_b)}{E_A (\tau_a)} = \frac{\sum_{b \in B} \left( \frac{\alpha}{1+(1-\alpha)\tau_b} \right)}{\sum_{a \in A} \left( \frac{\alpha}{1+(1-\alpha)\tau_a} \right)} \]  
(5.6)

Note that \( p_{\tau_a} > 0 \) i.e. an increase in \( \tau_a \) improves \( a \)'s terms of trade. Analogously, \( p_{\tau_b} < 0 \); an increase in country \( b \)'s tariff, \( \tau_b \), improves \( b \)'s terms of trade.

Finally, it is easily verified (and well known) that for the above set up world efficiency is maximized when all countries adopt free trade.

6. Characterization of Equilibrium

6.1. Characterization of equilibrium for one side of the market

In principle, despite the simplifications we have made, the tariff game described here is quite complex. Each country must decide its strategy not just against the countries on the
other side of the market but against the countries on its own side of the market as well. Following Amir and Bloch (2004), which builds in turn on Amir and Lambson (2000), we can simplify the problem by characterizing the tariff game played between countries on one side of the market, taking as given the tariffs of countries on the other side of the market. Given a simple characterization of this game, it is then straightforward to characterize the equilibrium of the tariff game between countries on both sides of the market.

In the symmetric Cournot oligopoly analyzed by Amir and Lambson, the optimal quantity choice of an oligopolist only depends on the total output of the \( (n - 1) \) remaining firms. Amir and Bloch extend this approach to a market game setting in which there are two sides to the market as in the model of this present paper. They show that the optimal quantity placed on the market by a player on one side of the market depends on the quantity choices of the other \( (n - 1) \) players on the same side of the market, taking as given the quantity choices of players on the other side of the market. Here in this present paper, we will show that the same approach as taken by Amir and Bloch can be extended to the present setting of a tariff game. We will hold the tariffs on one side of the market constant and analyze the tariff game played between countries on the other side of the market.

Formally, consider a symmetric tariff game, \( \Gamma(\tau_b) \), played by the manufacturers producers in \( A \), when the tariffs of primary product exporters in \( B \) are fixed at \( \tau_b \). Under Cobb-Douglas, the expression for preferences can be revised to

\[
\begin{align*}
  u^A(\tau_a; \tau_{-a}, \tau_b) &= u^A \left( 1 - e_a(\tau_a), e_a(\tau_a) \frac{E_B(\tau_b)}{e_a(\tau_a) + E_{A-a}(\tau_a)} \right) \\
  &= (1 - e_a(\tau_a))^{1-\alpha} \left( e_a(\tau_a) \frac{E_B(\tau_b)}{e_a(\tau_a) + E_{A-a}(\tau_a)} \right)^\alpha
\end{align*}
\]

where \( E_A(\tau_a) = e_a(\tau_a) + E_{A-a}(\tau_a) \). We may then define a manufacturer’s reaction function as

\[
\hat{\tau}_a(\tau_{-a}) \equiv \arg\max_{\tau_a} \left\{ u^A \left( 1 - e_a(\tau_a), e_a(\tau_a) \frac{E_B(\tau_b)}{e_a(\tau_a) + E_{A-a}(\tau_a)} \right) : \tau_a \in \mathbb{R}_+ \right\},
\]

where \( \hat{\tau}_a(\tau_{-a}) \) is a best response tariff. For any \( \tau_{-a} \) and \( E_{A-a}(\tau_{-a}) \), \( \hat{\tau}_a \) is chosen in (6.2) to obtain the welfare maximizing level of \( e_a \) in (6.1).
Note for future reference (particularly for Proposition 2) that it is equally valid to express the best response tariff \( \hat{\tau}_a \) as a function of \( E_{A-a}(\tau_{-a}) \), that is \( \hat{\tau}_a(E_{A-a}(\tau_{-a})) \), rather than more compactly as \( \hat{\tau}_a(\tau_{-a}) \). This will make it possible to model the relationship between a responding country’s exports, \( e_a \), and the exports of all the countries in \( A\setminus a \), expressed by the function \( e_a(\hat{\tau}_a(E_{A-a}(\tau_{-a}))) \).

In the first step towards the characterization of equilibrium, let us first characterize the best response tariff function. It is clear from (6.2) that the best response tariff \( \hat{\tau}_a(\tau_{-a}) \) is related to \( \tau_{-a} \) through the impact of a change of \( \tau_{-a} \) on \( E_{A-a}(\tau_{-a}) \) and, in turn, the effect of a change in \( \tau_a \) on \( e_a \). Therefore, a convenient way to characterize the best response tariff function of the \( \Gamma(\tau_b) \) game is in terms of exports.

**Lemma 1.** In the game \( \Gamma(\tau_b) \), for any \( \tau'_{-a} \neq \tau_{-a} \), we have

\[
\frac{e'_a(\hat{\tau}'_a(\tau'_{-a})) - e_a(\hat{\tau}_a(\tau_{-a}))}{E_{A-a}(\tau'_{-a}) - E_{A-a}(\tau_{-a})} > -1.
\]

This characterization of the reaction function says that \( e_a(\tau_a) \) increases in response to an increase in \( E_{A-a}(\tau_{-a}) \). And since both \( e_a(\tau_a) \) and \( E_{A-a}(\tau_{-a}) \) are decreasing in their arguments, the implication is that \( \hat{\tau}_a \) decreases in response to a decrease in \( \tau_{-a} \); \( \hat{\tau}_a(\tau_{-a}) \) is an increasing function. Thus, the tariff game between countries in \( A \) is supermodular.

The proof of Lemma 1 establishes that the condition depends on the normality of Goods 1 and 2 for manufacturers, which holds for Cobb-Douglas preferences as we noted above.

To see the intuition for Lemma 1, see (6.1). As \( E_{A-a}(\tau_{-a}) \) is increased, through a reduction in some element of \( \tau_{-a} \), this brings about a reduction in the terms-of-trade of all countries in \( A \), including country \( a \), and hence brings about a reduction in the purchasing power of \( a \)’s endowment. This may be thought of equivalently as a fall in \( a \)’s income. Since Goods 1 and 2 are both normal, country \( a \) demands less of both goods because its income has fallen, and therefore exports more of its endowment; \( e_a(\tau_a) \) is increased through a reduction in \( \tau_a \).

