Commutator Formula for Subnormal Tuple of Operators

Daoxing Xia

Abstract. Let $S = (S_1, \ldots, S_k)$ be a pure subnormal $k$-tuple of operators with minimal normal extension $N$ and defect space $\mathcal{M}$. Let $\Lambda_j = (S_j^*)^*$. We prove

$$[f(S^*)h(S), S_j]|_{\mathcal{M}} = \frac{1}{2\pi i} \int_{\mathcal{N}(N)} f(\bar{u})h(u)(u_j - \Lambda_j)e(du),$$

where $e(\cdot) = P_\mathcal{M}E(\cdot)|_{\mathcal{M}}$, $E(\cdot)$ is the spectral measure of $N$, $P_\mathcal{M}$ is the projection to $\mathcal{M}$, $f$ is any analytic function on $\times_{j=1}^k \sigma(S_j^*)$ and $h$ is any analytic function on $\times_{j=1}^k \sigma(S_j)$. If $\dim \mathcal{M} < \infty$, then this commutator equals to

$$\frac{1}{2\pi i} \int_A \mu_j(u_j)df(\bar{u}) \wedge dh(u),$$

where $A = \{\bar{u} : u = (u_1, \ldots, u_k)\}$ is in the joint spectrum of $S^*$, and $\mu_j(\cdot)$ is the mosaic of $S_j$.

Besides, some similar commutator formulas for a pure hyponormal operator associated with a quadrature domain are established.


Keywords. Subnormal operators, Commutator, Mosaic, Trace formula, Joint point spectrum, Hyponormal Operator, Quadrature domain, Schwarz function.

1. Introduction

The commutator formula is an important role in the study of operator theory. One example is the trace formula for the commutator of the semi-normal operators as in [4],[5],[13],[15]. In the present note, we focus on the commutator formula for subnormal $k$-tuple of operators, as well as hyponormal operators associated with quadrature domains.

Let $S$ be a pure subnormal operator on a Hilbert space $\mathcal{H}$ with minimal normal extension (m.n.e.) $N$ on $\mathcal{K} \supset \mathcal{H}$ (cf. [6]). Let $\mathcal{M} = \text{closure of } [S^*, S]\mathcal{H}$ be the defect space of $S$. In [24],[25], the author introduced the mosaic $\mu(z)$, $z \in \rho(N)$ for $S$:

$$\mu(z) \overset{\text{def}}{=} P_\mathcal{M}(N - SP_\mathcal{M})(N - z)^{-1}|_\mathcal{M}, z \in \rho(N),$$

where $P_\mathcal{M}$ is the projection from $\mathcal{K}$ to $\mathcal{M}$. This mosaic is an idempotent $L(\mathcal{M})$-valued analytic function on $\rho(N)$, i.e. $\mu(z)^2 = \mu(z)$, $z \in \rho(N)$. Besides, $\mu(z) \neq 0$ iff $z \in \sigma(S) \cap \rho(N)$. In [10], J.Gleason and C.R.Rosentrater established the following interesting commutator formula for pure subnormal operator $S$:

$$[S^m S^n, S]|_\mathcal{M} = \frac{1}{\pi} \int_C \frac{d^{m-1}z^m}{z^{n-1}} \mu(z) dA(z), \quad (1.1)$$

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where \( l \neq 0 \) and \( m \) are nonnegative integers, and \( A(\cdot) \) is the area measure. It is obvious that the integral in (1.1) makes sense only if the planar measure of \( \sigma(N) \) is zero and

\[
\int_C \|\mu(z)\| dA(z) < +\infty
\]

These two conditions are also sufficient.

If \( \dim \mathcal{M} < \infty \), it is easy to prove that these two conditions are satisfied. In [31], the author gives another proof of (1.1) in this special case.

In [10], it shows that through the well-known technique of the collapsing property of a trace bilinear form (cf. for example [3],[14],[23]) it shows that the commutator formula (1.1) implies the well-known trace formula (cf. [4],[16],[23],[24],[31])

\[
\text{tr}[f(S^*, S), h(S^*, S)] = \frac{1}{2\pi i} \int_{\sigma(S)} g(z) df(z, z) \wedge dh(z, z),
\] (1.2)

for the pure subnormal commutator \( S \) with trace self-commutator \( [S^*, S] \), where \( g(z) = tr\mu(z) \) is the Pincus principal function, \( f \) and \( h \) are analytic functions on \( \sigma(S^*) \times \sigma(S) \). Therefore the author feels that the commutator formula (1.1) may be considered as an “operator-lifting” of the trace formula (1.2).

The first aim of the present note is to study the commutator formula for the pure subnormal \( k \)-tuple of operators \( \mathcal{S} = (S_1, \ldots, S_k) \) on a Hilbert space \( \mathcal{H} \) with m.n.e. \( N = (N_1, \ldots, N_k) \) on \( \mathcal{K} \supset \mathcal{H} \) (cf.[7],[8],[26]). Let \( \mathcal{M} \overset{\text{def}}{=} \text{closure of } \vee_{j=1}^k [S_j^*, S_j] \mathcal{H} \) be the defect space of \( \mathcal{S} \) and \( P_{\mathcal{M}} \) be the projection to \( \mathcal{M} \). Let \( \Lambda_j \overset{\text{def}}{=} (S_j^*)^* \cdot \Lambda_j \cdot (S_j)^* \cdot \Lambda_j \). Let \( E(\cdot) \) be the spectral measure of \( N \) and \( \epsilon(\cdot) \) be the measure \( P_{\mathcal{M}} E(\cdot) |_{\mathcal{M}} \).

