THE ANALYTIC MODEL OF A SUBNORMAL OPERATOR

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The analytic model of a pure subnormal operator and its conjugate subnormal operator is obtained. A mosaic is introduced for subnormal operator. Some results in spectral analysis of subnormal operators are obtained by means of the analytic model and the mosaic. The form of pure subnormal operators with rank two self-commutator is determined.

§1. In this paper, $S$ is a pure subnormal operator on a separable Hilbert space $\mathcal{H}$ with minimal normal extension $N$ on a separable Hilbert space $\mathcal{K} \supset \mathcal{H}$. The operator $S'$ defined on $\mathcal{K}' = \mathcal{K} \ominus \mathcal{H}$ as

$$S' x = N^* x \quad \text{for } x \in \mathcal{H}'$$

is also pure subnormal and is said to be the conjugate subnormal operator of $S$.

In [6], [8], [9] the analytic model of a hyponormal operator with rank one self-commutator is introduced and some kernels and applications of the analytic model are also obtained.

By means of a kernel which is similar to $S(z, w)$ in [6], [8], [9], in the present paper (see §2) an analytic model of a pure subnormal operator and its conjugate operator is introduced.

In §3, a (parallel) projection-valued analytic function $\mu(\cdot)$ defined on $C \backslash \sigma(N)$ is introduced for a pure subnormal operator $S$ and is called the mosaic of $S$. This $\mu(\cdot)$ is unitarily invariant and is also a completely unitary invariance if $\sigma(N)$ doesn't have any interior point. This mosaic may have some relation with the mosaic introduced by Carey and Pincus [1] for hyponormal operator which is (orthogonal) projection-valued (cf. [2]) almost everywhere on $\sigma(S)$ and is always a completely unitary invariance but may not be analytic for a general pure subnormal operator. So these two mosaics are not the same. Our mosaic is closely related to the eigenvectors and the values of a function as the representation of a vector in the analytic model. It seems that this analytic model and $\mu(\cdot)$ may also have some connection with Cowen and Douglas theory [4].

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As an application of the analytic model and the mosaic \( \mu(\cdot) \) some theorems for the point-spectra and the eigenspaces of co-subnormal \( S^* \) and \( S'^* \) as well as the theorems for the spectra of \( S \) and \( S' \) are given also in \( \S 3 \).

In \( \S 4 \) a problem proposed in the book [3] by Conway is investigated. The problem is that what are the pure subnormal operators with finite rank self-commutators. Morrel [6] shows that every pure subnormal operator \( S \) with rank one \([S^*, S]\) must be \( \alpha U + \beta I \) where \( U \) is the unilateral shift of multiplicity 1 and \( \alpha, \beta \) are scalars. As an application of the analytic model and the mosaic \( \mu(\cdot) \), we determined the form of pure subnormal operators \( S \) with rank two \([S^*, S]\). It seems that the method used here may also apply to the case with any finite rank \([S^*, S]\).

\[ 
\text{§2. In this section, the analytic model of a pure subnormal operator is introduced. First, we have to introduce an analytic kernel } S(\cdot, \cdot) \text{ for } S. 
\]

**Lemma 1.** Let \( S \) be a subnormal operator on a separable Hilbert space \( \mathcal{H}, M = [S^*, S]\mathcal{H} \), and \( Q \) be the injection from \( M \) into \( \mathcal{H} \). Define an \( \mathcal{L}(M \rightarrow M) \)-valued analytic function

\[ S(z, w) = Q^*(\overline{w}I - S^*)(zI - S)^{-1}Q, \]

for \( z, w \in \rho(S) \). Then

\[ S(z, w) = ((\overline{w}I - \Lambda^*)(zI - \Lambda) - C)^{-1}, \]

for \( z, w \in \rho(S) \), where \( C = Q^*[S^*, S]|_M \) and \( \Lambda = (Q^*S^*|_M)^* \).

This function \( S(\cdot, \cdot) \) is said to be the determining function of the pure subnormal operator \( S \).

**Proof.** Let \( N \) be the minimal normal extension of \( S \) on a Hilbert space \( K \supseteq \mathcal{H} \), and \( \mathcal{H}' = K \ominus \mathcal{H} \). Denote by \( P \) and \( P' \) the orthogonal projections from \( K \) onto \( \mathcal{H} \) and \( \mathcal{H}' \) respectively.

Let \( A \) be the operator in \( \mathcal{L}(\mathcal{H}' \rightarrow \mathcal{H}) \) defined by

\[ Ax = PNx, \quad x \in \mathcal{H}', \]

and \( S' \) the operator in \( \mathcal{L}(\mathcal{H}' \rightarrow \mathcal{H}') \) defined by

\[ S'x = N^*x, \quad x \in \mathcal{H}'. \]

Note that \( S'x \in \mathcal{H}' \) for \( x \in \mathcal{H}' \), since \( \mathcal{H}' \) is an invariant subspace of \( N^* \).
The condition $NN^* = N^*N$ is equivalent to the following three identities

\[ [S^*, S] = AA^*, \]  
\[ [S'^*, S'] = A^*A. \]  

and

\[ S^*A = AS'. \]  

From (5), it follows that $M = A\overline{N}$. From (7), it follows that $M$ is an invariant subspace of $S^*$. We have to prove that

\[ \sigma(A^*) \subset \sigma(S^*). \]  

Let $\lambda \in \rho(S^*)$. Then $\lambda \in \rho(N^*)$, since $\sigma(S) \supset \sigma(N)$. For every $x \in \mathcal{H}'$, there is a unique pair of vectors $x_1 \in \mathcal{H}$ and $x_2 \in \mathcal{H}'$ such that

\[ x = (\lambda I - N^*)(x_1 + x_2) = (\lambda I - S^*)x_1 + (-A^*x_1 + (\lambda I - S')x_2). \]

Therefore

\[ (\lambda I - S^*)x_1 = Px = 0, \]

and hence $(\lambda I - S')x_2 = x$. Thus

\[ Ax = (\lambda I - S^*)Ax_2. \]

Since $(\lambda I - S^*)A = A(\lambda I - S')$ by (7). This shows that $(\lambda I - S^*)M$ is a dense subset of $M$ and hence it equals $M$ since $\lambda I - S^*$ is invertible. In conclusion, $\lambda \in \rho(A^*)$, which proves (8). From (8), it follows immediately that

\[ \rho(S) \subset \rho(A) \]  

According to the definition of $C$, it is obvious that

\[ [S^*, S] = QCQ^* \]  

Define another $\mathcal{L}(M \rightarrow M)$-valued analytic functions

\[ T(z, w) = Q^*(zI - S)^{-1}(\overline{w}I - S^*)^{-1}Q \]  

for $z, w \in \rho(S)$. From the commutational relation and the identity (10), it follows that

\[ [(\overline{w}I - S^*)^{-1}, (zI - S)^{-1}] = (\overline{w}I - S^*)^{-1}(zI - S)^{-1}QCQ^*(zI - S)^{-1}(\overline{w}I - S^*)^{-1}. \]
Hence the functions \( S(\cdot, \cdot) \) and \( T(\cdot, \cdot) \) satisfy the following identity

\[
S(z, w) - T(z, w) = T(z, w)CS(z, w),
\]

for \( z, w \in \rho(S) \). The identity (12) is similar to some relations between operator-valued functions in hyponormal operator theory (cf. [2], [9] and [10]).

Now, let us determine \( T(\cdot, \cdot) \). If \( \lambda \in \rho(S) \), then \( \lambda \in \rho(A) \) (see (9)). Hence \( \lambda \in \rho(A^*) \) and

\[
(\lambda I - \Lambda^*)^{-1}x = (\lambda I - S^*)^{-1}x, \quad x \in M.
\]

Therefore

\[
(T(z, w)u, v) = ((\lambda I - S^*)^{-1}u, (\lambda I - S^*)^{-1}v) = ((\lambda I - \Lambda^*)^{-1}u, (\lambda I - \Lambda^*)^{-1}v) = ((\lambda I - \Lambda)^{-1}(\lambda I - \Lambda^*)^{-1}u, v),
\]

for \( u, v \in M \) and \( z, w \in \rho(S) \). Thus

\[
T(z, w) = (\lambda I - \Lambda)^{-1}(\lambda I - \Lambda^*)^{-1}, \quad z, w \in \rho(S).
\]

From (12) and (13), it follows that

\[
((\lambda I - S^*)(\lambda I - \Lambda) - C)S(z, w) = I, \quad z, w \in \rho(S).
\]

Similarly, we also have

\[
S(z, w)((\lambda I - \Lambda^*)(\lambda I - \Lambda) - C) = I, \quad z, w \in \rho(S)
\]

Hence the operator \((\lambda I - \Lambda^*)(\lambda I - \Lambda) - C\) is invertible and (2) holds.

**LEMMA 2.** Under the conditions of Lemma 1, let \( E(\cdot) \) be the spectral measure of the normal operator \( N \), and \( e(\cdot) = Q^*E(\cdot)Q \) be the \( L(M \to M) \)-valued positive measure. Then

\[
e(\sigma(N)) = I, \quad \int \frac{uI - \Lambda}{u - z}e(du) = 0.
\]

for \( z \in \rho(S) \), and

\[
\int_F ((\lambda I - \Lambda^*)(uI - \Lambda) - C)e(du) = 0
\]

for every Borel set \( F \subset \sigma(N) \).
PROOF. From (1) and (2), it follows that

$$\int \frac{e(du)x}{\gamma (\overline{w}-\overline{u})(z-u)} = ((zI - N)^{-1}x, (wI - N)^{-1}y)$$

$$= ((zI - S)^{-1}x, (wI - S)^{-1}y)$$

$$= (((\overline{w}I - \Lambda^*)(zI - \Lambda) - C)^{-1}x, y)$$

for \(x, y \in M\) and \(z, w \in \rho(S)\), where \(\gamma = \sigma(N)\). Hence

$$e(\sigma(N)) + \int_{\gamma} \frac{(\overline{w}I - \Lambda^*)e(du)}{\overline{w} - \overline{u}} + \int_{\gamma} \frac{(uI - \Lambda)e(du)}{z - u}$$

$$+ \int_{\gamma} \frac{((\overline{w}I - \Lambda^*)(uI - \Lambda) - C)e(du)}{(\overline{w} - \overline{u})(z-u)} = ((\overline{w}I - \Lambda^*)(zI - \Lambda) - C) \int_{\gamma} \frac{e(du)}{(\overline{w} - \overline{u})(z-u)} = I. \quad (16)$$

It is obvious that \(e(\gamma) = I\). Putting \(w \to \infty\) in the left-most-hand side of (16), we get (14).