The next result shows that if the \( \Gamma(\tau_b) \) game is supermodular then the Nash equilibria of the \( \Gamma(\tau_b) \) game are symmetric.

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Proposition 1. For any number of manufacturers, \( m \), all Nash equilibria of the game \( \Gamma(\tau_b) \) are symmetric.

The principle established in Lemma 1, that \( e_a(\hat{\tau}_a(\tau-a)) \) increases with \( E_{A-a}(\tau-a) \), underpins the symmetry of equilibrium established in Proposition 1. To see why the equilibrium must be symmetric, suppose to the contrary that an equilibrium exists in which Countries 1 and 2 in \( A \) set different tariffs; \( \hat{\tau}_1 \neq \hat{\tau}_2 \). And suppose without loss of generality that \( \hat{\tau}_1 < \hat{\tau}_2 \). Then it follows by (5.5) and the definition of \( E_{A-a}(\tau-a) \) that

\[
E_{A-1}(\tau-1) < E_{A-2}(\tau-2).
\]

We also require that, by definition, \( E_{A-1}(\tau-1) + e_1(\hat{\tau}_1) = E_{A-2}(\tau-2) + e_2(\hat{\tau}_2) = E_A \), which in turn implies that \( e_a(\hat{\tau}_2) < e_1(\hat{\tau}_1) \). But this contradicts the fact that, by Lemma 1, \( e_a(\tau_a) \) is increasing in \( E_{A-a}(\tau-a) \) through the choice of best-response tariffs. Of course, a symmetrical result holds for the primary product producers in \( B \).

The next result is stronger. It shows that the equilibrium of the tariff game between countries in \( A \) has a unique Nash equilibrium tariff \( \hat{\tau}_a \) under Cobb-Douglas preferences, and that \( \hat{\tau}_a \) is decreasing in \( m \).

Proposition 2. For any number of manufacturers, \( m \), the tariff game \( \Gamma(\tau_b) \) admits a unique Nash equilibrium. Furthermore, the unique equilibrium tariff, \( \hat{\tau}_a \), is decreasing in \( m \) and total exports \( E_a \) are increasing in \( m \).

In a symmetrical equilibrium, it must be the case that \( e_a(\hat{\tau}_a) = E_{A-a}(\tau-a) / (m-1) \). Because (5.5) is a differentiable function, and because (6.1) is twice continuously differentiable, by implicit differentiation we may evaluate \( \partial e_a(\hat{\tau}_a(E_{A-a}(\tau-a))) / \partial E_{A-a}(\tau-a) \). If there exists more than one equilibrium, then it must be the case that \( \partial e_a(\hat{\tau}_a(E_{A-a}(\tau-a))) > 1 / (m-1) \) at one or more equilibrium. The proof shows that when Good 1 is normal then this cannot happen because \( e_a(\hat{\tau}_a(E_{A-a}(\tau-a))) \) is increasing in \( E_{A-a}(\tau-a) \) but at a decreasing rate. Recall that an increase in \( E_{A-a}(\tau-a) \) reduces the purchasing power of country \( a \), and hence causes \( a \) to demand less of Good 1 and hence export more. But also observe that Good 2 is now relatively more expensive, causing \( a \)’s demand for Good 2 to fall at the margin. Hence, while the amount that \( a \) exports in order to obtain Good 2 increases, the rate of increase declines. Therefore, at equilibrium it must be the case that \( \partial e_a(\hat{\tau}_a(E_{A-a}(\tau-a))) / \partial E_{A-a}(\tau-a) < 1 / (m-1) \), ruling out the possibility that there can
be more than one equilibrium.

The result that \( E_a \) is increasing in \( m \) is easy to see, once it is realized that (starting at the unique equilibrium) country \( a \) responds to an increase in exports by all other countries in \( A \) by lowering its own tariff so that its own exports increase. It does not matter to country \( a \) whether the increase in \( E_{A-a}(\tau_{-a}) \) comes about because one (or more) existing country in \( A \setminus a \) increases its exports or because an additional country is added to \( A \) and that country’s exports are positive.

This concludes our characterization of equilibrium for one side of the market. Our results have been derived for manufacturers, but all results extend directly to primary product producers as well. So we may now proceed to characterize equilibrium in both sides of the market simultaneously, thus characterizing general equilibrium.

### 6.2. Characterization of equilibrium for both sides of the market

So far we have defined a tariff game \( \Gamma (\tau_b) \), played by all countries in \( A \) taking as given the tariff vector \( \tau_b \). For this game, using the payoff function defined by (6.1), we have obtained a unique equilibrium tariff for the countries in \( A \), \( \hat{\tau}_a \), and shown that this tariff must be declining in the number of countries, \( m \), in \( A \).

Symmetrically, we may define a tariff game \( \Gamma (\tau_a) \), played by all countries in \( B \) and taking as given the tariff vector \( \tau_a \). From Propositions 1 and 2, for any \( \tau_a \) there must exist a unique equilibrium tariff \( \hat{\tau}_b \), which is decreasing in \( n \).

Since we know that each side of the market sets a unique tariff in equilibrium, we may define a best response tariff function for each side of the market in terms of a unique tariff set by the other side of the market. In addition, we know that each tariff is a decreasing function of the number of countries on its own side of the market, and independent of the number of countries on the other side of the market. Thus, in general we have overall tariff reaction functions \( \hat{\tau}_a (\tau_b, m) \) and \( \hat{\tau}_b (\tau_a, n) \).

We are now able to solve explicitly for these equilibrium tariffs using the payoff functions (5.1) and (5.2).
Proposition 3. The minimal equilibrium tariff of countries in $A$ is

$$\hat{\tau}_a = \frac{1}{m-1},$$

and the minimal equilibrium tariff of countries in $B$ is

$$\hat{\tau}_b = \frac{1}{n-1}.$$  

If $m = 1$ and/or $n = 1$ then no trading equilibrium exists. If $m \geq 2$ and $n \geq 2$ then there is a unique trading equilibrium.

It is well known that there is a continuum of Nash equilibria of a tariff game in which no trade takes place. Clearly, if $m = 1$ and/or $n = 1$ then the tariff of one or both countries is prohibitive and there is no trading equilibrium. For $m \geq 2$ and $n \geq 2$, it is straightforward to solve for the trading equilibria of the tariff game on each side of the market and, in the process of doing so, verify that the equilibrium is unique. The general characterization of equilibrium is presented in the appendix. Let us here take a look at the specific solution for a country’s best-response tariff function, and see how this gives rise to the equilibrium tariffs presented in Proposition 3.