In section 2, the author proves that

\[
[f(S^*)h(S), S_j]|_{\mathcal{M}} = \frac{1}{2\pi i} \int_{sp(N)} f(\bar{u})h(u)(u_j - \Lambda_j)e(du), \quad j = 1, \ldots, k,
\] (1.3)

where \( f \) and \( h \) are analytic functions on \( \times_{j=1}^k \sigma(S_j^*) \) and \( \times_{j=1}^k \sigma(S_j) \) respectively. If \( \dim \mathcal{M} < \infty \), then in section 4, we prove that

\[
[f(S^*)h(S), S_j]|_{\mathcal{M}} = \frac{1}{2\pi i} \int_{sp_{jp}(S^*)^*} \mu_j(u)h(u)(\bar{u})d\mu(u_j), \quad j = 1, \ldots, k,
\] (1.4)

where \( \mu_j(\cdot) \) is the mosaic of \( S_j \), and

\[
sp_{jp}(S^*)^* \overset{\text{def}}{=} \{ \bar{u} : u = (u_1, \ldots, u_k) \text{ is a joint eigenvalue of } S^* \}. \]

In order to prove (1.4), in section 4 it studies the structure of \( sp_{jp}(S^*)^* \) and its relation with \( \mu_j(\cdot), \quad j = 1, \ldots, k. \)

Besides, in the section 5 of this note, the author studies the commutator formula for the pure hyponormal operator \( H \) on \( \mathcal{H} \) associated with a quadrature domain (cf. [2]) i.e. its Pincus principal function \( g(z) \equiv 1 \) on \( \sigma(H) \) and \( \sigma(H) \) is the closure of a quadrature domain (cf.[20],[21],[22],[28],[29],[31]). The author proves that

\[
[f(H^*)h(H), H]|_{K} = \frac{1}{2\pi i} \int_{\partial(\sigma(H))} f(\bar{u})h(u)\mu(u)du
\]

\[
= \frac{1}{2\pi i} \int_{\sigma(H)} \mu(u)h(u)(\bar{u})d\mu(u),
\]

where \( f \) and \( h \) are analytic functions on \( \sigma(H^*) \) and \( \sigma(H) \) respectively, but \( h(\cdot) \) must have certain zeros in \( \sigma(H) \), and \( K \overset{\text{def}}{=} \vee_{j=0}^{\infty} \{ H^j[H^*, H] \mathcal{H} \} \) which actually is of finite dimension.
2. Commutator Formula for Subnormal Tuple of Operators

Let $S = (S_1, \ldots, S_k)$ be a pure subnormal $k$-tuple operator on a Hilbert space $\mathcal{H}$ with m.n.e. $N$ on a Hilbert space $K$ containing $\mathcal{H}$ as a subspace (cf. [6], [8], [26], [31]). Let $\mathcal{M} \overset{\text{def}}{=} \text{closure of } \bigvee_{j=1}^{k} [S_j^*, S_j] \mathcal{H}$. Let $E(\cdot)$ be the spectral measure of $N$. Define $e(\cdot) = P_M E(\cdot)|_M$ as an $L(M)$-valued measure on the $sp(N)$, where $P_M$ is the projection from $K$ to $\mathcal{M}$. Define operators

$$C_m = [S_1^*, S_m]|_M$$

and $A_j = (S_j^*|_M)^*$ on $\mathcal{M}$. By some formulas in [26], [31] we may prove the following:

**Theorem 2.1.** Let $S = (S_1, \ldots, S_k)$ be a pure subnormal $k$-tuple of operators on $\mathcal{H}$ with m.n.e. $N$ on $K \supset \mathcal{H}$. Then for every pair of analytic functions $f$ and $h$ on $\times_{j=1}^{k} \sigma(S_j^*)$ and $\times_{j=1}^{k} \sigma(S_j)$ respectively,

$$[f(S^*) h(S), S_l] = \int_{sp(N)} f(\bar{u}) h(u) (u_l - \Lambda_l) e(du),$$

(2.1)

where $l = 1, \ldots, k$ and $u = (u_1, \ldots, u_k) \in sp(N)$.

**Proof.** Without loss of generality, we may assume that

$$f = \prod_{j=1}^{k} (\bar{\lambda}_j - u_j)^{-1}$$

and

$$h = \prod_{j=1}^{k} (\xi_j - u_j)^{-1}.$$

where $\lambda_j$ and $\xi_j \in \rho(S_j), j = 1, \ldots, k$. Then $f(S^*) = \prod_{j=1}^{k} (\bar{\lambda}_j - S_j^*)^{-1}$ and $h(S) = \prod_{j=1}^{k} (\xi_j - S_j)^{-1}$. It is obvious that

$$[f(S^*) h(S), S_l] = \prod_{j=1}^{k} (\bar{\lambda}_j - S_j^*)^{-1}, S_l] h(S), l = 1, \ldots, k,$$

(2.2)

since $[h(S), S_l] = 0$.