From (14) and (16), it follows that

$$\int_{\gamma} \frac{((\overline{u}I - \Lambda^*)(uI - \Lambda) - C)e(du)}{(\overline{w} - \overline{u})(z-u)} = 0. \quad (17)$$

By Stone-Weierstrass Theorem, it is easy to prove that every continuous function on \(\sigma(S)\) may be approximated uniformly by linear combination of functions \((z - (\cdot))^{-1}(\overline{w} - (\cdot))^{-1}\) for \(z, w \in \rho(S)\). Thus (17) implies that

$$\int_{\sigma(S)} f(u)(\overline{u}I - \Lambda^*)(uI - \Lambda) - C)e(du) = 0. \quad (18)$$

From (18), it is easy to prove (15), since \(\sigma(N)\) is closed subset of \(\sigma(S)\).

**THEOREM 1.** Let \(S\) be a pure subnormal operator on a separable Hilbert space \(\mathcal{H}\), \(M = [S^*, S]\mathcal{H}\), and \(N\) be the minimal normal extension of \(S\) on a Hilbert space \(\mathcal{K} \supset \mathcal{H}\). Then there exists an \(\mathcal{L}(M \to M)\)-valued positive measure \(e(\cdot)\) on \(\sigma(N)\) satisfying (14) and (15) where \(\Lambda, C \in \mathcal{L}(M)\) and \(C\) is a positive operator such that the operator \(U\) defined by

$$Uf(N)\alpha = f(\cdot)\alpha, \quad (19)$$

...
for every bounded Borel function \( f \) and \( \alpha \in M \), extends a unitary operator from \( K \) onto \( \mathcal{L}^2(e) \) satisfying

\[
(USU^{-1}f)(u) = uf(u), \quad u \in \sigma(N),
\]

and

\[
(US^*U^{-1}f)(u) = \overline{u}(f(u) - f(\Lambda)) + \Lambda^*f(\Lambda), \quad u \in \sigma(N).
\]

for \( f \in U\mathcal{H} \), where

\[
f(\Lambda) = \int_{\sigma(N)} e(du)f(u) = UQ^*U^{-1}f,
\]

\( \mathcal{L}^2(e) \) is the Hilbert space of all measurable \( M \)-valued functions \( f \) satisfying

\[
(f, f) = \int_{\sigma(N)} (e(du)f(u), f(u)) < +\infty,
\]

where \( f \) and \( g \) are considered as a same vector if \( \|f - g\| = 0 \), and \( U\mathcal{H} \) is the closure of the set of all linear combinations of vectors \( (\lambda - (\cdot))^{-1}\alpha \), for \( \lambda \in \rho(S) \) and \( \alpha \in M \).

PROOF. We only have to prove (21) and (22), since the existence of \( e(\cdot) \) is proved in Lemma 1 and the unitarity of \( U \) and (20) are obvious.

It is easy to see that

\[
(Q^*(zI - S)^{-1}\alpha, \beta) = (\alpha, (\overline{z}I - S^*)^{-1}\beta) = (\alpha, (\overline{z}I - \Lambda^*)^{-1}\beta)
\]

\[
= ((zI - \Lambda)^{-1}\alpha, \beta),
\]

for \( \alpha, \beta \in M \), and \( z \in \rho(S) \). Therefore

\[
Q^*(zI - S)^{-1}\alpha = (zI - \Lambda)^{-1}\alpha,
\]

for \( \alpha \in M \). On the other-hand, it is obvious that

\[
S^*(zI - S)^{-1} - (zI - S)^{-1}S^* = (zI - S)^{-1}QCQ^*(zI - S)^{-1}
\]

by (10). Therefore

\[
S^*(zI - S)^{-1}\alpha = (zI - S)^{-1}\Lambda^*\alpha + (zI - S)^{-1}C(zI - S)^{-1}\alpha
\]

for \( \alpha \in M \) and \( z \in \rho(S) \). Thus for every analytic function \( f \) on \( \sigma(S) \)

\[
(US^*U^{-1}f\alpha)(u) = \Lambda^*f(u)\alpha + C(uI - \Lambda)^{-1}(f(u)I - f(\Lambda))\alpha, \quad u \in \sigma(N),
\]

(24)
where \( \alpha \in M \).

It is obvious that

\[
(UN^*U^{-1}f)(u) = \overline{u}f(u)
\]  

(25)

for \( f \in L^2(\varepsilon) \), and by (3)

\[
A^*x = N^*x - S^*x,
\]  

(26)

for \( x \in \mathcal{H} \).

From (24), (25), and (26), it follows that

\[
(UA^*U^{-1}f\alpha)(u) = (\overline{u}I - \Lambda^*)f(u)\alpha - C(uI - \Lambda)^{-1}(f(u)I - f(\Lambda))\alpha,
\]  

for every analytic function \( f \) on \( \sigma(S) \) and \( \alpha \in M \). It is easy to see that by (15)

\[
(g, UA^*U^{-1}f\alpha) = \int (g(u), e(du)(\overline{u}I - \Lambda^*)f(\Lambda)\alpha
\]  

\[
+ \int (g(u), e(du)((\overline{u}I - \Lambda^*)(uI - \Lambda) - C)(uI - \Lambda)^{-1}(f(u)I - f(\Lambda))\alpha
\]  

\[
= \int (g(u), e(du)(\overline{u}I - \Lambda^*)f(\Lambda)\alpha
\]  

(27)

for every \( g \in L^2(\varepsilon) \). Hence as a vector in the Hilbert space \( L^2(\varepsilon) \),

\[
(UA^*U^{-1}f\alpha)(u) = (\overline{u}I - \Lambda^*)f(\Lambda)\alpha.
\]  

(28)

From (24), (25), and (28) it follows

\[
(US^*U^{-1}f\alpha)(u) = \overline{u}(f(u)I - f(\Lambda))\alpha + \Lambda^*f(\Lambda)\alpha
\]  

(29)

where \( f \) is an analytic function on \( \sigma(S) \) and \( \alpha \in M \). From (14), it is easy to see that

\[
f(\Lambda)\alpha = \int_{\sigma(N)} e(du)f(u)\alpha,
\]  

for every analytic function on \( \sigma(S) \) and \( \alpha \in M \), which implies (21) and (22).

**COROLLARY 1.** Under the conditions of Theorem 1,

\[
(UA^*U^{-1}f)(u) = (\overline{u}I - \Lambda^*)f(\Lambda),
\]  

(30)

for \( f \in U\mathcal{H} \),

\[
\text{for } f \in U\mathcal{H},
\]
for $f \in \mathcal{L}^2(e) \ominus U\mathcal{H}$,

$$\langle U[S^*, S]U^{-1}f, g \rangle = Cg(\Lambda) \quad (32)$$

for $f \in U\mathcal{H}$,

$$\langle US'U^{-1}f, g \rangle = \bar{u}f(u) \quad (33)$$

and

$$\langle U[S'^*, S']U^{-1}f, g \rangle = (\bar{u}I - \Lambda^*) \int (vI - \Lambda)e(dv)f(v) \quad (35)$$

for $f \in \mathcal{L}^2(e) \ominus U\mathcal{H}$, where the operators $A$ and $S'$ are defined by (3) and (4) respectively.

**PROOF.** From (28), it follows (30). From

$$\langle UA^{-1}f, g \rangle = \langle g, UA^*U^{-1}g \rangle = \int (e(du)f(u), (\bar{u}I - \Lambda^*)g(\Lambda))$$

$$= \int (uI - \Lambda)e(du)f(u), g(\Lambda))$$

$$= \int (e(dv) \int (uI - \Lambda)e(du)f(u), g(v))$$

for $f \in \mathcal{L}^2(e) \ominus U\mathcal{H}$ and $g \in U\mathcal{H}$, it follows (31). The identity (32) is a consequence of (10) and (22). It is evident that (4) implies (33), and the identity

$$S'^*x = Nx - Ax \quad \text{for } x \in \mathcal{H}'$$

and (30) imply (34). The identity (35) is a direct consequence of (33) and (34). Corollary 1 is proved.

**COROLLARY 2.** Under the conditions of Theorem 1, the pair of operators $\{A, C\}$ is a completely unitary invariance of $S$.