We know from (5.5) that, rather than choose $\tau_a$ and obtain a resulting value of $e_a$, we can instead choose a value of $e_a$ and solve for a value of $\tau_a$ that would implement $e_a$. More generally, for any feasible value of total exports $E_A$, we may solve for a vector of tariffs $\tau_a$ that would implement $E_A$. This property of the model is useful in deriving the best-response tariff function $\hat{\tau}_a(\tau_{-a})$ for the game $\Gamma(\tau_b)$. The same holds for the derivation of the best-response tariff function $\hat{\tau}_b(\tau_{-b})$ for the game $\Gamma(\tau_a)$.

To derive the best-response tariff function $\hat{\tau}_a(\tau_{-a})$ of the game $\Gamma(\tau_b)$, assume that we have an arbitrary but feasible level of exports $E_B$ and $E_{A-a}$, with corresponding tariff vectors $\tau_{-a}$ and $\tau_b$ that would implement these levels of exports. Fix the tariff vectors $\tau_{-a}$ and $\tau_b$ in the payoff function (6.1). To obtain the best response function, differentiate (6.1) and set this equal to zero in order to obtain the first order condition for the problem. We thus obtain

$$\hat{\tau}_a = \frac{\sqrt{E_{A-a}(\tau_{-a}) (E_{A-a}(\tau_{-a}) + 4\alpha (1-\alpha)) - E_{A-a}(\tau_{-a})}}{2(1-\alpha) E_{A-a}(\tau_{-a})}$$

\[12\] See Dixit (1987).
This root satisfies $\hat{\tau}_a \in \mathbb{R}_+$ and it is the unique positive root. Thus we have a unique best response function $\hat{\tau}_a (\tau_a)$. Also note from this solution that it does not depend on the vector of tariffs $\tau_b$. Using (5.5) and the fact that in a symmetrical equilibrium $E_{A-a} = (m - 1)e_a$, we can solve for the unique equilibrium tariff $\hat{\tau}_a = 1 / (m - 1)$. The equilibrium solution for $\hat{\tau}_b$ is obtained analogously.

There are two features of the symmetric equilibrium tariffs that are worth highlighting. First, $\hat{\tau}_a$ depends only on $m$ and $\hat{\tau}_b$ depends only on $n$. Thus, the effects of a change in the number of countries on one side of the market can be analyzed in a very tractable way. Second, as $m$ is increased the countries in $A$ behave in an increasingly competitive fashion, and indeed as $m \to \infty$ the equilibrium tariff approaches free trade. Countries in $B$ respond in a corresponding way to an increase in $n$.

6.3. Entry of countries on one side of the market

Let us now focus on the trading equilibrium characterized above. We can perform comparative statics on the equilibrium, focusing in particular on the effect of changes in the number of countries on either side of the market. The basic framework for analysis is set up by substituting the equilibrium tariffs into the expressions for payoffs and terms of trade. As both these expressions are continuous, we can then perform comparative statics on them, differentiating in terms of $m$ and $n$ and evaluating the signs of the resulting expressions.

Substituting (symmetric) equilibrium tariffs $\hat{\tau}_a = 1 / (m - 1)$ and $\hat{\tau}_b = 1 / (n - 1)$ in (5.6), we obtain

$$p(m, n) = p(\hat{\tau}_a(m), \hat{\tau}_b(n)) = \frac{n(n - 1)(m - \alpha)}{m(m - 1)(n - \alpha)}$$

Using this expression and equilibrium tariffs in (6.1), we have

$$u^A(m, n) = u^A(\hat{\tau}_a(m), \hat{\tau}_b(n)) = \frac{m(1 - \alpha) \left( \frac{n(n - 1)\alpha}{m(n - \alpha)} \right)^{\alpha} \left( \frac{m(1 - \alpha)}{m - \alpha} \right)^{-\alpha}}{m - \alpha}$$

The properties of equilibrium presented in the next result are easily obtained by performing comparative statics on $u^A(m, n)$. 

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Proposition 4. The unique trading equilibrium exhibits the following comparative statics properties:

(i) The aggregate export volume $E_A(\hat{\tau}_a; m)$ increases monotonically with $m$ and converges to the efficient level as $m \to \infty$;

(ii) The level of $E_B(\hat{\tau}_b; n)$ increases monotonically with $n$ and converges to the efficient level as $n \to \infty$;

(iii) $du^A(m, n)/dm < 0$ and is decreasing in the ratio of $n$ to $m$;

(iv) $du^A(m, n)/dn > 0$ and is decreasing in the ratio of $n$ to $m$.

An increase in $n$ has the effect of increasing exports of Good 2 to the world market and bringing about a reduction in $\hat{\tau}_b$. By inspection of $p(m, n)$, both of these effects improve country $a$’s terms-of-trade and hence its welfare; see $u^A(m, n)$. While the effect of an increase in $n$ on $u^A(m, n)$ is positive, it impact diminishes as $n$ increases. This effect is easy to see by inspection of $u^A(m, n)$, and makes intuitive sense when it is realized that the effect is driven by an increase of $x_{a2}$ in $u^A$, which in turn is valued less highly at the margin as $n$ increases.

An increase in $m$ has the opposite effect, of increasing exports of Good 1 to the world market and bringing about a reduction of $\hat{\tau}_a$. Both of these effects contribute to a reduction in $a$’s terms-of-trade and hence welfare. The negative impact on welfare increases with $m$, since it is driven by a decrease of $x_{a2}$ in $u^A$, and $x_{a2}$ is in turn valued more highly at the margin as $m$ increases.

6.4. Entry of countries on both sides of the market

We are now in a position to study the effects of simultaneous entry of countries on both sides of the market. Because the international market has two sides in our model, we can define any international market in terms of the ratio of countries on one side of the market to countries on the other side. For example, say that $m = 4$ and $n = 6$. Then we have $r = 3/2$. Now if we fix $r$ then we can study the replication of the international economy by doubling $m$. Where the response of the economy in equilibrium to replication is monotonic, we can define replication simply in terms of an increase in $m$. 