It is not difficult to calculate that

$$\prod_{j=1}^{k} (\bar{\lambda}_j - S_j^*)^{-1}, S_l] h(S)|_M$$

$$= \sum_{j=1}^{k} \prod_{m=1}^{j-1} (\bar{\lambda}_m - S_m^*)^{-1} [(\bar{\lambda}_j - S_j^*), S_l] \prod_{m=j+1}^{k} (\bar{\lambda}_m - S_m^*)^{-1} h(S)|_M,$$

where $\prod_{m=1}^{0} (\bar{\lambda}_m - S_m^*)^{-1} = \prod_{m=k+1}^{k} (\bar{\lambda}_m - S_m^*)^{-1} = I$. From [26] and [31] we have

$$[(\bar{\lambda}_j - S_j^*)^{-1}, S_l] = (\bar{\lambda}_j - S_j^*)^{-1} C_{jl} P_M (\bar{\lambda}_j - S_j^*)^{-1},$$


$$\prod_{m=1}^{j} (\bar{\lambda}_j - S_m^*)^{-1}|_M = \prod_{m=1}^{j} (\bar{\lambda}_j - \Lambda_m^*)^{-1}$$

and

$$P_M \prod_{m=j}^{k} (\bar{\lambda}_j - S_m^*)^{-1} h(S)|_M = \int_{sp(N)} \frac{e(du) h(u)}{\prod_{m=j}^{k} (\bar{\lambda}_m - \bar{u}_m)},$$

where $u = (u_1, \ldots, u_k)$. Thus

$$[\prod_{j=1}^{k} (\bar{\lambda}_j - S_j^*)^{-1}, S_l] h(S)|_M = \sum_{j=1}^{k} \prod_{m=1}^{j} (\bar{\lambda}_m - \Lambda_m^*)^{-1} \int \frac{C_{jl} e(du) h(u)}{\prod_{m=j}^{k} (\bar{\lambda}_m - \bar{u}_m)}.$$  

(2.3)
From \( C_{j\ell}h(du) = (\bar{u}_j - \Lambda^*_j)(u_\ell - \Lambda_\ell)h(du) = (\bar{\lambda}_j - \Lambda^*_j - (\bar{\lambda}_j - \bar{u}_j))(u_\ell - \Lambda_\ell)h(du), \) we have
\[
\int \frac{C_{j\ell}h(du)}{\prod_{m=1}^{j}(\bar{\lambda}_m - \bar{u}_m)} = (\bar{\lambda}_j - \Lambda^*_j) \int \frac{(u_\ell - \Lambda_\ell)h(du)}{\prod_{m=1}^{j}(\bar{\lambda}_m - \bar{u}_m)} - \int \frac{(u_\ell - \Lambda_\ell)h(du)}{\prod_{m=j+1}^{k}(\bar{\lambda}_m - \bar{u}_m)}, \tag{2.4}
\]
\[j = 1, \ldots, k, \text{ where } \prod_{m=k+1}^{k}(\bar{\lambda}_m - \bar{u}_m) \text{ means } 1. \] From (2.4), the right-hand side of (2.3) equals to
\[
\sum_{j=1}^{k} \left( \prod_{m=1}^{j}(\bar{\lambda}_m - \Lambda^*_m)^{-1} \right) \int \frac{(u_\ell - \Lambda_\ell)h(du)}{\prod_{m=1}^{j}(\bar{\lambda}_m - \bar{u}_m)} - \prod_{m=1}^{j}(\bar{\lambda}_m - \Lambda^*_m)^{-1} \int \frac{(u_\ell - \Lambda_\ell)h(du)}{\prod_{m=j+1}^{k}(\bar{\lambda}_m - \bar{u}_m)}. \]

However, \( f(u_\ell - \Lambda_\ell)h(du) = 0, \) since \((u_\ell - \Lambda_\ell)h(du)\) is an analytic function on \( x_{j=1}^{k} \sigma(S_j) \). Therefore
\[
\left[ \prod_{j=1}^{k}(\bar{\lambda}_j - \sigma(S_j)^{-1}) - S_j \right]h(S)|_{M} = \int_{sp(N)} \frac{(u_\ell - \Lambda_\ell)h(du)}{\prod_{m=1}^{k}(\bar{\lambda}_m - \bar{u}_m)},
\]
which proves (2.1).
\[\square\]

Remark 2.2. If \( [S_j^*, S_j], j = 1, \ldots, k \) are in the trace class, then it is easy to calculate that
\[
\text{tr}[f(S^*)h(S), l(S)] = \sum_{j=1}^{k} \text{tr}[f(S^*)h(S) \frac{\partial l}{\partial u_j}(S), S_j], \tag{2.5}
\]
where \( f \) is an analytic function on \( x_{j=1}^{k} \sigma(S_j^*) \), and \( h \) and \( l \) are analytic functions on \( x_{j=1}^{k} \sigma(S_j) \). The commutator formula (2.1) with trace formula (2.5) may lead to the trace formula:
\[
\text{str}[f(S^*, S), h(S^*, S)] = \int_{sp(N)} f(\bar{u}, u)duh(\bar{u}, u), \tag{2.6}
\]
in [30] and [31], where \( f \) and \( h \) are analytic functions on \((x_{j=1}^{k} \sigma(S_j^*)) \times (x_{j=1}^{k} \sigma(S_j)), [S_j^*, S_j]^{1/2}, j = 1, \ldots, k \) are in the trace class. Therefore the commutator formula (2.1) is an "operator-lifting" of the trace formula (2.6).