**PROOF.** It is obvious that $\{A, C\}$ is a unitary invariance of $S$. 

\[ UA^{-1}f = \int (uI - \Lambda)e(du)f(u) \quad (31) \]
Suppose now $S_1$ is a pure subnormal operator in $\mathcal{H}_1, \{\Lambda_1, C_1\}$ is the pair of operators on $M_1$ corresponding to $S_1$ and $V$ is a unitary operator from $M$ onto $M_1$ such that
\[ VAV^{-1} = \Lambda_1 \quad \text{and} \quad VCV^{-1} = C_1. \tag{36} \]
We have to prove that there is a unitary operator $W$ from $\mathcal{H}$ onto $\mathcal{H}_1$ such that
\[ WSW^{-1} = S_1 \tag{37} \]
Let $S_1(z, w)$ be the determining function of the operator $S_1$. For (1) and (36) it follows that
\[ VS(z, w)V^{-1} = S_1(z, w) \tag{38} \]
Define a linear operator $W$ from the linear manifold $M$ spanned by the set \( \{S^n\alpha : n \geq 0, \alpha \in M\} \) to the linear manifold $M_1$ spanned by the set \( \{S^n\alpha_1 : n \geq 0, \alpha_1 \in M_1\} \) by
\[ WS^nS^nV\alpha = S^nS^nS^n\alpha \]
for $\alpha \in M, m, n = 0, 1, \ldots$. By a method in [11] and (3.8) we can prove that $W$ is an isometry from $M$ to $M_1$. Since $S$ and $S_1$ are pure, $M = \mathcal{H}$ and $M_1 = \mathcal{H}_1$. Thus $W$ extends a unitary operator from $\mathcal{H}$ onto $\mathcal{H}_1$, satisfying (37).

The following theorem is the converse of Theorem 1.

**THEOREM 2.** Let $M$ be an auxiliary Hilbert space, $e(\cdot)$ an $L(M \rightarrow M)$-valued positive measure on a compact support set $\gamma \subset C$ satisfying $e(\gamma) = I$,
\[ \int_{\gamma} \frac{(uI - \Lambda)}{z - u} e(du) = 0 \tag{39} \]
for $z$ in the unbounded component of $C \setminus \gamma$ and
\[ \int_{F} ((uI - \Lambda)(uI - \Lambda) - C)e(du) = 0 \tag{40} \]
for every Borel set $F \subset \gamma$, where $\Lambda, C \in L(M \rightarrow M)$ and $C$ is positive. Let $D$ be the set of $z \in C \setminus \gamma$ on which (39) holds, $\mathcal{H}$ be the closure in $L^2(e)$ of all linear combinations of the functions $(\lambda - (\cdot))^{-1}$, for $\lambda \in D$. Define
\[ (Sf)(u) = uf(u), \quad u \in \gamma, \tag{41} \]
for \( f \in \mathcal{H} \), then \( S \) is a pure subnormal in \( \mathcal{H} \) with the minimal normal extension

\[
(Nf)(u) = \langle f \cdot \rangle, \quad f \in L^2(\mathcal{E}),
\]

and the adjoint operator of \( S \) is

\[
(S^*f)(u) = \bar{u}(f(u) - f(\Lambda)) + \Lambda^* f(\Lambda), \quad u \in \gamma,
\]

for \( f \in \mathcal{H} \), where

\[
f(\Lambda) = \int_{\gamma} e(du)f(u).
\]

**Proof.** It is obvious that the operator \( S \) is subnormal since it has a normal extension

\[
(Nf)(u) = uf(u), \quad u \in \gamma,
\]

for \( f \in L^2(\mathcal{E}) \).

To verify (42), note that

\[
\int \frac{(uI - \Lambda)e(du)}{\lambda - u} = 0
\]

for \( \lambda \in D \). Hence

\[
\int (e(du)(\bar{u}I - \Lambda^*)x,g(u)) = (x, \int (uI - \Lambda)e(du)g(u)) = 0,
\]

for \( g \in \mathcal{H} \). Therefore

\[
((\cdot)(f(\cdot)I - f(\Lambda)) + \Lambda^* f(\Lambda), g(\cdot))
\]

\[
= \int (e(du)f(u), ug(u)) + \int e(du)(\bar{u}I - \Lambda^*)f(\Lambda), g(u))
\]

\[
= (f, Sg).
\]

Hence

\[
S^*f = P((\cdot)(f(\cdot)I - f(\Lambda)) + \Lambda^* f(\Lambda))
\]

for \( f \in \mathcal{H} \), where \( P \) is the projection from \( L^2(\mathcal{E}) \) to \( \mathcal{H} \). Similar to the calculation in (27), we may prove that as a vector in \( L^2(\mathcal{E}) \)

\[
\bar{u}(f(u)I - f(\Lambda)) + \Lambda^* f(\Lambda) = \Lambda^* f(u) + C(uI - \Lambda)^{-1}(f(u) - f(\Lambda)),
\]
if \( f(\cdot) = (\lambda - (\cdot))^{-1}x \) for \( \lambda \in D \) and \( x \in M \). Hence the vector \((\cdot)(f(\cdot)I - f(\lambda)) + \Lambda^*f(\lambda) \in \mathcal{H}\) for \( f \in \mathcal{H} \) which proves (42). From (41) and (42), it follows that

\[
\]

Identify the vector \( \alpha \in M \) with the constant function \( \alpha \in \mathcal{L}^2(e) \) and consider \( M \) as a subspace of \( \mathcal{L}^2(e) \), then \([S^*, S]M = M\) since \( C \) is positive. Hence \( S \) is pure.

Thus the operator \( S \) and \( S^* \) in (41) and (42) respectively are called the analytic model of a subnormal operator and its adjoint. From now one, in this paper, we always assume the pure subnormal and its adjoint are in the form of analytic model and when we use the formulas in Theorem 1 and its proof we always omit the operator \( U \) and \( U^{-1} \) appeared there.

It is easy to see that the operator \( S' \) on \( \mathcal{H}' \) is also a subnormal operator, it is pure if \( S \) is pure and its minimal normal extension of \( S' \) is \( N^* \). The subnormal operator \( S' \) is said to be the conjugate of \( S \). Now let us calculate the determining function \( S'(z, w), z, w \in \rho(S') \) of \( S' \).

First of all, it is easy to see that

\[
\int |u|^2 e(du) = C^2 + \Lambda \Lambda^*\tag{44}
\]

by (14) and (15). From (14) and (44), it follows that

\[
\|((\cdot)I - \Lambda^*)a\| = \|C^{\frac{1}{2}}a\|.
\]

Hence the operator \( \Omega \) defined by

\[
\Omega a = ((\cdot)I - \Lambda^*)C^{-\frac{1}{2}}a, \quad \text{for} \quad a \in C^{\frac{1}{2}}M
\]

is an isometry and it extends a unitary operator from \( M \) onto \([S'^*, S']M' = M'\). Let \( Q' \) be the injection from \( M' \) into \( \mathcal{H}' \). Then by definition

\[
S'(z, w) = Q'^* (\overline{w}I - S'^*)(zI - S')^{-1}Q'. \tag{45}
\]

It is easy to see that

\[
[S'^*, S']\Omega a = \Omega Ca, \quad \text{for} \quad a \in M,
\]

by (35). Hence the operator \( C' \) corresponding to \( S' \) is

\[
C' = Q'^*[S'^*, S']|_{M'} = \Omega C\Omega^{-1} \tag{46}
\]
Let us calculate \( (\Omega^* S'^* |_{M'})^* \). For \( a \in C^\frac{1}{2} M \), from (31), it follows that

\[
S'^* \Omega a = ((\bar{u} I - \Lambda^*) u - C) C^{-\frac{1}{2}} a = \Omega C^\frac{1}{2} \Lambda C^{-\frac{1}{2}} a
\]

by (15). Hence \( C^{\frac{1}{2}} \Lambda C^{-\frac{1}{2}} \) extends a bounded operator on \( M \) and

\[
\Omega^* S'^* |_{M'} = \Omega C^\frac{1}{2} \Lambda C^{-\frac{1}{2}} \Omega^{-1}.
\]  

(47)

Applying Lemma 1 to operator \( S' \) and using (45), (46) and (47), we get

\[
S'(z, w) = \Omega(((\bar{u} I - C^\frac{1}{2} \Lambda C^{-\frac{1}{2}})(z I - (C^\frac{1}{2} \Lambda C^{-\frac{1}{2}})^*) - C)^{-1} \Omega^{-1}
\]

(48)

for \( z, w \in \rho(S') \). Hence, for symmetry, we redefine the determining function of \( S' \) as

\[
S'(z, w) = \Omega^{-1} Q'^* ((\bar{w} I - S'^*)^{-1}(z I - S')^{-1} Q' \Omega
\]

(49)

then

\[
S'(z, w) = ((\bar{w} I - \Lambda'^*)(z I - \Lambda') - C)^{-1}
\]

(50)

by (48) and (49), where

\[
\Lambda' = (C^\frac{1}{2} \Lambda C^{-\frac{1}{2}})^*.
\]

(51)

From (49) and (51) it follows that

\[
\int \frac{C^{-\frac{1}{2}}(u I - \Lambda)e(du)(\bar{u} I - \Lambda^*) C^{-\frac{1}{2}}}{(u - z)(\bar{u} - \bar{w})} = ((z I - C^\frac{1}{2} \Lambda C^{-\frac{1}{2}})(\bar{w} I - (C^\frac{1}{2} \Lambda C^{-\frac{1}{2}})^*) - C)^{-1}.
\]

(52)

§3. For the subnormal operator \( S \), define an \( \mathcal{L}(M \rightarrow M) \)-valued analytic function

\[
\mu(z) = \int (u - z)^{-1}(u I - \Lambda)e(du), \quad \text{for } z \in C \setminus \sigma(N).
\]

This function \( \mu(\cdot) \) is said to be the mosaic of \( S \).