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Proposition 5. Assume an initial trading equilibrium for which \( m \geq 2, n \geq 2 \) and \( r = n/m \).

World efficiency implications of replication: The higher is \( m \) then the higher is world welfare, and world welfare is maximized as \( m \to \infty \).

Distributional implications of replication: There exists a value \( r' > 1 \) such that if \( r = r' \) then a given increase in \( m \) will leave \( u^A(m, rm) \) unchanged, if \( r < r' \) then an increase in \( m \) will bring about an increase of \( u^A(m, rm) \) and if \( r > r' \) then an increase in \( m \) will bring about a decrease of \( u^A(m, rm) \).

From the results that have already been established, the implications for world welfare of replication follow naturally. By Proposition 4, as the international trading economy is replicated, export volumes increase monotonically as equilibrium tariffs fall. Trade flows increase monotonically from one side of the market to the other and world welfare increases monotonically as well. Eventually, as \( m \) becomes large, equilibrium tariffs \( \hat{\tau}_a = 1/(m - 1) \) and \( \hat{\tau}_b = 1/(rm - 1) \) tend towards zero, and in the limit the outcome of world efficiency (free trade) is attained.

The distributional implications are more surprising but they can be understood as follows. The higher the value of \( r \), the more scarce are manufactures (endowments in \( A \)) relative to primary products (endowments in \( B \)). Country \( a \) is able to exploit this scarcity because the import elasticity of demand for its export is relatively low in equilibrium. Consequently, country \( a \) sets a relatively high tariff in equilibrium compared to country \( b \), and as a result \( u^A(m, rm) > u^B(rm, m) \). If \( r \) is relatively high then the terms-of-trade effects of its relatively high tariff may be sufficient to ensure that \( u^A(m, rm) \) is above its free trade level.

As \( m \) is increased this reduces the relative scarcity of country \( a \)'s good, reducing \( a \)'s equilibrium tariff. This has two effects on \( u^A(m, rm) \). The static efficiency gains of tariff reduction increase \( u^A(m, rm) \). On the other hand, the tariff reduction may also reduce \( a \)'s terms-of-trade, reducing \( u^A(m, rm) \). As \( m \) is increased, \( u^A(m, rm) \) converges to its efficient (free trade) level. If at low levels of \( m, r < r' \) then \( u^A(m, rm) \) converges to its efficient free trade level from below; the static efficiency effects of tariff reduction dominate. But, if \( r > r' \) then \( u^A(m, rm) \) converges to its free trade level from above (while
$u^B(rm,m)$ converges to its efficient (free trade) level from below). Note that this result may depend on the Cobb-Douglas functional form. But it is not special to Cobb-Douglas. It holds for the quasi-linear preferences that we will look at in the next section.

7. Suggestions for Extension

The results of the analysis have been established for a Cobb-Douglas example because this yields a tractable form to work with, in which the general functional form for exports $e_a(p(\tau_a, \tau_{-a}, \tau_b), \tau_a)$ simplifies to $e_a(\tau_a)$. We will now show that the results may well generalize by showing that they appear to hold for a more general example in which exports function is of the form $e_a(p(\tau_a, \tau_{-a}, \tau_b), \tau_a)$.

As for the Cobb-Douglas example, we work out the problem of a type $A$ country as the problem of a type $B$ country is analogous. The preferences of the representative consumer in country $a \in A$ are of the following quasi-linear form:

$$u_a = x_{a1} + \frac{\sigma}{\sigma - 1} (x_{a2})^{\frac{1}{\sigma - 1}}.$$  \hspace{1cm} (7.1)

with $\sigma > 1$, and where $\sigma$ measures the elasticity if substitution. The optimization problem gives demands for the two goods:

$$x_{a2} = \left[ \frac{p_2(1 + \tau_a)}{p_1} \right]^{-\sigma};$$ \hspace{1cm} (7.2)

$$x_{a1} = 1 + \frac{R_1}{p_1} - \frac{p_2(1 + \tau_a)x_{a2}}{p_1} = 1 + \frac{R_a}{p_1} - \left[ \frac{p_2(1 + \tau_a)}{p_1} \right]^{1-\sigma}.$$ \hspace{1cm} (7.3)

Using the fact that $R_a = p_2 \tau_{a2} x_{a2}$ we have

$$x_{a1} = 1 - \frac{p_2}{p_1} \left( \frac{p_2(1 + \tau_{a2})}{p_1} \right)^{-\sigma}.$$  

Choosing $p_1 = 1$ and letting $p_2 = p$ as above, and solving for exports, we have

$$e_a(p, \tau_a) = p(1 + \tau_{a2})^{-\sigma}.$$  

Given that $p$ is a function of the tariff vector $\tau$, we can see that for the quasi-linear functional form exports are a function of all tariffs, $e_a(p(\tau_a, \tau_{-a}, \tau_b), \tau_a)$, not just $\tau_a$ as
under Cobb-Douglas. Nevertheless, we shall now see that it is possible to solve for tariffs under quasi-linear preferences in the same manner as for Cobb-Douglas.

We now introduce the fact that \( p \) is a function of tariffs; \( p = p(\tau) \). As for the Cobb-Douglas example, the indirect utility for the representative household \( a \in A \) is derived by substituting (7.3) and (7.2) back into (7.1) to get

\[
\begin{align*}
  u^A(p(\tau), \tau_{a2}) &= \frac{1}{\sigma - 1} \left[ p(\tau) (1 + \tau_{a2}) \right]^{1 - \sigma} + p(\tau) \tau_{a2} \left[ p(\tau) (1 + \tau_i^a) \right]^{-\sigma} \\
  \text{(7.4)}
\end{align*}
\]

We now need to calculate how \( p \) changes with the tariff vector \( \tau = (\tau_1, ..., \tau_m, \tau_{m+1}, ..., \tau_{n+m}) \). We now derive an expression for world prices \( p \) strictly in terms of tariffs. We have

\[
\begin{align*}
  p(\tau_1, ..., \tau_m, \tau_{m+1}, ..., \tau_{n+m}) &= \frac{E_B(\tau_b)}{E_A(\tau_a)} \\
  &= \frac{p^{1-\sigma} \sum_{b \in B} (1 + \tau_b)^{-\sigma}}{1/p^{1-\sigma} \sum_{a \in A} (1 + \tau_a)^{-\sigma}}
\end{align*}
\]

Solving for \( p \) strictly in terms of \( \tau \):

\[
p(\tau) = \left( \frac{\sum_{a \in A} (1 + \tau_a)^{-\sigma}}{\sum_{b \in B} (1 + \tau_b)^{-\sigma}} \right)^{\frac{1}{2\sigma - 1}}.
\]

Note that \( p_{\tau_a} > 0 \) i.e. an increase in the tariff \( \tau_a \) always improves \( a \)'s terms of trade. Similarly, \( p_{\tau_b} < 0 \) i.e. an increase in the tariff \( \tau_b \) for \( b \in B \) always improves \( b \)'s terms of trade.