3. Decomposition of the Mosaics of a Pure Subnormal Tuple of Operators with Finite Dimensional Defect Space

Let \( S = (S_1, \ldots, S_k) \) be a pure subnormal \( k \)-tuple of operators \( H \) with m.n.e. \( N \) on \( \mathcal{K} \supset H \). Let \( e_j(\cdot) \) be the measure \( e(\cdot) \) for the subnormal operator \( S_j \) and \( \mu_j(\cdot) \) be the mosaic of \( S_j \). Let \( sp_p(S^*) \) be the set of all joint eigenvalues \((\lambda_1, \ldots, \lambda_k) \), i.e. there is a vector \( a \in H, a \neq 0 \) satisfying
\[
S_j^*a = \lambda_ja, j = 1, \ldots, k.
\]
Let \( sp_p(S^*)^{*} = \{(\bar{\lambda}_1, \ldots, \bar{\lambda}_k) : (\lambda_1, \ldots, \lambda_k) \in sp_p(S^*) \} \). It is obvious that \( sp_p(S^*)^{*} \subset sp_p(S) \), the right spectrum of \( S \).

Assume that \( \dim M < \infty \), where \( M \) is the defect space of \( S \). Define
\[
P_m(z, w) = \text{det}(\bar{w}_m - \Lambda^*_m)(z_m - \Lambda_m) - C_{lm}), l, m = 1, \ldots, k.
\]
it is easy to see that \( P_m(z, w) = P_m(w, z) \). Define \( R_m(z) = C_{lm}(z - \Lambda_m)^{-1} + \Lambda^*_l, \) for \( z \in \rho(\Lambda_l) \).

Lemma 3.1. ([30],[31]) Let \( S = (S_1, \ldots, S_k) \) be a pure subnormal \( k \)-tuple of operators on \( H \) with finite dimensional defect space \( M \) and with m.n.e. \( N = (N_1, \ldots, N_k) \) on \( \mathcal{K} \supset H \). Corresponding to \( S_j, j = 1, \ldots, k \), there are some finite union of qudrature domain \( D_j \) in the Riemann surfaces with boundary \( \iota_j \), Schwarz function \( S_j(\cdot) \) and projection \( \psi_j(\cdot) \) satisfying \( \psi_j(D_j) = D_j, \psi_j \) is locally univalent on \( L_j \), where the closure of \( D_j \) equals \( \sigma(S_j) \), \( \psi(\mathcal{L}_j) = L_j \) the essential spectrum of \( N_j \).
There is a meromorphic \( L(M) \)-valued function \( \nu_j(\zeta), \zeta \in \mathcal{D}_j \) with continuous boundary value on \( \mathcal{L}_j \), satisfying \( \nu_j(\cdot)^2 = \nu_j(\cdot) \), \( \nu_j(\zeta_1)\nu_j(\zeta_2) = \nu_j(\zeta_2)\nu_j(\zeta_1) = 0 \) for \( \Psi(\zeta_1) = \Psi(\zeta_2) \) and \( \zeta_1 \neq \zeta_2 \),

\[
\nu_j(\zeta)R_{jj}(\Psi(\zeta)) = R_{jj}(\Psi(\zeta))\mu_j(\zeta) = S_j(\zeta)\nu_j(\zeta), \zeta \in \mathcal{D}_j,
\]

and

\[
\mu_j(z) = \sum_{\Psi_j(\zeta) = z} \nu_j(\zeta), \zeta \in \mathcal{D}_j.
\]

There is an orientation of \( \mathcal{L}_j \) such that the boundary value of \( \nu_j(\cdot) \) and \( \Psi(\cdot) \) satisfy

\[
\nu_j(\zeta) = 2\pi i (\Psi_j(\zeta) - \Lambda_j) e_j(d\Psi_j(\zeta))/d\Psi_j(\zeta), \text{ for a.e. } \zeta \in \mathcal{L}_j.
\]

By a method in [17], we firstly consider the case that \( S \) has the property (A): the restriction of projection from \( \mathbb{C}^k \) to \( \mathbb{C} \):

\[
P_j(z_1, \ldots, z_k) \overset{\text{def}}{=} z_j
\]

is one to one from \( \mathrm{sp}(\mathbb{N}) \) to \( \sigma(N_j) \), for \( j = 1, \ldots, k \).

**Lemma 3.2.** Under the condition of Lemma 3.1, assume that \( S \) has the property (A). Then there are analytic mappings \( g_{lm}(\cdot) \) from \( \mathcal{D}_m \) to \( \mathcal{D}_l \) such that

\[
R_{lm}(\Psi_m(\zeta))\nu_m(\zeta) = \nu_m(\zeta)R_{lm}(\Psi_m(\zeta)) = \overline{\mathcal{S}_l(g_{lm}(\zeta))}\nu_m(\zeta), \text{ for } \zeta \in \mathcal{D}_m,
\]

for \( l, m = 1, \ldots, k \).