**Lemma 3.** Let \( z \notin \sigma(N) \), then for every \( f \in \mathcal{H} \), there is a unique vector in \( M \) which is denoted by \( f(z) \) and is said to be the value of \( f \) at \( z \) such that

\[
\frac{f(\cdot) - f(z)}{\cdot - z} \in \mathcal{H} \quad \text{and} \quad \frac{f(z)}{\cdot - z} \in \mathcal{H}.
\]

(53)
The value of a vector possesses the following properties:

(i) \((\alpha f + g\beta)(z) = \alpha f(z) + \beta g(z)\), for \(f, g \in \mathcal{H}\) and \(\alpha, \beta \in \mathbb{C}\),

(ii) \((\lambda I - S)^{-1}f(z) = (\lambda - z)^{-1}f(z)\) for \(\lambda \in \rho(S), f \in \mathcal{H}\),

(iii) if \(\{f_n\} \subset \mathcal{H}, f \in \mathcal{H}\) and \(\|f_n - f\| \to 0\), then

\[\|f_n(z) - f(z)\| \to 0.\]

(iv) \(f(z) = \mu(z)f(z)\), \(\text{for } f \in \mathcal{H}\).

and

(v) \(f(z)\) is an analytic function of \(z \in \mathbb{C}\backslash\sigma(N)\),

\[f(z) = \int (u - z)^{-1}(uI - \Lambda)e(du)f(u), \quad \text{for } f \in \mathcal{H}.\] \hspace{1cm} (54)

PROOF. Let

\[f_1 = P(N - zI)^{-1}f \quad \text{and} \quad f_2 = P'(N - zI)^{-1}f,
\]

then \(f = (N - zI)f_1 + (N - zI)f_2 = ((S - zI)f_1 + Af_2) + (S' - zI)f_2\) by (3) and (4). From (20), (31) and (33), it follows that

\[f(u) = (u - z)f_1(u) + \int (uI - \Lambda)e(du)f_2(v)\] \hspace{1cm} (55)

and

\[(u - z)f_2(u) - \int (uI - \Lambda)e(du)f_2(v) = 0.\] \hspace{1cm} (56)

Denote \(f(z) = \int (uI - \Lambda)e(du)f_2(u)\), then (55) and (56) imply (53), which proves the existence of \(f(z)\) satisfying (53).

If there is a vector \(a \in \mathcal{M}\) satisfying

\[\frac{f(\cdot) - a}{(\cdot) - z} \in \mathcal{H} \quad \text{and} \quad \frac{a}{(\cdot) - z} \in \mathcal{H}'.\] \hspace{1cm} (57)

then from (53) and (57), it follows that

\[\frac{f(z) - a}{(\cdot) - z} \in \mathcal{H} \cap \mathcal{H}'.\]

that means \(f(z) = a\) which proves the uniqueness of \(f(z)\).
From (56), it follows that
\[ f_2(u) = (u - z)^{-1} f(z). \]
Thus
\[ f(z) = \int (uI - \Lambda) e(du) f_2(u) = \int (u - z)^{-1} (uI - \Lambda) e(du) f(z) \]
which proves (iv).

It is obvious that
\[ \int \frac{uI - \Lambda}{u - z} e(du) (f(u) - f(z)) = 0 \quad \text{for } f \in \mathcal{H}, \tag{58} \]
since
\[ \left( \int \frac{uI - A}{u - z} e(du) (f(u) - f(z)), b \right) = \left( \frac{f(\cdot) - f(z)}{\cdot - z}, ((\cdot)I - \Lambda^*) b \right) = 0 \]
for every \( b \in M \) by (53). From (58) and (iv), it follows (54). (i) and (iii) are direct consequence of (v).

To prove (ii), note that
\[ \frac{(\lambda - (\cdot))^{-1} f(\cdot) - (\lambda - z)^{-1} f(z)}{(\cdot) - z} = (\lambda - (\cdot))^{-1} \frac{f(\cdot) - f(z)}{(\cdot) - z} \]
and \((\lambda - z)^{-1}((\cdot) - z)^{-1} f(z) \in \mathcal{H}'\) by (53). Therefore (iii) holds.

Denote \( M_z = \{ b \in M : ((\cdot) - z)^{-1} b \in \mathcal{H} \} \) and \( M'_z = \{ b \in M : ((\cdot) - z)^{-1} b \in \mathcal{H}' \} \).

**THEOREM 3.** Let \( \mu(\cdot) \) be the mosaic for the subnormal operator \( S \), and \( z \in \mathbb{C} \setminus \sigma(N) \). Then

(i) \( \mu(z) = \mu(z)^2 \),

(ii) \( \mu(z)M = M'_z \),

(iii) \( (I - \mu(z))M = M_z \),

(iv) for every \( a \in M \), \( \mu(z)a \) is the value of \( a \) at \( z \); and

(v) \( M'_z \) is the space of values at \( z \) of all vectors in \( \mathcal{H} \).
PROOF. For every $a$, let $a(z)$ be the value of $a$ at $z$ in the sense in Lemma 3, then $a(z)$ is the unique vector satisfying
\[ a(z) \in M'_z \quad \text{and} \quad a - a(z) \in M_z \]
by (53). Thus
\[ M = M'_z + M_z, \]
and $a \mapsto a(z)$ and $a \mapsto a - a(z)$ are parallel projections from $M$ onto $M'_z$ and $M_z$ respectively.

By Lemma 3, (iv), it is obvious that
\[ a(z) = \mu(z)a(z), \quad \text{for} \quad a \in M. \]

On the other-hand, we have
\[ (v - z)^{-1}(a - a(z)) \in \mathcal{H}, \]
Since $a - a(z) \in M_z$. Therefore
\[ (\mu(z)(a - a(z)), b) = \left( \frac{a - a(z)}{v - z}, ((i)I - \Lambda')b \right) = 0 \]
for every $b \in M$. Hence
\[ \mu(z)a = \mu(z)a(z) = a(z). \]

Thus $\mu(z)$ is the parallel projection from $M$ onto $M'_z$.

Therefore (i)-(iv) in this lemma are proved. (v) of Lemma 4 is an immediate consequence of Lemma 3, (iv) and Lemma 4 (iv).

Let $B$ be an operator on $\mathcal{H}$. If \( \{f_j : j \in J\} \) is a set of vectors in $\mathcal{H}$ satisfying the condition that the set \( \{(\lambda I - B)^{-1}f_j : j \in J, \lambda \in \rho(B)\} \) spans $\mathcal{H}$, then \( \{f_j : j \in J\} \) is said to be a set of generators for $B$. The smallest cardinal number of the set of generators for $B$ is said to be the multiplicity of $B$. If the set of generators is a single set \( \{f\} \) then $B$ is said to be cyclic and $f$ is said to be a cyclic vector for $B$.

COROLLARY 3 Let $S$ be a subnormal operator in $\mathcal{H}$ with minimal normal extension $N$. If the multiplicity of $S$ is $\mu$ then $\text{rank}(\mu(z)) \leq n$ for all $z \in C\setminus\sigma(N)$, where $\mu(\cdot)$ is the mosaic of $S$.

PROOF. We use the analytic model of $S$. Let \( \{f_j : j \in J\} \) be the set of generators for $S$ with cardinal number $n$ and $M(z)$ be the subspace of $M$ spanned
by \( \{f_j(z); j \in J\} \). Then \( \dim M(z) \leq n \). By means of Lemma 3 (i), (ii), if \( h \) is a linear combination of vectors \( (\lambda_j I - S)^{-1} f_j; \lambda \in \rho(S) \), then

\[
h(z) \in M(z). \tag{59}
\]

Since \( \{f_j; j \in J\} \) is a set of generators, for every vector \( f \in \mathcal{H} \) and natural number \( m \), there is a vector \( h_m \) which is a linear combination of vectors \( (\lambda_j I - S)^{-1} f_j; \lambda_j \in \rho(S) \) such that \( \|f - h_m\| < \frac{1}{m} \). From (59), it follows that

\[
h_m(z) \in M(z).
\]

By means of Lemma 3, (iii), \( \|f(z) - h_m(z)\| \to 0 \). Hence \( f(z) \in M(z) \). Thus \( M'_z \subset M(z) \) by Theorem 3 (v), which proves the Corollary 3.

For the space \( \mathcal{H}' \) and operator \( S' \), we have the following Lemma 4.

**Lemma 4.** Let \( z \notin \sigma(N) \), then for every \( f \in \mathcal{H}' \), there is a unique vector \( m_f \in M \) such that

\[
((\cdot) - \bar{z})^{-1}((\cdot)I - \Lambda^*)m_f \in \mathcal{H} \quad \text{and} \quad ((\cdot) - \bar{z})^{-1}(f(\cdot) - ((\cdot)I - \Lambda^*)m_f) \in \mathcal{H}'. \tag{60}
\]

The proof of Lemma 4 is similar to that of Lemma 3. We only point out that the vector \( m_f \) in (60) is

\[
m_f = f_1(\Lambda)
\]

where \( f_1 = P(\mathcal{N}^* - \bar{z}I)^{-1} f \), and omit the details of the proof.

Let

\[
M^*_z = \{a \in M : ((\cdot) - \bar{z})^{-1}((\cdot)I - \Lambda^*)a \in \mathcal{H}\}
\]

and

\[
M'^*_z = \{a \in M : ((\cdot) - \bar{z})^{-1}((\cdot)I - \Lambda^*)a \in \mathcal{H}'\}.
\]

By means of Lemma 4 and the similar method in the proof of Theorem 3, we may prove the following lemma.

**Lemma 5.** Let \( z \notin \sigma(N) \), then operators \( \mu(z)^* \) and \( I - \mu(z)^* \) are parallel projections from \( M \) onto \( M^*_z \) and \( M'^*_z \) respectively.