Since we do not necessarily know that Propositions 1-3 necessarily hold for (7.1), let us solve for the optimal tariff \( \hat{\tau}_a \) of country \( a \in A \) in the conventional way.\(^{13}\) From (7.4) we have:

\[
\begin{align*}
  \frac{\partial u^A(\tau_a, p)}{\partial \tau_a} &= -p^2 \tau_a (p (1 + \tau_a))^{-1 - \sigma} \\
  \frac{\partial u^A(\tau_a, p)}{\partial p} &= - (p (1 + \tau_a))^{-\sigma} (1 + \tau_a \sigma) \\
  \frac{\partial p(\tau)}{\partial \tau_a^2} &= - \frac{\sigma (1 + \tau_a^2)^{-\sigma - 1}}{(2\sigma - 1) \sum_{a \in A} (1 + \tau_a^2)^{-\sigma}} \cdot p(\tau)
\end{align*}
\]

Then, solving

\[
\begin{align*}
  \frac{du^A(\tau_a, p)}{d\tau_a} = \frac{\partial u^A(\tau_a, p)}{\partial \tau_a} + \frac{\partial u^A(\tau_a, p)}{\partial p} \frac{\partial p(\tau)}{\partial \tau_a^2} = 0
\end{align*}
\]

\(^{13}\)Note that this conventional method of solving for optimal tariffs can be used to obtain the same solution as presented in Proposition 3 for the tariff game under Cobb-Douglas.
in terms of \( \tau_a \), and setting \( \hat{\tau}_a = \hat{\tau}_{-a} \), we obtain the following solution:

\[
\hat{\tau}_a = \frac{1}{(\sigma - 1) + (a - 1)(2\sigma - 1)}.
\]

An analogous procedure obtains the following solution for \( \hat{\tau}_b \):

\[
\hat{\tau}_b = \frac{1}{(\sigma - 1) + (b - 1)(2\sigma - 1)}.
\]

As \( a \) is increased \( \hat{\tau}_a \) falls. Under quasi-linear preferences, unlike for Cobb-Douglas, when \( a = 1 \), \( \hat{\tau}_a \) is not prohibitively high and there is trade in equilibrium. As for Cobb-Douglas, as \( a \) becomes large, \( \hat{\tau}_a \) converges towards the free trade level \( \hat{\tau}_a = 0 \). The tariff \( \hat{\tau}_b \) behaves in the same way with respect to \( b \). Notice that, as for Cobb-Douglas, \( \hat{\tau}_a \) depends only on \( a \) and \( \hat{\tau}_b \) depends only on \( b \). It can be shown that the properties established for Cobb-Douglas in Propositions 4 and 5 hold for equilibrium under quasi-linear preferences (7.1) as well.

8. Conclusions

To be added.

A. Appendix

A.1. Proof of Propositions

**Proof of Lemma 1.** Through the change of variable, \( E_A(\tau_a) = e_a(\tau_a) + E_{A-a}(\tau_{-a}) \), we may view the objective of country \( a \) as being to choose \( \tau_a \) in order to set \( E_A \in [E_{A-a}(\tau_{-a}), E_{A-a}(\tau_{-a}) + 1] \) instead of \( e_a \in [0,1] \). The corresponding payoff is thus given by

\[
\max \{ \tilde{u}^A(\tau_a, \tau_{-a}) \}
= \tilde{u}^A \left( 1 - E_A(\tau_a) + E_{A-a}(\tau_{-a}), \left( 1 - \frac{E_A(\tau_a)}{E_{A-a}(\tau_{-a})} \right) E_B \right),
\]

\( \tau_a \in \mathbb{R}_+ \).

Solving for the value of \( \tau_a \) that maximizes \( \tilde{u}^A \), denote the optimal response by \( \hat{\tau}_a(\tau_{-a}) \) and the corresponding level of total exports by \( \hat{E}_A(\hat{\tau}_a(\tau_{-a})) \). Since

\[
\hat{E}_A(\hat{\tau}_a(\tau_{-a})) = e_a(\hat{\tau}_a(\tau_{-a})) + E_{A-a}(\tau_{-a})
\]
we have that
\[ \frac{e'_a (\hat{\tau}_a (\tau_{-a})) - e_a (\hat{\tau}_a (\tau_{-a}))}{E_{A-a} (\tau_{-a}) - E_{A-a} (\tau_{-a})} > -1 \]
if and only if \( \hat{\tau}_a (\tau_{-a}) \) is strictly increasing. To see why, first note that \( \hat{E}_A (\hat{\tau}_a (\tau_{-a})) \) is only increasing in \( E_{A-a} (\tau_{-a}) \) if \( e_a (\hat{\tau}_a (\tau_{-a})) \) is increasing in \( E_{A-a} (\tau_{-a}) \). Let \( \partial \tau_{-a} \) denote a change of a single element of the vector \( \tau_{-a} \) and let \( \partial E_{A-a} (\tau_{-a}) / \partial \tau_{-a} \) denote the change in \( E_{A-a} (\tau_{-a}) \) that results from a change in a single element of the vector \( \tau_{-a} \). Then observe that, by (5.5), both \( \partial e_a (\tau_a) / \partial \tau_a < 0 \) and \( \partial E_{A-a} (\tau_{-a}) / \partial \tau_{-a} < 0 \). Therefore, if \( \hat{\tau}_a (\tau_{-a}) \) is strictly increasing then a reduction in \( \tau_{-a} \) will bring about an increase in \( E_{A-a} (\tau_{-a}) \) and a reduction in \( \hat{\tau}_a \), which in turn will bring about an increase in \( e_a (\hat{\tau}_a (\tau_{-a})) \).