**Proof.** From [26],[30] and [31], we have

\[
((\bar{u}_i - \Lambda^*_i)(u_m - \Lambda_m) - C_{lm})e(du) = e(du)(((\bar{u}_i - \Lambda^*_i)(u_m - \Lambda_m) - C_{lm}) = 0
\]

and

\[
P_{lm}(u_m, u_l) = 0, \text{ for } (u_1, \ldots, u_k) \in \mathrm{sp}(\mathbb{N}).
\]

From property (A), there are functions \( f_{lm}(\cdot) \) such that

\[
\mathrm{sp}(\mathbb{N}) = \{(f_{1m}(z), \ldots, f_{km}(z)) : z \in \sigma(N_m)\}, m = 1, \ldots, k,
\]

and \( f_{mm}(z) = z \). From (3.5), we have

\[
R_{lm}(\Psi(\zeta))\nu_m(\zeta) = \overline{\mathcal{S}_l(g_{lm}(\zeta))}\nu_m(\zeta),
\]

for \( \zeta \in \mathcal{L}_m \) and \( l, m = 1, \ldots, k \). From (3.6), we have

\[
h_{lm}(\zeta) = (R_{lm}(\Psi(\zeta))\nu_m(\zeta))a, b)/(\nu_m(\zeta)a, b), \zeta \in \mathcal{L}_m,
\]

is independent of \( \{a, b\} \). From (3.7), \( h_{lm}(\zeta) \) extends to a meromorphic function on \( \mathcal{D}_m \). On the other hand, the projection \( \Psi_m(\cdot) \) is univalent on a neighborhood of each point of \( \mathcal{L}_m \). Therefore there exist mappings \( g_{lm} \) from \( \mathcal{L}_m \) onto \( \mathcal{L}_l \) such that

\[
f_{lm}(\Psi_m(\zeta)) = \overline{\Psi_l(\zeta)}, \zeta \in \mathcal{L}_m.
\]

However \( S_l(\zeta) = \overline{\Psi_l(\zeta)} \) for \( \zeta \in \mathcal{L}_l \). From (3.8) we have

\[
h_{lm}(\zeta) = S_l(g_{lm}(\zeta)) = \overline{f_{lm}(\Psi_m(\zeta))}.
\]

It is easy to see that \( f_{lm} = f_{ml}^{-1} \). Therefore \( g_{lm} = g_{ml}^{-1} \). Thus \( S_l^{-1} \circ h_{lm} = h_{lm} \circ S_m \). Therefore \( g_{lm}(\zeta) = S_l^{-1}(h_{lm}(\zeta)) \) extends an injective analytic mapping from \( \mathcal{D}_m \) to \( \mathcal{D}_l \). From (3.7), (3.8) and (3.9) it implies (3.4).

Let \( \mathcal{E} = \{(\zeta_1, \ldots, \zeta_k) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_k : \alpha_m(\zeta_m) = \zeta_l\} \). Then \( \mathcal{E} \) is a finite union of domains in some analytic manifolds. Let \( \mathcal{E} = \{\left(\Psi_1(\zeta_1), \ldots, \Psi_k(\zeta_k)\right) \in \mathbb{C}^k : (\zeta_1, \ldots, \zeta_k) \in \mathcal{E}\} \). Then \( \mathcal{E} \) is a finite union of domains in some analytic manifolds with boundary \( \mathrm{sp}(\mathbb{N}) \).

\[\square\]
4. Structure of $sp_{jp}(S^*)^*$ and Commutator Formula for Pure Subnormal $k$-Tuple of Operators with Finite Dimensional Defect Space

First, we have the following:

**Lemma 4.1.** ([30],[31]) Let $S = (S_1, \ldots, S_k)$ be a pure subnormal $k$-tuple of operator on $K$ with m.n.e. $N = (N_1, \ldots, N_k)$ on $K \supset H$ and with finite dimensional defect space $M$. Let $w_j \in \sigma(S_j) \cap \rho(N_j), j = 1, \ldots, k$. Suppose there are $c_i \in C, l = 1, \ldots, k$ and vector $a \in M, a \neq 0$ such that

$$\mu_j(w_j)^*a = a, j = 1, \ldots, k$$

and

$$R_{l_j}(w_j)^*a = c_l a, j = 1, \ldots, k.$$  

Let $f_j(u) \defeq (\bar{u}_j - \bar{w}_j)^{-1}(u_j - \Lambda_j^*)a, u \in sp(N), j = 1, \ldots, k$. Then $f_1 = \cdots = f_k$ as a vector in $H$ and $S_j^*f_j = \bar{w}_j f_j$

i.e. $(w_1, \ldots, w_k) \in sp_j(S^*)^*$.

**Theorem 4.2.** Let $S = (S_1, \ldots, S_k)$ be a pure subnormal $k$-tuple of operators with finite dimensional defect space. Then

$$sp_{jp}(S^*)^* \setminus sp(N) = E \setminus sp(N).$$  

**(4.1)**

**Proof.** First let us assume that $S$ has the property (A). For $F \subset sp(N)$, we have

$$e(F) = \frac{1}{2\pi i} \int_{\Psi_j(\zeta_j) \in F_j} (\Psi_j(\zeta_j) - \Lambda_j)^{-1}\nu_j(\zeta_j)d\Psi_j(\zeta_j),$$

for $j = 1, \ldots, k$. Therefore

$$(\Psi_1(\zeta_1) - \Lambda_1)^{-1}\nu_1(\zeta_1)\frac{d\Psi_1(\zeta_1)}{d\Psi_m(\zeta_m)} = (\Psi_m(\zeta_m) - \Lambda_m)^{-1}\nu_m(\zeta_m),$$  

**(4.2)**

for $\zeta = g_{lm}(\zeta_m), \zeta_m \in L_m$. But $g_{lm}(\cdot)$ is an analytic mapping from $D_m \cup L_m$ to $D_l \cup L_l$. Therefore (4.2) holds good also for $\zeta = g_{lm}(\zeta_m), \zeta_m \in D_m$. Thus

$$\nu_l(\zeta)^*M = \nu_m(\zeta_m)^*M, \text{ for } (\zeta_1, \ldots, \zeta_k) \in E.$$