**Proof.** By Lemma 4, for every \( a \in M \), there is a vector \( m_f \in M \) such that (60) holds where

\[
f(\cdot) = ((\cdot)I - \Lambda^*)a.
\]
Since \((\xi - \bar{z})^{-1}(\xi I - \Lambda^*)m_f \in \mathcal{H}\) and \((\xi I - \Lambda^*)b \in \mathcal{H}'\) for \(b \in M\), we have

\[
\left(\int (\bar{u} - \bar{z})^{-1}(u I - \Lambda) e(du)(\bar{u} I - \Lambda^*)m_f, b \right) = ((\xi - \bar{z})^{-1}(\xi I - \Lambda^*)m_f, (\xi - \Lambda^*)b) = 0,
\]

for \(b \in M\), i.e.

\[
\int (\bar{u} - \bar{z})(u I - \Lambda)e(du)(\bar{u} I - \Lambda^*)m_f = 0.
\]

Hence

\[
C\mu(z)^*m_f = \int (\bar{u} - \bar{z})^{-1}(\bar{u} I - \Lambda^*)(\bar{u} I - \Lambda)e(du)(\bar{u} I - \Lambda^*)m_f
= \int (\bar{u} I - \Lambda)e(du)(\bar{u} I - \Lambda^*)m_f
+ (\bar{z} I - \Lambda^*) \int (\bar{u} - \bar{z})^{-1}(u I - \Lambda)e(du)(\bar{u} I - \Lambda^*)m_f
= Cm_f.
\]

Therefore \(m_f = \mu(z)^*m_f\). On the other-hand

\[
(\mu(z)^*(a - m_f), b) = ((\xi - \bar{z})^{-1}(f(\xi) - (\xi I - \Lambda^*)m_f), b) = 0
\]

for \(b \in M\), by (60). Therefore

\[
m_f = \mu(z)^*m_f = \mu(z)^*a.
\]

Thus

\[
\mu(z)^*a \in M_z^* \quad \text{and} \quad (I - \mu(z)^*)a \in M_z^*
\]

for \(a \in M\) which proves the lemma.

For a pure subnormal operator \(S\), let

\[
\tau_p(S) = \{z \in \mathbb{C} : [S - z I]\mathcal{H} \cap [S^*, S]^{\frac{1}{2}} \mathcal{H} \neq \{0\}\}.
\]

**THEOREM 4.** Let \(S\) be a pure subnormal operator, \(N\) its minimal normal extension, and \(\mu(z), z \in \mathbb{C}\backslash \sigma(N)\) be the mosaic of \(S\). Then

\[
\sigma_p(S^*)\backslash \sigma(N^*) = \{z \in \mathbb{C}\backslash \sigma(N^*) : \mu(z) \neq 0\},
\]

(61)
and if $\bar{z} \in \sigma_p(S^*)\setminus \sigma(N^*)$ then the eigen space of $S^*$ corresponding to eigenvalue $\bar{z}$ is

$$R_{\bar{z}}^* = \{((\cdot) - \bar{z})^{-1}((\cdot)I - \Lambda^*)\mu(z)^*a : a \in M\},$$

where $f(\Lambda) = \int e(du)f(u) = \mu(z)^*f(\Lambda)$.

Moreover,

$$\tau_p(S')\setminus \sigma(N) = \{z \in C\setminus \sigma(N) : \mu(z) \neq I\}$$

and

$$R_{\bar{z}}^* \oplus R_{\bar{z}}'^* = (N^* - \bar{z}I)^{-1}[S'^*, S']^{1/2}\mathcal{H}'.$$

for $z \notin \sigma(N)$, where

$$R_{\bar{z}}'^* = \{((\cdot) - \bar{z})^{-1}((\cdot)I - \Lambda^*)(I - \mu(z)^*)a : a \in M\} = (S' - \bar{z}I)^{-1}[S'^*, S']^{1/2}\mathcal{H}'.$$

PROOF. Let $S_{\bar{z}} = S - zI$ and $f \in \ker(S_{\bar{z}}^*)$, then

$$f(u) = (\bar{u} - \bar{z})^{-1}(\bar{u}I - \Lambda^*)f(\Lambda)$$

where

$$f(\Lambda) = \int e(du)f(u) = \mu(z)^*f(\Lambda).$$

If $\bar{z} \in \sigma_p(S)\setminus \sigma(N)$, then we may choose $f \neq 0$, so that $f(\Lambda) \neq 0$ and hence $\mu(z) \neq 0$. In this case $\ker(S_{\bar{z}}^*)$ is exactly the set (62).

Conversely, if $\mu(z) \neq 0$, then choose $a \in \mu(z)^*M$, $a \neq 0$, and

$$f(u) = (\bar{u} - \bar{z})^{-1}(\bar{u}I - \Lambda^*)a.$$ 

It shows that $f \neq 0$ and $f \in \ker(S_{\bar{z}}^*)$. Therefore $\bar{z} \in \sigma_p(S)\setminus \sigma(N)$.

The next step is to show that

$$[S'^*, S']^{1/2}\mathcal{H}' = \{((\cdot)I - \Lambda^*)a : a \in M\}.$$  

From (46) it is easy to see that

$$[S'^*, S']^{1/2}\mathcal{H}' = C'^{1/2}M' = \Omega C^{1/2}M$$

which proves (66).

If $\bar{z} \in \tau_p(S')$ and $f \in \mathcal{H}'$, $f \neq 0$, satisfies

$$(S' - \bar{z}I)f \in [S'^*, S']^{1/2}\mathcal{H}'$$
then there is a vector \( a \in M, a \neq 0 \), such that
\[
(S' - \bar{z}I)f = ((\cdot)I - \Lambda^*)a.
\] (67)

by (66). From (33) and (66), it follows that \( a \in M'_z \). Thus \( a \in (I - \mu(z)^*)M \) by Lemma 5. This means \( \mu(z) \neq I \). Hence
\[
\tau_p(S') \sigma(N) \subset \{ \bar{z} \in \mathbb{C} \setminus \sigma(N) : \mu(z) \neq I \}.
\] (68)

Besides, we have also proved that
\[
(S' - \bar{z}I)^{-1}[S'^*, S']^{1/2} \mathcal{H} \subset \{((\cdot) - \bar{z})^{-1}(\cdot) - \Lambda^*)(I - \mu(z)^*)a : a \in M \}.
\] (69)

The opposite inclusive relations for (68), (69) may be proved similarly. Thus (65) and (63) hold.

COROLLARY 4. Under the condition of Theorem 4, if \( z \not\in \sigma(N) \) and
\[
\nu_p(S) = \{ z \in \mathbb{C} : (S - zI)\mathcal{H} \cap [S*, S] \mathcal{H} \neq \{0\} \}
\]
introduce an inner product
\[
\left( \int \frac{(uI - \Lambda)\epsilon(du)(\bar{u}I - \Lambda^*)}{|u - z|^2} a, b \right), \quad a, b \in M
\] (70)
to \( M \) then
\[
M = M_z^* \oplus M'_z
\]
with respect the inner product (70).

For a pure subnormal operator \( S \), let
\[
\nu_p(S) = \{ z \in \mathbb{C} : (S - zI)\mathcal{H} \cap [S*, S] \mathcal{H} \neq \{0\} \}
\]

For a set \( F \) in \( \mathbb{C} \). Let \( F^* = \{ \bar{z} \in \mathbb{C} : z \in F \} \).

THEOREM 5. Under the condition of Theorem 4,
\[
\sigma_p(S'^*) \setminus \sigma(N) = \sigma_p(S^*) \setminus \sigma(N)
\] (71)

and
\[ \nu_p(S) \sigma(N) = \tau_p(S')^* \sigma(N). \quad (72) \]

Let \( R_z = (S - zI)^{-1}[S^*, S]^H \) and \( R'_z = \ker(S'^* - zI) \) then
\[ R_z = \{(i - z)^{-1}(I - \mu(z))a : a \in M\}, \quad (73) \]
\[ R'_z = \{(i - z)^{-1}\mu(z)a : a \in M\}, \quad (74) \]
and
\[ R_z \oplus R'_z = (N - zI)^{-1}[S^*, S]^H. \quad (75) \]

**PROOF.** By the method of proving (62) and (65), we may prove (73), (74),
\[ \sigma_p(S'^*) \sigma(N) = \{z \in C \sigma(N) : \mu(z) \neq 0\} \]
and
\[ \nu_p(S) \sigma(N) = \{z \in C \sigma(N), \mu(z) \neq I\} \]
which also proves (71) and (72) by (61) and (63). The proof of (75) is similar to the proof of (64).

**COROLLARY 5.** Under the condition of Theorem 4,
\[ \dim(\ker(S^*_z)) = \text{rank } (\mu(z)^*), \]
and
\[ \dim(\ker(S'^*_z)) = \text{rank } (\mu(z)) \]
for \( z \in \sigma(S) \sigma(N) \).

Hence
\[ \dim(\ker(S^*_z)) + \dim(\ker(S'^*_z)) = \dim[S^*, S]^H. \]

**COROLLARY 6.** Under the condition of Theorem 4,
\[ \rho(S) = \{z \in C \sigma(N) : \mu(z) = 0\}. \quad (76) \]
\[ \sigma(S) = \sigma(N) \cup \sigma_p(S'^*) \quad (77) \]
and
\[ \sigma(S'^*) = \sigma(S) \quad (78) \]
where \( S' \) is the conjugate of \( S \).
\textbf{PROOF.} First, we have to prove that
\begin{equation}
\{z \in C \setminus \sigma(N) : \mu(z) = 0\} \subset \rho(S). \tag{79}
\end{equation}
If fact, if \( z \in C \setminus \sigma(N) \) and \( \mu(z) = 0 \), then \( f(z) = 0 \) for all \( f \in \mathcal{H} \) by Theorem 3 (v) and hence
\begin{equation}
g(\cdot) = (\cdot - z)^{-1}f(\cdot) \in \mathcal{H} \tag{80}
\end{equation}
by (53). Thus for every \( f \in \mathcal{H} \), there is a \( g \in \mathcal{H} \) in (80) such that
\[(S - zI)g = f \]
i.e. \((S - zI)\mathcal{H} = \mathcal{H} \). On the other-hand \( \sigma_p(S) = \phi \) since \( S \) is pure. Thus \( z \in \rho(S) \), which proves (79).