To establish that \( \hat{\tau}_a (\tau_{-a}) \) is strictly increasing, we begin by obtaining the first order condition for \( \tilde{\tau}^A \):
\[
\frac{\partial \tilde{\tau}^A (E_A (\tau_a))}{\partial \tau_a} = \frac{\partial e_a (\tau_a)}{\partial \tau_a} \left( \frac{E_{A-a} (\tau_{-a}) \frac{E_B}{E_A (\tau_a)^2} u_{x2} - u_{x1}}{E_A (\tau_a)} \right) = 0
\]
Then
\[
\frac{\partial^2 \tilde{\tau}^A (E_A (\tau_a))}{\partial \tau_a \partial \tau_{-a}} = \frac{\partial e_a (\tau_a)}{\partial \tau_a} \frac{\partial E_{A-a} (\tau_{-a})}{\partial \tau_{-a}} \times \left( - \frac{E_A (\tau_a) u_{x1}^A}{E_B} u_{xx}^A + u_{xy}^A + \frac{E_{A-a} (\tau_{-a}) u_{xy}^A + u_{y1}^A}{E_A (\tau_a)} - \frac{E_{A-a} (\tau_{-a}) E_B u_{xy}^A}{[E_A (\tau_a)]^2} \right)
\]
Evaluating along the first order condition, this reduces to
\[
\left[ \frac{\partial \tilde{\tau}^A (E_A (\tau_a))}{\partial \tau_a \partial \tau_{-a}} \right]_{\partial \tilde{\tau}^A (E_A (\tau_a))/\partial \tau_a = 0} = \frac{\partial e_a (\tau_a)}{\partial \tau_a} \frac{\partial E_{A-a} (\tau_{-a})}{\partial \tau_{-a}} \times \left[ \frac{E_{A-a} (\tau_{-a})}{E_A (\tau_a)} \right] \left[ u_x^A (u_x^A u_{xy} - u_y^A u_{xx}) \right]
\]
By (5.5) the first two terms on the right hand side are negative, and so their product is positive. As pointed out above, for the Cobb-Douglas function, (5.1),
\[
(u_x^A u_{xy} - u_y^A u_{xx}) > 0,
\]
\[
(u_y^A u_{xx} - u_x^A u_{yy}) > 0,
\]

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and \( u^A_y > 0 \). Therefore the right hand side is positive. □

**Proof of Proposition 1.** Existence of a pure-strategy Nash equilibrium of the game \( \Gamma (\tau_b) \) follows from standard arguments. The domain, \( \mathbb{R}_+ \), of \( u^A (\tau_a; \tau_{-a}, \tau_b) \), is a compact convex set, and \( u^A (\tau_a; \tau_{-a}, \tau_b) \) is a continuous function from \( \mathbb{R}_+ \) into itself. Hence, the reaction functions are continuous single-valued functions and a pure-strategy Nash equilibrium exists by Brower’s fixed point theorem.

Suppose, contrary to the proposition, that the game \( \Gamma (\tau_b) \) admits an asymmetric equilibrium; without loss of generality, say that countries 1 and 2 set equilibrium tariffs \( \hat{\tau}_1 \) and \( \hat{\tau}_2 \) respectively, where \( \hat{\tau}_1 \neq \hat{\tau}_2 \), giving rise to equilibrium exports \( e_1 (\hat{\tau}_1) \neq e_2 (\hat{\tau}_2) \) and total exports \( E_A \). Then clearly, \( E_A - e_1 (\hat{\tau}_1) \neq E_A - e_2 (\hat{\tau}_2) \). However,

\[
\hat{E}_A (\hat{\tau}_2 (\tau_{-2})) = e_a (\hat{\tau}_1 (\tau_{-1})) + E_{A-a} (\tau_{-1}) = e_a (\hat{\tau}_2 (\tau_{-2})) + E_{A-a} (\tau_{-2}) = \hat{E}_A (\hat{\tau}_2 (\tau_{-2})) = E_A.
\]

This is a contradiction to Lemma 1. □

**Lemma 2.** (Lemma 3, Amir and Bloch 2004) Let \( b \geq a \) and \( f : [0, a] \rightarrow [0, a] \) and \( g : [0, b] \rightarrow [0, b] \) be two continuous functions. Let \( \overline{x}_f \) and \( \underline{x}_f \) be the largest and smallest fixed points of \( f \), and let \( \overline{x}_g \) and \( \underline{x}_g \) be the largest and smallest fixed points of \( g \). If \( f (x) \leq g (x) \) for all \( x \in [0, a] \), then \( \overline{x}_g \geq \overline{x}_f \) and \( \underline{x}_g \geq \underline{x}_f \).

**Proof of Lemma 2.** We show that \( \overline{x}_g \geq \overline{x}_f \) (the proof of \( \underline{x}_g \geq \underline{x}_f \) is similar and is thus omitted). Since \( f \leq g \), we have \( g (\overline{x}_f) \geq f (\overline{x}_f) = \overline{x}_f \). Consider the function \( G (x) = g (x) - x \) on the restricted domain \( [\overline{x}_f, b] \) and \( g (b) \leq b \), we have \( G (\overline{x}_f) = g (\overline{x}_f) - \overline{x}_f \geq 0 \) and \( G (b) = g (b) - b \leq 0 \). By the intermediate value theorem applied to the continuous function \( G \) on \( [\overline{x}_f, b] \), there is some \( \tilde{x} \in [\overline{x}_f, b] \) such that \( G (\tilde{x}) = 0 \). This is equivalent to \( g (\tilde{x}) = \tilde{x} \). Since \( \tilde{x} \geq \overline{x}_f \) and \( \overline{x}_g \) is postulated to be the largest fixed point of \( g \) we have, a fortiori, \( \overline{x}_g \geq \tilde{x} \geq \overline{x}_f \). □

**Proof of Proposition 2.** Since the pure strategy Nash equilibria of the game \( \Gamma (\tau_b) \) are symmetric, in an equilibrium every country \( a \in A \) sets the same equilibrium tariff \( \hat{\tau}_a \) and, by (5.5), every country has the same level of exports \( e_a (\hat{\tau}_a) \). Then at a symmetric equilibrium it must be the case that \( e_a (\hat{\tau}_a, E_{A-a} (\hat{\tau}_{-a})) = E_{A-a} (\hat{\tau}_{-a}) / (m - 1) \). Indeed,
this is a necessary condition; only at this point is the responding country’s exports equal to every other country’s exports. And by (5.5), for \( e_a (\hat{\tau}_a, E_{A-a} (\hat{\tau}_-a)) \) to be equal across all countries \( a \in A \) it must be the case that the equilibrium tariff \( \hat{\tau}_a \) must be the same for all \( a \in A \).