Let us define

$$M_\zeta = \nu(\zeta)^*M, \text{ for } \zeta = (\zeta_1, \ldots, \zeta_k) \in E, m = 1, \ldots, k.$$

From $\nu_l(\zeta_l)(I - \mu_l(\Psi_\zeta)) = (I - \mu_l(\Psi_\zeta))\nu_l(\zeta_l) = 0$, we may conclude that

$$\mu_l(\Psi_\zeta)^*a = a, \text{ for } a = M_\zeta, \zeta = (\zeta_1, \ldots, \zeta_k), l = 1, k.$$

From (3.5), $(R_{lm}(\Psi_m(\zeta_m))^* - S_l(\zeta_l))\nu_m(\zeta_m)^* = 0, \text{ for } (\zeta_1, \ldots, \zeta_k) \in E$. Thus

$$R_{lm}(\Psi_m(\zeta_m))^*a = S_l(\zeta_l)a, \text{ for } (\zeta_1, \ldots, \zeta_k) \in E.$$

From Lemma 4.1, we have $(\Psi_1(\zeta_1), \ldots, \Psi_k(\zeta_k)) \in sp_{jp}(S^*)^*$, for $(\zeta_1, \ldots, \zeta_k) \in E$, which proves $E \setminus sp(N) \subset sp_{jp}(S^*)^*$.

Now suppose $(z_1, \ldots, z_k) \in sp_{jp}(S^*)^* \setminus sp(N)$. From the property (A), we may conclude that $z_j \notin \sigma_j(N_j)$. There is an $f \in H, f \neq 0$ such that

$$S_j^*f = z_jf, j = 1, \ldots, k.$$

From [24], [25],[30] and [31], we have

$$f = \frac{\bar{u}_j - \Lambda_j^*}{\bar{u}_j - \bar{z}_j}\mu_j(z_j)^*a, j = 1, \ldots, k,$$

**(4.3)**

where $a = f(\Lambda) = \int_{sp(N)} e(du)f(u)$. From (4.3) it is easy to see that

$$\mu_j(z_j)^*a = a, \text{ for } j = 1, \ldots, k.$$  

**(4.4)**
From (3.2) and (4.4), we have \( a = \sum_{j,l} a_{j}^{(l)} \), where \( a_{j}^{(l)} = \nu_{j}(\zeta_{j}^{(l)}a) \), where \( \Psi_{j}(\zeta_{j}^{(l)}) = z_{j} \). But from (3.5), we have
\[
\left(R_{n}(z_{j})^{*} - S_{n}(\zeta_{j}^{(l)})\right)a_{j}^{(l)} = 0, n, l, j = 1, \ldots, k. \tag{4.5}
\]
From [30] and [31], we have
\[
\int \frac{e(du)(u_{j} - \Lambda_{j}^{*})}{(u_{n} - z)(u_{j} - z_{j})} = \int \frac{e(du)}{u_{n} - z} + (z - R_{n}(z_{j})^{*})^{-1}(I - \mu_{j}(z_{j})^{*}) + \int \frac{e(du)}{u_{n} - z} (R_{n}(z_{j})^{*} - z)^{-1}
\]
From (4.5), we have
\[
\int \frac{e(du)}{u_{n} - z} f(u) = \int \frac{e(du)(u_{j} - \Lambda_{j}^{*})}{(u_{n} - z)(u_{j} - z_{j})} a
\]
\[
= \int \frac{e(du)}{u_{n} - z} a + \mu_{n}(z)(R_{n}(z_{j})^{*} - z)^{-1}a, \tag{4.6}
\]
where \( \mu_{n}(z) = \int e(du)(u_{n} - \Lambda_{n})(u_{n} - z)^{-1} \). From (4.5) and (4.6) we have
\[
\int \frac{e(du)}{u_{n} - z} a = \int \frac{e(du)}{u_{n} - z} a = \mu_{n}(z)\sum_{l} \frac{a_{j}^{(l)}}{S_{n}(\zeta_{j}^{(l)}) - z}
\]
for all \( z \in \rho(N_{n}), n = 1, \ldots, k \). Therefore \( a_{j}^{(l)} \) should be independent of \( j \). Denote \( a_{j}^{(l)} \) by \( a^{(l)} \). Thus
\[
a = \sum_{l} a^{(l)}.
\]
Let \( f_{j}^{(l)}(u) = (u_{j} - z_{j})^{-1}(u_{j} - \Lambda_{j}^{*}) a^{(l)} \), then
\[
\mu_{j}(z_{j})^{*}a^{(l)} = a^{(l)} \text{ and } R_{n}(z_{j})^{*}a^{(l)} = S_{n}(S_{n}(\zeta_{j}^{(l)}) - z)^{-1}a^{(l)},
\]
for \( j = 1, \ldots, k \) and \( n = 1, \ldots, k \). From Lemma 4.1, we may conclude that \( f_{j}^{(l)} = f_{m}^{(l)} \) for \( j \neq m \). We denote this vector by \( f^{(l)} \). Therefore \( (\zeta_{j}^{(l)}, \ldots, \zeta_{k}^{(l)}) \in E \) and then \( (z_{1}, \ldots, z_{k}) \in E \). Thus \( sp_{p}(S_{n}) \notin sp(N) \subset E \), which proves this theorem in the case that \( S \) has property (A).