On the other-hand, if \( z \in \rho(S) \), then \( z \in \sigma(N) \), besides,
\[(\cdot - z)^{-1}f(\cdot) = ((S - zI)^{-1}f)(\cdot) \in \mathcal{H} \]
Hence \( f(z) = 0 \) for every \( f \in \mathcal{H} \). That means \( \mu(z) = 0 \) by Theorem 3 (v), which proves (76).

Since
\[\sigma_p(S^*)^* = \{z \in C \setminus \sigma(N) : \mu(z) \neq 0\}\]
by (61), (76) implies (77). Finally, (78) is a direct consequence of (71) and (77).

By this lemma, for every \( f \in \mathcal{H} \),
\[f(z) = 0, \quad \text{if} \quad z \in \rho(S).\]

\textbf{COROLLARY 7.} Under the conditions of Theorem 4 the boundary of \( \sigma(S) \) must be a subset of \( \sigma(N) \).

\textbf{PROOF.} Suppose on contrary, there is a boundary point \( z \in C \setminus \sigma(N) \) of \( \sigma(S) \). The function \( \mu(\cdot) \) is regular at \( z \), since \( z \in C \setminus \sigma(N) \). There is a sequence \( \{z_n\} \subset \rho(S) \) such that \( z_n \rightarrow z \), since \( z \) is a boundary point of \( \sigma(S) \). Hence
\[\mu(z) = \lim_{n \rightarrow \infty} \mu(z_n) = 0.\]
Thus \( z \not\in \sigma(S^*_p)^* \). Therefore \( z \in \rho(S) \) which contradicts the fact that \( z \) is a boundary point of \( \sigma(S) \). Hence, the corollary is proved.
The operator function $\mu(\cdot)$ is a unitary invariant, i.e. if $S$ and $S_1$ are two subnormal operators on $\mathcal{H}$ and $\mathcal{H}_1$ with minimal normal extension $N$ and $N_1$ respectively and there is a unitary operator $U$ from $\mathcal{H}$ on to $\mathcal{H}_1$ such that

$$S_1 = USU^{-1} \quad \text{and} \quad N_1 = UNU^{-1} \tag{81}$$

then there is a unitary operator $V$ from $M = [S^*, S]\mathcal{H}$ on to $M_1 = [S_1^*, S_1]\mathcal{H}_1$ such that the mosaics $\mu(\cdot)$ and $\mu_1(\cdot)$ of $S$ and $S_1$ respectively, are unitarily equivalent, i.e.

$$\mu_1(z) = V\mu(z)V^{-1}, \quad \text{for} \quad z \in \mathbb{C}\setminus\sigma(N). \tag{82}$$

In fact, we may choose $V = U|_M$.

Under certain condition, the converse of above statement is also true, i.e. $\mu(\cdot)$ is a completely unitary invariant in the following sense.

**THEOREM 6.** Let $S$ and $S_1$ be two subnormal operators on $\mathcal{H}$ and $\mathcal{H}_1$ with minimal normal extension $N$ and $N_1$ respectively. Suppose $\sigma(N) = \sigma(N_1)$ has no interior point. Let $\mu(\cdot)$ and $\mu_1(\cdot)$ be the mosaics of $S$ and $S_1$ respectively. If there is a unitary operator $V$ from $M = [S^*, S]\mathcal{H}$ onto $M_1 = [S_1^*, S_1]\mathcal{H}_1$ such that (82) holds, then there is a unitary operator $U$ from $\mathcal{H}$ on to $\mathcal{H}_1$ such that (81) holds.

**PROOF.** For simplicity, we only consider the analytic model of $S$ and $S_1$. The operators and operator-valued measures in Theorem 2 corresponding to $S$ and $S_1$ are $C, \Lambda, \varepsilon(\cdot)$ and $C_1, \Lambda_1, \varepsilon_1(\cdot)$ respectively. From (82) it follows

$$\int_{\sigma(N)} \frac{(uI - \Lambda_1)e_1(du)}{u - z} = V \int_{\sigma(N)} \frac{(uI - \Lambda)e(du)}{u - z} V^{-1}. \tag{83}$$

Since $\sigma(N)$ has no interior point, every continuous function $f(\cdot)$ on $\sigma(N)$ may approximated by linear combination of functions

$$((\cdot) - z)^{-1}, \quad z \in \mathbb{C}\setminus\sigma(N).$$

Thus from (83), it follows

$$\int_{\sigma(N)} f(u)(uI - \Lambda_1)e_1(du) = \Lambda \int_{\sigma(N)} f(u)(uI - \Lambda)e(du)\Lambda^{-1}$$

for every continuous function $f(\cdot)$ on $\sigma(N)$. Hence

$$\int_{\sigma(N)} \frac{(uI - \Lambda_1)e_1(du)}{(u - z)(\overline{u} - \overline{w})} = V \int_{\sigma(N)} \frac{(uI - \Lambda)e(du)}{(u - z)(\overline{u} - \overline{w})} V^{-1} \tag{84}$$
for $z, w$ in the neighborhood 0 of infinity. However

$$
\int \frac{(uI - A)e(du)}{(u - z)(\bar{u} - \bar{w})} = \int \frac{e(du)}{u - w} + (zI - A)S(z, w)
$$

$$
= (\Lambda^* - \bar{w}I) + (zI - A)((\bar{w}I - \Lambda^*)(zI - A) - C)^{-1}
$$

by (2). Hence (84) implies

$$
((\Lambda^*_1 - \bar{w}I) + (zI - \Lambda_1)((\bar{w}I - \Lambda^*_1)((zI - \Lambda_1) - C_1)^{-1})
$$

$$
= V((\Lambda^* - \bar{w}I) + (zI - \Lambda)((\bar{w}I - \Lambda^*)(zI - A) - C)^{-1}V^{-1}. \tag{85}
$$

Putting $w \to \infty$ in (85), we get $\Lambda^*_1 = VA^*V^{-1}$ and $\Lambda^*_1 = VAV^{-1}$. Thus from (85), it follows $C_1 = VCV^{-1}$ which proves Theorem 6.

For $z, w \not\in \sigma(N)$, define

$$
S(z, w) = \int \frac{e(du)}{(u - z)(\bar{u} - \bar{w})}.
$$

It is obvious that, if $z, w \not\in \sigma(S)$ then this function $S(z, w)$ equals the determining function in (1). The following lemma may be useful somewhere else.

**LEMMA 6.** If $z, w \not\in \sigma(N)$, and $(\bar{w}I - \Lambda^*)(zI - A) - C$ is invertible, then

$$
S(z, w) = ((\bar{w}I - \Lambda^*)(zI - A) - C)^{-1} - \mu(w)((\bar{w}I - \Lambda^*)(zI - A) - C)^{-1}.
$$

**PROOF.** By a simple calculation, it is easy to see that

$$
((\bar{w}I - \Lambda^*)(zI - A) - C)S(z, w)
$$

$$
= \int ((\bar{w}I - \Lambda^*)(zI - A) - (\bar{w}I - \Lambda^*)(uI - A))e(du)(u - z)^{-1}(\bar{u} - \bar{w})^{-1}
$$

$$
= I - \mu(z) - \int (\bar{w}I - \Lambda^*)e(du)(\bar{u} - \bar{w})^{-1}, \tag{86}
$$

Since $(\bar{w}I - \Lambda^*)(uI - A)e(du) = Ce(du)$ by (15), and

$$(\bar{w}I - \Lambda^*)(zI - A) - (\bar{w}I - \Lambda^*)(uI - A) = (u - z)(\bar{u} - \bar{w})I - (uI - A)(\bar{u} - \bar{w}) - (\bar{w}I - \Lambda^*)(u - z).
$$

On the other hand, we have

$$
\int \frac{\bar{w}I - \Lambda^*}{\bar{u} - \bar{w}} e(du)((\bar{w}I - \Lambda^*)(zI - A) - C) = ((\bar{w}I - \Lambda^*)(zI - A) - C)\mu(w)^*, \tag{87}
$$

since
\[ \int \frac{\overline{u} I - \Lambda^*}{\overline{u} - \overline{w}} e(du)(\overline{w} I - \Lambda^*) = (\overline{w} I - \Lambda^*)\mu(w)^* \]

and
\[ \int \frac{\overline{u} I - \Lambda^*}{\overline{u} - \overline{w}} e(du)((\overline{w} I - \Lambda^*)\Lambda + C) = \int \frac{\overline{u} I - \Lambda^*}{\overline{u} - \overline{w}} e(du)((\overline{w} - \overline{u})\Lambda + (\overline{u} I - \Lambda^*)u) \]
\[ - ((\overline{w} I - \Lambda^*)\Lambda + C)\mu(w)^*. \]

It is obvious that (86) and (87) imply Lemma 6.

By means of the method of proving (52), we get a formula corresponding to \( S' \):
\[ \int \frac{(u I - \Lambda)e(du)(\overline{u} I - \Lambda^*)}{(u - z)(\overline{u} - \overline{w})} = (1 - \mu(z))C^\frac{1}{2}((z I - C^\frac{1}{2}\Lambda C^{-\frac{1}{2}})(\overline{w} I - (C^\frac{1}{2}\Lambda C^{-\frac{1}{2}})^*) - C)^{-1}C^\frac{1}{2} \]
\[ - C^\frac{1}{2}((z I - C^\frac{1}{2}\Lambda C^{-\frac{1}{2}})(\overline{w} I - (C^\frac{1}{2}\Lambda C^{-\frac{1}{2}})^*) - C)^{-1}C^\frac{1}{2}\mu(w)^*. \]

The following lemma is useful for \( \S 4 \).