In view of the symmetry of every Nash equilibrium (Lemma 1) and the differentiability of (6.2) (by (5.5) and the implicit function theorem), to show the uniqueness of Nash equilibrium it is sufficient to show that at every Nash equilibrium

\[
\frac{\partial e_a (\hat{\tau}_a, E_{A-a} (\hat{\tau}_-a))}{\partial E_{A-a} (\hat{\tau}_-a)} < \frac{1}{m-1}.
\]

Since \( e_a (\hat{\tau}_a, E_{A-a} (\hat{\tau}_-a)) \) is continuous, for it to be the case that \( e_a (\hat{\tau}_a, E_{A-a} (\hat{\tau}_-a)) = E_{A-a} (\hat{\tau}_-a) / (m-1) \) at more than one symmetric equilibrium, it must be true that

\[
\frac{\partial e_a (\hat{\tau}_a, E_{A-a} (\hat{\tau}_-a))}{\partial E_{A-a} (\hat{\tau}_-a)} > 1/(m-1) \text{ at one or more equilibrium. For brevity, write } \hat{e}_a \text{ instead of } e_a (\hat{\tau}_a, E_{A-a} (\hat{\tau}_-a)).
\]

But by the implicit function theorem, we have

\[
\frac{\partial e_a (\hat{\tau}_a, E_{A-a} (\tau_-a))}{\partial E_{A-a} (\tau_-a)} = \frac{E_b \left( 2E_a^2E_{A-a}u_y - [E_A]^2 (\hat{e}_a u_{xy} + u_y) + \hat{e}_a E_{A-a} E_b u_{yy} \right)}{[E_A]^4 u_{xx} - 2 [E_A]^2 E_b E_{A-a} u_{xy} - 2E_A E_b E_{A-a} u_y + [E_b]^2 [E_{A-a}]^2 u_{yy}} + \frac{\partial \hat{e}_a}{\partial \hat{\tau}_a} \left( [E_A]^4 u_{xx} - 2 [E_A]^2 E_b E_{A-a} u_{xy} - 2E_A E_b E_{A-a} u_y + [E_b]^2 [E_{A-a}]^2 u_{yy} \right).\]

The first order condition implies that \( E_B = [E_A]^2 u_x / (E_{A-a} u_y) \), and at a symmetric equilibrium \( E_{A-a} = (m-1) \hat{e}_a \) and \( E_A = m\hat{e}_a \). Using these facts, we see that

\[
\frac{\partial e_a (\hat{\tau}_a, E_{A-a} (\tau_-a))}{\partial E_{A-a}} < 1/(m-1) \text{ if and only if } -u_{xx} + \frac{u_x}{u_y} u_{xy} + \frac{1}{\hat{e}_a} u_x > 0,
\]

which is implied by \( (u_x^A u_{xy} - u_y^A u_{xx}) > 0 \). Hence there is a unique (and symmetric) Nash equilibrium for every \( m \).

To prove that the unique equilibrium tariff \( \hat{\tau}_a \) is decreasing in \( m \), and that total exports \( E_A \) are increasing in \( m \), consider the mapping \( ER_m : \mathbb{R}_+ \to \mathbb{R}_+ \) defined by

\[
ER_m (\tau_-a) = \frac{m-1}{m} [e_a (\hat{\tau}_a, E_{A-a} (\tau_-a)) + E_{A-a} (\tau_-a)].
\]

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It is easy to verify that $ER_m(\tau-a)$ maps $\mathbb{R}_+$ into itself and that $ER_m$ is increasing in $m$ for each $\tau-a$ and decreasing in $\tau-a$ for each $m$ (by Lemma 2 and its proof). It can be shown that the fixed points of $ER_m(\tau-a)$ are the Nash equilibria of the game $\Gamma(\tau-b)$ and vice versa. Hence, by the foregoing $ER_m(\tau-a)$ has a unique fixed point. Moreover, by Lemma 3, this fixed point increases with $m$. Thus, $E_{\lambda-a}(\hat{\tau}_a)$ increases with $m$. For this to happen, by (5.5), each element of $\hat{\tau}_a$ must decrease with $m$ (where all elements are identical in equilibrium, $\hat{\tau}_a$). Since $E_{\lambda-a}(\hat{\tau}_a)$ increases with $m$, for each $m$ it follows by Lemma 1 that $c_a(\hat{\tau}_a, E_{\lambda-a}(\hat{\tau}_a))$ is increasing with $m$, and so by definition $E_{\lambda}$ in equilibrium must increase with $m$. $\square$

**Proof of Proposition 3.** Given Proposition 2, we can define, for any $\tau_b$, the single valued mapping $\hat{\tau}_a(\tau_b)$ that assigns to each tariff vector $\tau_b$ the equilibrium tariff $\hat{\tau}_a$ of countries in $A$ in the game $\Gamma(\tau_b)$. Also, write $\hat{\tau}_a(\tau_b)$ for the equilibrium tariff of the game $\Gamma(\tau_b)$ in which all tariffs in the vector $\tau_b$ are equal at $\tau_b$. Given that $u^A$ is strictly concave in $\tau_a$, and jointly continuous in all tariffs, the mapping $\hat{\tau}_a(\tau_b)$ is a continuous function. Similarly, we may define $\hat{\tau}_b(\tau_a)$ as the equilibrium tariffs of countries in $B$ when the countries in $A$ set the tariffs listed in the vector $\tau_a$. Also, write $\hat{\tau}_b(\tau_a)$ for the equilibrium tariff in the case where all tariffs in $\tau_a$ are equal at $\tau_a$. Now, consider the mapping $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ where $\tau = \hat{\tau}_a \circ \hat{\tau}_b$. As $\hat{\tau}_b$ is independent of $m$ and $\hat{\tau}_a$ is non-increasing in $m$, $\tau$ is also non-increasing in $m$. Given that $\tau$ is continuous and that its domain is of the form $\mathbb{R}_+$, we can invoke Lemma 2 to conclude that the extremal fixed points of $\tau$ are non-increasing in $m$.