Now, we have to release the restriction: property (A) of \( S \). For any \( S \) with m.e. \( N \) and finite dimensional defect space, the \( sp(N) \) consists of finite set of algebraic arcs and a possible finite set of isolated points. There exists a non-singular linear transformation \( \pi \) on \( C^{k} \): for \( (c_{1}, \ldots, c_{k}) \in C^{k} \),
\[
\pi(c_{1}, \ldots, c_{k}) = \left( \sum_{j} \pi_{j}^{(1)}c_{j}, \ldots, \sum_{j} \pi_{j}^{(k)}c_{j} \right),
\]
such that for the set \( \pi(sp(N)) \), the projections to its coordinate planes are one to one. By this \( \pi \), we define \( \hat{S}_{n} = \sum_{j} \pi_{j}^{(n)}s_{j} \) and \( \hat{N}_{n} = \sum_{j} \pi_{j}^{(n)}N_{j} \). Then \( \hat{S} = (\hat{S}_{1}, \ldots, \hat{S}_{k}) \) is a pure subnormal \( k \)-tuple of operators with m.e. \( (\hat{N}_{1}, \ldots, \hat{N}_{k}) \) and the same defect space \( M \). It is obvious, this \( \hat{S} \) has property (A). Therefore the conclusion of this theorem is true for \( \hat{S} \). By the linear transform \( \pi^{-1} \), we may prove this theorem for the original \( S \). \( \square \)

**Theorem 4.3.** Let \( S = (S_{1}, \ldots, S_{k}) \) be a pure subnormal \( k \)-tuple of operators on \( H \) with finite dimensional defect space \( M \). Then
\[
[f(S^{*})h(S), S_{l}]_{M} = \frac{1}{2\pi i} \int_{sp_{p}(S^{*})} h(u)\mu_{l}(u)df(u) \wedge du_{l}, \tag{4.7}
\]
for \( l = 1, \ldots, k \), where \( \mu_{l}(\cdot) \) is the mosaic of \( S_{l} \), and \( f \) and \( h \) are analytic functions on \( \times_{j=1}^{k} \sigma(S_{j}^{*}) \) and \( \times_{j=1}^{k} \sigma(S_{j}) \) respectively.
Proof. By Lemma 3.1, we have

\[(u_i - \Lambda_i)e(du) = \frac{1}{2\pi i} \nu_i(\zeta_i)d\Psi(\zeta_i),\]

for \(u = (\psi_1(\zeta_1), \ldots, \psi_k(\zeta_k)) \in sp_{ess}(\mathbb{N})\). By the Cartan's formula \(\int_{\partial D} \omega = \int_D dw\), the technique of removing the contribution of \(e(\cdot)\) measure on the \(sp_{p}(\mathcal{N})\) in the proof of Theorem 2.4.1 in [31], and letting \(\Psi(\zeta) \triangleq (\psi_1(\zeta_1), \ldots, \psi_k(\zeta_k))\), for \(\zeta = (\zeta_1, \ldots, \zeta_k) \in \mathcal{E} \cup \partial \mathcal{E}\), we have

\[
\int_{sp(\mathcal{N})} f(\bar{u})h(u)(u_i - \Lambda_i)e(du) = \frac{1}{2\pi i} \int_{\partial \mathcal{E}} f(\bar{\Psi}(\zeta))h(\bar{\Psi}(\zeta))\nu_i(\zeta_i)d\Psi_i(\zeta_i)
\]

\[
= \frac{1}{2\pi i} \int_{\partial \mathcal{E}} h(\Psi(\zeta))\nu_i(\zeta_i)d\bar{\Psi}(\zeta) \wedge \Psi_i(\zeta),
\]

(4.8)

where \(\bar{\Psi}(\zeta) = (\bar{\psi}_1(\zeta_1), \ldots, \bar{\psi}_k(\zeta_k))\). From (3.2), we have

\[
\int_{\mathcal{E}} h(\Psi(\zeta))\nu_i(\zeta_i)d\bar{\Psi}(\zeta) \wedge d\Psi_i(\zeta) = \int_{\mathcal{E}} h(u)\mu_i(u)d(\bar{u}) \wedge du,
\]

(4.9)

However the \(\partial \mathcal{E} = sp(\mathcal{N})\) is a finite union of algebraic arcs with finite set of points. Thus from (2.1), (4.8) and (4.9), we have (4.7).

Remark 4.4. First, by the technique in Remark 2.2, the trace formula (4.9) leads to the trace formula in [17] and [19]. By the same technique the commutator formula (4.7) leads to the following trace formula in [31]:

\[
tr[f(S^*, S)] = \frac{1}{2\pi i} \int_{sp_{p}(S^*)} m(u)d(\bar{u}, u) \wedge dh(\bar{u}, u),
\]

(4.10)

where \(m(u)\) is the dimension of the subspace of the joint eigenvectors of \(S^*\) corresponding to the joint eigenvalues \(\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)\). Thus the commutator formula (4.7) can be considered as an “operator-lifting” of the trace formula (4.10). This formula (4.10) is also related to the theory of the principal current in [5].