**Lemma 7.** Under the conditions of Theorem 1, if \( z \in \rho(\Lambda) \setminus \sigma(N) \), then
\[ [\mu(z), \ C(z I - \Lambda)^{-1} + \Lambda^*] = 0. \quad (88) \]

**Proof.** We only have to prove that
\[ (C + \Lambda^*(z I - \Lambda)) \int \frac{e(du)}{u - z} = (z I - \Lambda) \int \frac{e(du)}{u - z}(C(z I - \Lambda)^{-1} + \Lambda^*). \quad (89) \]

It is easy to see that the left-hand side of (89) equals
\[ \int (u - z)^{-1}((\overline{u} I - \Lambda^*)(u I - \Lambda) + \Lambda^*(z I - \Lambda))e(du) \]
\[ = \int \overline{u}(u - z)^{-1}(u I - \Lambda)e(du) - \Lambda^* \]
\[ = (z I - \Lambda) \int \overline{u}(u - z)^{-1}e(du), \]

by (14) and (15). On the other hand,
\[ \int \frac{e(du)}{u - z}(\overline{u} I - \Lambda^*) = \int \frac{e(du)}{u - z}(\overline{u} I - \Lambda^*)(u I - \Lambda - (u - z)I)(z I - \Lambda)^{-1} \]
\[ = \int \frac{e(du)}{u - z}C(z I - \Lambda)^{-1}. \]
Therefore (89) and hence (88) hold.

§4 The case of rank \([S^*, S] = 2\).

Let \(U\) denote the unilateral shift of multiplicity 1. Morrel shows that every pure subnormal operator \(S\) with rank one self-commutator \([S^*, S]\) must be \(\alpha U + \beta I\) where \(\alpha \neq 0\) and \(\beta\) are scalars. In this section, the form of pure subnormal operator \(S\) with rank two \([S^*, S]\) is determined. The method in this section may be generalized to the case of certain kind of subnormal operators with any finite rank self-commutators.

Let \(T = \{u \in \mathbb{C} : |u| = 1\}\). \(v\) be a measure on \(T_a = T \cup \{a\}\) where \(|a| < 1\) such that

\[
\frac{dv(e^{i\theta})}{2\pi} = d\theta
\]

and \(v(\{a\}) = v_a > 0\), and \(H^2(v)\) be the Hilbert space completion of the space of all polynomials with respect to the inner product

\[
(f, g) = \int_{T_a} f(u)\overline{g(u)}dv(u).
\]

Let \(U_v\) be the operator \((U_v f)(u) = uf(u)\).

**THEOREM 7.** Let \(S\) be a pure subnormal operator in a separable Hilbert space \(\mathcal{H}\) with rank two self-commutator \([S^*, S]\). Then (i) \(S\) is reducible iff \(\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\), \(\mathcal{H}_j\) reduces \(S\) and \(S|_{\mathcal{H}_j} = \alpha_j U + \beta_j, \alpha \neq 0\) and \(\beta_j\) are scalars.

(ii) \(S\) is irreducible and cyclic iff either (iia) there is a \(U_v\) on \(H^2(v)\) and scalars \(\alpha \neq 0\) and \(\beta\) such that \(S\) is unitarily equivalent to \(\alpha U_v + \beta\) or (iib) \(S = \psi(U)\) where

\[
\psi(u) = \frac{\alpha u^2 + \beta u + \gamma}{u + \delta},
\]

(\(\alpha, \beta, \gamma, \delta\) are scalars) is regular and univalent on \(|u| \leq 1\) and is not degenerated to a linear function.

And (iii) \(S\) is irreducible and non-cyclic, iff \(S = \psi(U)\) where \(\psi\) is a function in (90), regular on \(|u| < 1\), non-univalent on \(|u| < 1\) and is not degenerated to a linear function.

**PROOF.** The proof of case (i) and the "if" part in the case (ii) and (iii) is not difficult, we omit the details. So we suppose that \(S\) is irreducible.
**Lemma 8.** Under the conditions of Theorem 1, if $\sigma(S)$ is disconnected, then $S$ is reducible.

**Proof.** Let $D_\infty$ be the unbounded component of $C \setminus \sigma(S)$, then $C \setminus D_\infty$ is the union of two disjoint compact sets $\sigma_1$ and $\sigma_2$ satisfying

$$\sigma_j \cap \sigma(S) \neq \emptyset \quad \text{for } j = 1, 2.$$  

There exists a smooth contour $\gamma$ in $D_\infty$ such that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{d\lambda}{\lambda - u} = 1, \quad \text{for } u \in \sigma_1,$$

and

$$\frac{1}{2\pi i} \int_{\gamma} \frac{d\lambda}{\lambda - u} = 0, \quad \text{for } u \in \sigma_2.$$  

Let $\mathcal{H}_1 = Q \mathcal{H}$ and $\mathcal{H}_2 = (I - Q) \mathcal{H}$ where

$$Q = \frac{1}{2\pi i} \int_{\gamma} (\lambda I - S)^{-1} d\lambda.$$  

is a parallel projection. Then $\mathcal{H}_1 \neq \{0\}$ and $\mathcal{H}_2 \neq \{0\}$. Note that $\mathcal{H}_1 \perp \mathcal{H}_2$, since

$$f(u) = 0 \quad \text{for } u \in \sigma_2 \cap \sigma(N) \quad \text{and} \quad f \in \mathcal{H}_1,$$

and

$$f(u) = 0 \quad \text{for } u \in \sigma_1 \cap \sigma(N) \quad \text{and} \quad f \in \mathcal{H}_2.$$  

Thus $Q$ is an orthogonal projection. From (91), it follows that $\mathcal{H}_1$ is an invariant subspace of $S$. Therefore $\mathcal{H}_1$ reduces $S$, i.e., $S$ is reducible. Lemma 7 is proved.

By Lemma 8, we only have to consider the case of connected $\sigma(S)$.

Since $[S^*, S] \mathcal{H}$ is two dimensional, $M = [S^*, S] \mathcal{H}$ is invertible and $\sigma(N)$ is a subset of algebraic curve

$$L = \{u \in M : \det(C - (uI - \Lambda^*)(uI - \Lambda)) = 0\}.$$  

By adding $aI$ to $S, a \in \mathbb{C}$, we may assume that $0 \in \sigma(\Lambda)$. Choose a basis in $M$, so that

$$\Lambda = \begin{pmatrix} 0 & 0 \\ \lambda & \eta \end{pmatrix}, \quad \lambda, \eta \in \mathbb{C}.$$  

Denote
\[ C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} \]

Then \( L \) is the set of all vectors \( u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \) satisfying

\[(|u|^2 + |\lambda|^2 - C_{11})(|u - \eta|^2 - C_{22}) = |\lambda(u - \eta) + C_{12}|^2. \quad (92)\]

Let \( \gamma \) be the boundary of \( \rho(S) \). Then \( \gamma \subset L \) and \( \gamma \) must be union of finite collection of simple Jordan curves which consists of finite pieces of analytic arcs and at most two points. Let \( u = u(s), 0 \leq s \leq t \) be the equation of \( \gamma \), where \( s \) is the arc length such that \( \rho(S) \) is on the right of \( \gamma \). By Plemelj's formula

\[ \mu(u(s)) = 2\pi i (u(s)I - \Lambda) \frac{de(u(s))}{du(s)} \quad (93) \]

for almost \( s \in [0, t] \), where

\[ \mu(u(s)) = \lim_{z \to u(s)} \mu(z), \]

as \( z \in \sigma(s) \setminus \gamma \) approaches \( u(s) \) along any non-tangential path, since \( \mu(z) = 0 \) for \( z \in \rho(S) \).

Let

\[ D_j = \{ z \in \sigma(s) \setminus \sigma(N) ; \quad \text{rank} \, \mu(z) = j \}, \quad j = 1, 2. \]

Note that \( z \in D_1 \) is equivalent to \( z \in \sigma(S) \setminus \sigma(N) \) and \( \det \mu(z) = 0 \), Therefore

\[ \bar{D}_1 \cap \sigma(S) \setminus \sigma(N) \subset D_1. \quad (94) \]

On the other hand \( z \in D_j \) is equivalent to \( \mu(z) = I \), since \( \dim M = 2 \). Therefore

\[ \bar{D}_2 \cap \sigma(S) \setminus \sigma(N) \subset D_2. \quad (95) \]

Since \( \sigma(S) \setminus \sigma(N) \) is open by Corollary 7, from (94) and (95) it follows that \( D_1 \) and \( D_2 \) are two components of the open set \( \sigma(S) \setminus \sigma(N) \).

There is no open arc in \( \gamma \) which lies on the boundary of \( D_2 \). In fact, if \( u = u(s), a < s < b \), is such arc, then \( \mu(u(s)) = 1 \) for \( a < s < b \) and hence

\[ de(u(s)) = \frac{1}{2\pi i} (u(s)I - \Lambda)^{-1} du(s) \]
for a.e. \( s \in (a, b) \), by (93). But (40) implies that

\[
((u(s)I - \Lambda^*)(u(s)I - \Lambda) - C)(u(s)I - \Lambda)^{-1} = 0 \tag{96}
\]

for a.e. \( s \in (a, b) \). From (96), it follows \( C = \Lambda = 0 \). This is a contradiction.

Thus the boundary \( \gamma_1 \) of \( D_2 \) must be in the boundary of \( D_1 \). Let \( u = u_1(s), 0 \leq s \leq l \) be the equation of \( \gamma_1 \), the direction of \( \gamma_1 \) be chosen such that \( D_1 \) is on the left of \( \gamma_1 \). Let \( \mu(u_1(s)) \) be the limit of \( \mu(z) \) as \( z \) approaches to \( u_1(s) \) in \( D_1 \) along non-tangential path. Then by Plemelj’s formula again.

\[
I - \mu(u_1(s)) = 2\pi i(u_1(s)I - \Lambda) \frac{de(u_1(s))}{du_1(s)} \tag{97}
\]

for a.e. 0 \( \leq s \leq l_1 \), since \( \mu(z) = I \) for \( z \in D_1 \).