As a fixed point $\tau_0$ of $\tau$ satisfies $\hat{\tau}_a(\tau_0) = \hat{\tau}_a(\tau_0)$, it is clear that the pair $\{\hat{\tau}_a, \hat{\tau}_b(\tau_0)\}$ are equilibrium tariffs. Conversely, every Nash equilibrium of the game is a fixed point of $\tau$. Thus, the maximal fixed point of $\tau$ must induce an equilibrium outcome in which there is autarky. By standard arguments, there exists at least one fixed point of $\tau$ between the maximal fixed point and the zero tariff vector (free trade). The minimal fixed point of $\tau$, call it $\underline{\tau}$, is interior and decreasing in $m$ by the argument stated above.

The remainder of the proof is presented in the body of the paper. $\square$
Proof of Proposition 4. Differentiation of $u^A(\hat{\tau}_a(m), \hat{\tau}_b(n))$ with respect to $m$ yields

$$\frac{du^A(\hat{\tau}_a(m), \hat{\tau}_b(n))}{dm} = - \frac{(1 + m - 2\alpha) \alpha (1 - \alpha) \left(\frac{n(n-1)\alpha}{m(n-\alpha)}\right)^{\alpha - 1} \left(\frac{1-\alpha m}{m-\alpha}\right) - \alpha}{(m - \alpha)^2} < 0$$

By inspection, $du^A(\hat{\tau}_a(m), \hat{\tau}_b(n))/dm$ is decreasing in $m$ if and only if $n(n - 1)\alpha/(m(n - \alpha))$ is increasing in $n$, which holds for all feasible values of $\alpha$, $m$ and $n$.

Differentiation of $u^A(\hat{\tau}_a(m), \hat{\tau}_b(n))$ with respect to $n$ yields

$$\frac{du^A(\hat{\tau}_a(m), \hat{\tau}_b(n))}{dn} = \frac{(1 - \alpha) \alpha^2 \left(\frac{n(n-1)\alpha}{m(n-\alpha)}\right)^{\alpha - 1} \left(\frac{1-\alpha m}{m-\alpha}\right) - \alpha}{(m - \alpha)(n - \alpha)^2} (n^2 - 2n\alpha + \alpha) > 0.$$

By inspection or by taking the second derivative, $du^A(\hat{\tau}_a(m), \hat{\tau}_b(n))/dn$ is decreasing in $n$. □

Proof of Proposition 5. Efficiency implications of replication. In a trading equilibrium, by symmetry of the equilibrium world welfare is given by

$$mu^A(m, rm) + rmu^B(rm, m) =$$

$$m \left(\frac{1 - \alpha}{m - \alpha}\right)^{\alpha} \left(\frac{m(1-\alpha)}{m-\alpha}\right)^{-\alpha} + rm \left(\frac{r(m-1)\alpha}{rm-\alpha}\right)^{-\alpha}$$

We will show that the above expression is globally increasing in $m$. Differentiating, we get

$$\frac{d}{dm} \left(mu^A(m, rm) + rmu^B(rm, m)\right) =$$

$$\frac{1}{(m - \alpha)^2 (rm - \alpha)^2} \times$$

$$\left(\frac{m(1 - \alpha)}{r(\frac{rm - 1}{rm - \alpha})} \right)^{\alpha} \left(\frac{m(1-\alpha)}{m-\alpha}\right)^{-\alpha} \Theta(m; r, \alpha)$$

$$+ r \left(\frac{(rm - 1)\alpha}{rm - \alpha}\right)^{-\alpha} \left(\frac{(m - 1)\alpha}{r(m - \alpha)}\right)^{\alpha - 1} \Phi(m; r, \alpha)$$

where

$$\Theta(m; r, \alpha) = \left(r^2 m^3 + m\alpha (1 + 2r) - (2 - \alpha) \alpha^2 - rm^2 \left(1 + r (2 - \alpha) + \alpha^2\right)\right)$$

$$\Phi(m; r, \alpha) = \left(\frac{r^2 m^3}{m - \alpha} + m\alpha \left(1 + 2r\right) - (2 - \alpha) \alpha^2 - rm^2 \left(1 + r (2 - \alpha) + \alpha^2\right)\right)$$
and
\[
\Phi(m; r, \alpha) = (rm^3 + m\alpha (1 + 2r) - (2 - \alpha)\alpha^2 - m^2 (r + 2\alpha + (r - 1)\alpha^2))
\]
The result is established by verifying that \(\Theta(m; r, \alpha) > 0\) and \(\Phi(m; r, \alpha) > 0\) for all feasible \(m, r\) and \(\alpha\).

We will show the existence of a value \(r' > 1\) for which \(du_A (rm_k) / dm \geq 0\) for \(r \leq r'\). To do so, first observe that
\[
\frac{du_A (rm_k)}{dm} = \frac{du_A (m_k, rm_k)}{dm} + r \frac{du_A (m_k, rm_k)}{dn}
\]
\[
= r (1 - \alpha)^2 \alpha^2 (m (1 - m (1 - r))r - \alpha) \left( \frac{r(r-1)}{rm-\alpha} \right)^{\alpha-1} \left( \frac{m(1-\alpha)}{m-\alpha} \right)^{-\alpha}
\]
\[
= \frac{r (1 - \alpha)^2 \alpha^2 (m^2 + \alpha - 2m\alpha) \left( \frac{m(1-\alpha)}{m-\alpha} \right) \left( \frac{(m-1)\alpha}{m-\alpha} \right)^{\alpha-1}}{(m-\alpha)^3}
\]
We can see by inspection that \(du_A (rm_k) / dm\) is monotonically decreasing in \(r\). Now if we fix \(r = 1\) we find that
\[
\frac{du_A (rm_k)}{dm} = \frac{(1 - \alpha)^2 \alpha^2 (m^2 + \alpha - 2m\alpha) \left( \frac{m(1-\alpha)}{m-\alpha} \right) \left( \frac{(m-1)\alpha}{m-\alpha} \right)^{\alpha-1}}{(m-\alpha)^3} > 0
\]
So there must exist a value \(r' > 1\) for which a doubling of \(m\) has no effect on \(du_A (m, rm)\).
\(\Box\)

References