5. Commutator Formula for Hyponormal Operator associated with a Quadrature Domain

Let \(H\) be a pure hyponormal operator on a Hilbert space \(\mathcal{H}\) with rank one self-commutator \([H^*, H]\). If the Pincus principal function \(g_H(\cdot)\) of \(H\) satisfies the condition that \(g_H(z) \equiv 1\), for \(z \in \sigma(H)\) and \(\sigma(H)\) is the closure of a quadrature domain \(D\), then \(H\) is said to be a hyponormal operator associated with a quadrature domain. In this case

\[
K \triangleq \sqrt{[H^*, H] \mathcal{H} : n = 0, 1, 2, \ldots}
\]

(5.1)

is of finite dimension. Define

\[
C = [H^*, H]|_K \quad \text{and} \quad \Lambda = (H^*_k)\mathcal{N}.
\]

In [29] and [31], let \(P(z, w) \triangleq \det((\bar{w} - \Lambda^*)(z - \Lambda) - C)\) and \(Q_D(z) \triangleq \det(z - \Lambda). \) Define \(P_{\bar{w}} = \frac{\partial}{\partial \bar{w}} P(z, w)\). Let

\[
D_0 \triangleq \{z \in D : z \in \rho(\Lambda), S(z) \in \rho(\Lambda^*), P_{\bar{w}}(z, S(z)) \neq 0\},
\]

where \(S(\cdot)\) is the Schwarz function of \(D\). Then \(D \setminus D_0\) is a finite set. As in [29] and [31], the \(L(K)\)-valued analytic function

\[
\mu(z) \triangleq (S(z) - \Lambda^*)^{-1}C(z - \Lambda)^{-1}(S(z) - \Lambda^*)^{-1}R(z), z \in D_0,
\]

where \(k(z) \triangleq P_{\bar{w}}(z, S(z))^{-1}Q_D(S(z)), \) is defined as the mosaic of \(H\). Then \(\mu(z)^2 = \mu(z), z \in D\) (cf. [29] and [31]). Let \(R(z) = C(z - \Lambda)^{-1} + \Lambda^*\), then

\[
R(z)\mu(z) = \mu(z)R(z) = S(z)\mu(z).
\]

(5.2)
There exists a unique polynomial $r(\cdot)$ with leading coefficient 1 and minimal degree such that $r(u)\mu(u)$ is analytic on $\sigma(H)$ and

$$\int_L \|r(u)\mu(u)\|du < +\infty,$$

where $L = \partial D$. From Lemma 5.5.1 of [31], we may prove that

$$\frac{1}{2\pi i} \int_L f(u)(\bar{u} - \bar{\lambda})^{-1}(u - \Lambda)^{-1}\mu(u)du = P_K(\bar{u} - H^*)^{-1}f(H)|_K$$

(5.3)

if $f(\cdot)r(\cdot)^{-1}$ is analytic on $\sigma(H)$, where $P_K$ is the projection from $\mathcal{H}$ to $K$.

**Theorem 5.1.** Let $H$ be a pure hyponormal operator associated with a quadrature domain on $\mathcal{H}$. Let $h$ be an analytic function on $\sigma(H)$ satisfying that $hr^{-1}$ is also analytic on $\sigma(H)$. Let $f$ be an analytic function on $\sigma(H^*)$. Then

$$[f(H^*)h(H), H]|_K = \frac{1}{2\pi i} \int_{\partial \sigma(H)} f(\bar{u})h(u)\mu(u)du$$

(5.4)

$$= \frac{1}{2\pi i} \int_{\sigma(H)} h(u)\mu(u)df(\bar{u}) \wedge du.$$  

(5.5)

**Proof.** Let $\lambda \in \rho(H)$, then

$$[((\bar{\lambda} - H^*)^{-1}h(H), H)|_K = [((\bar{\lambda} - H^*)^{-1}, H)h(H)|_K$$

$$= (\bar{\lambda} - H^*)^{-1}CP_K(\bar{\lambda} - H^*)h(H)|_K.$$  

(5.6)

From (5.3), the right-hand side of (5.6) equals to

$$(\bar{\lambda} - \Lambda^*)^{-1}C\frac{1}{2\pi i} \int_L \frac{h(u)(u - \Lambda)^{-1}}{\bar{\lambda} - \bar{u}}\mu(u)du.$$  

(5.7)

From (5.2), we have $C(u - \Lambda)^{-1}\mu(z) = (\bar{u} - \Lambda^*)\mu(u)$. Thus (5.7) equals to

$$= (\bar{\lambda} - \Lambda^*)^{-1}\int_L \frac{(\bar{u} - \Lambda^*)h(u)\mu(u)}{\bar{\lambda} - \bar{u}}du$$

$$= \frac{1}{2\pi i} \int_L \frac{f(u)\mu(u)du}{\bar{\lambda} - \bar{u}} - \frac{1}{2\pi i} \int_L h(u)\mu(u)du.$$  

(5.8)

But $h(u)\mu(u) = (h(u)r(u)^{-1})(r(u)\mu(u))$ is analytic on $L \cup D$. Therefore the second term in the most right-hand side of (5.8) is zero, which proves (5.4) for $f(\bar{u}) = (\bar{\lambda} - \bar{u})^{-1}$ and hence for any analytic function $f$ on $\sigma(H^*)$.

By Cartan’s formula, (5.4) implies (5.5), since $\int_{\sigma(H)} \|h(u)\mu(z)\|dA < +\infty$. □

**References**


Daoxiong xia
Department of Mathematics
Vanderbilt University
Nashville, TN 37240, USA
e-mail: daoxing.xia@Vanderbilt.edu