From (5), it is easy to see that

\[
(C - (\overline{u}(s)I - \Lambda^*)(u(s)I - \Lambda)) \frac{de(u(s))}{du(s)} = 0. \tag{98}
\]

(93) and (98) imply

\[
(C(u(s)I - \Lambda)^{-1} - (u(s)I - \Lambda^*))\mu(u(s)) = 0, \tag{99}
\]

for a.e. \( s \in [0, l] \).

On the other-hand, for \( z \in D_1 \setminus \sigma(\Lambda) \), from Lemma 6, \( \mu(z) \) is a projection to a 1-dimensional eigenspace of \( C(zI - \Lambda)^{-1} + \Lambda^* \). Let \( \phi(z) \) be the eigenvalue, then

\[
(C(zI - \Lambda)^{-1} - (\phi(z)I - \Lambda^*))\mu(z) = 0. \tag{100}
\]

It is easy to see that \( \phi(z) \) is an analytic function on \( D_1 \setminus \sigma(\Lambda) \) and may have simple pole at the point of \( D_1 \cap \sigma(\Lambda) \). Comparing (99) and (100) we know that the analytic function \( \phi(z) \) has boundary value

\[
\phi(u) = \overline{u} \tag{101}
\]

on \( \gamma \setminus \sigma(\Lambda) \).

Let \( \gamma_0 \) be an analytic arc in \( L \) satisfying the condition that each point of \( \gamma \) is an interior point of \( \gamma_0 \cup D_1 \). Let \( u = u(s), 0 \leq s \leq l \) be the equation of \( \gamma_0 \). If \( u(s) \in \sigma(N) \) then let \( \mu_+(s) \) and \( \mu_-(s) \) be the limit of \( \mu(z) \) as \( z \) approached \( u(s) \) from the left and right along a non-tangential path in \( D_1 \). Then by Plemelj’s formula again

\[
\mu_+(s) - \mu_-(s) = 2\pi i(u(s)I - \Lambda) \frac{de(u(s))}{du(s)}
\]
for a.e. \( s \in [0, l] \). By (5),

\[
(C(u(s)I - \Lambda)^{-1} - (\overline{u(s)}I - \Lambda^*))((\mu_+(s) - \mu_-(s))
\]

\[
= 2\pi i (C - (\overline{u(s)}I - \Lambda^*)(u(s)I - \Lambda)) \frac{d\epsilon(u(s))}{du(s)} = 0.
\]

Let \( \phi_{\pm}(s) \) be the corresponding limits of \( \phi(z) \). Then

\[
(\phi_+(s) - u(s))\mu_+(s) = (\phi_-(s) - \overline{u(s)})\mu_-(s),
\]

Since \( \mu_\pm(\cdot) \) are boundary values of non-zero analytic function on \( D_1, \mu_\pm(s) \neq 0 \) for a.e.s. If \( \phi_-(s) - \overline{u(s)} = 0 \) then \( \phi_+(s) - u(s) = 0 \) and vice versa. If \( \phi_-(s) - \overline{u(s)} \neq 0 \) and \( \phi_+(s) - u(s) \neq 0 \), then

\[
(\phi_+(s) - u(s))^2 \mu_+(s) = ((\phi_+(s) - u(s))\mu_+(s))^2
\]

\[
= ((\phi_-(s) - \overline{u(s)})\mu_-(s))^2
\]

\[
= (\phi_-(s) - \overline{u(s)})^2 \mu_-(s)
\]

Therefore \( \phi_+(s) = \phi_-(s) \) for \( s \in [0, l] \). Since \( C(zI - \Lambda)^{-1} + \Lambda^* \) can not equal to \( \phi(z)I \) on \( \gamma \setminus \sigma(\Lambda) \) identically \( \mu_+(\cdot) \) and \( \mu_-(\cdot) \) are parallel projections onto 1-dimensional eigenspaces corresponding to \( \phi(u(s)) \), so we have

\[
\mu_+(s) = \mu_-(s).
\]

Therefore

\[
(\gamma \setminus \sigma(\Lambda)) \cap \sigma(N) = \emptyset.
\]

If \( D_2 = \phi \) then \( D_1 \) must be a region whose boundary consists of piece-wise analytic Jordan curve and subset \( R \) of \( \sigma(\Lambda) = \{0, \eta\} \).

Suppose the boundary \( \gamma \) of \( D_1 \) consists of an Jordan curve and a point \( \{a\} \subset R \). Then it is easy to see that \( \phi(z) \) is also regular on the boundary \( \gamma \) and has possible simple poles at \( z = \eta \) or \( z = 0 \), since \( \phi(z) \) satisfies the equation

\[
0 = \det(C(zI - \Lambda)^{-1} + \Lambda^* - \phi(z)I)
\]

or more explicitly

\[
(C_{11} \frac{1}{z} + C_{12} \frac{\lambda}{z - \eta} - \phi(z))(C_{22} \frac{1}{z - \eta} + \bar{\eta} - \phi(z))
\]

\[
= (C_{12} \frac{1}{z - \eta} + \bar{\epsilon})(\bar{C}_{12} \frac{1}{z} + C_{22} \frac{\lambda}{z(z - \eta)}).
\]
By (101) the mapping \( w = \phi(z) \) maps the interior \( D_1 \cup \{a\} \) of \( \gamma \) to the exterior \( D_\infty \) of \( \gamma \) and maps \( a \) to \( \infty \). Since \( w = \phi(z) \) is one-to-one from \( \gamma \) to \( \gamma^* \) and \( \phi(\cdot) \) is conformal mapping, so \( w = \phi(u) \) is also one to one from \( D_1 \cup \{a\} \) onto \( D_\infty \).

Let \( z = \psi(\zeta) \) be the conformal mapping from \( |\zeta| < 1 \) onto \( D_1 \cup \{a\} \) satisfying \( \psi(0) = a \). Then \( \psi(\cdot) \) is also regular on \( \{\zeta : |\zeta| = 1\} \) and

\[
\phi(\psi(\zeta)) = \psi(\zeta)
\]

by (101). Define a function

\[
\psi_0(\zeta) = \phi(\psi(\frac{1}{\zeta}))
\]
on \( |\zeta| > 1 \). The function \( \psi_0(\zeta) \) is analytic on \( 1 < |\zeta| < \infty \) with a simple pole at \( \zeta = \infty \) and is a one-to-one mapping from \( |\zeta| > 1 \) onto \( D_\infty \). From (102)

\[
\psi_0(\zeta) = \psi(\zeta)
\]
for \( |\zeta| = 1 \). Therefore \( \psi_0(\zeta) \) is the analytic continuation of \( \psi(\zeta) \) on \( |\zeta| > 1 \). Hence there must be a scalar \( b \neq 0 \) such that

\[
\psi(\zeta) = a + b\zeta
\]

From (102), it follows that

\[
\phi(b\zeta + a) = \frac{1}{\zeta} + a
\]
Therefore

\[
\phi(\zeta) = \frac{\overline{a}u - |a|^2 + |b|^2}{u - a}.
\]

Hence the curve \( \gamma : \phi(\zeta) = \overline{u} \) is a circle

\[
|u - a|^2 = |b|^2.
\]

However, it must be a part of \( L \).

Now, we consider the case \( a = 0 \). All the points \( u : |u| = |b| \) must satisfy (92), i.e.

\[
(|b|^2 + |\lambda|^2 - C_{11})(|b|^2 + |\eta|^2 - C_{22} - 2R\overline{\eta}u) = |C_{12} - \overline{\lambda}|^2 + |\lambda|^2|b|^2 + 2R(C_{12} - \lambda\overline{\eta})\overline{u}u.
\]
Therefore

\[
(|b|^2 + |\lambda|^2 - C_{11})(|b|^2 + |\eta|^2 - C_{22}) = |C_{12} - \overline{\lambda}\eta|^2 + |\lambda|^2|b|^2.
\]
and
\[ \eta(C_{11} - |\lambda|^2 - |b|^2) = \lambda(C_{12} - \bar{\lambda} \eta). \]  
(106)

It is evident that (106) is equivalent to
\[ \eta C_{11} - \lambda C_{12} = \eta |b|^2. \]  
(107)

It is easy to calculate, that in this case the equation (92) of \( L \) may be simplified as
\[ (|u|^2 - |b|)(|u - \eta|^2 + |b|^2 - C_{11} - C_{22}) = 0. \]

Thus if there is another point \( u_0 \in \sigma(N) \setminus \{ u : |u|^2 - |b|^2 = 0 \} \), then \( u_0 \) must satisfy the equation
\[ |u_0 - \eta|^2 + |b|^2 - C_{11} - C_{22} = 0. \]  
(109)

It is obvious that \( |u_0| < |b| \).

Solving \( \mu(\cdot) \) from (100), we find that \( \mu(\cdot) \) has a pole \( u_0 \) in \( |\mu| < |b| \) which belongs to \( \sigma(N) \) and is a root of the polynomial
\[ \bar{\eta}u^2 + (C_{11} + C_{22} - |\eta|^2 - 2|b|^2)u + |b|^2 \eta = 0. \]

Determine \( de(u) \) for \( u = |b|e^{i\theta} \) from (93) and
\[ E(\{u_0\}) = I - \int_{|u| = |b|} de(u) \]

Then it is easy to calculate that \( S \) is unitarily equivalent to the operator \( |b|U_v + a \) where \( v \) has a mass at \( u_0/|b| \).

The situation is similar if \( \eta \) is a simple pole for \( \phi(\cdot) \).

Now, suppose \( 0, \eta \) are simple poles for \( \phi(\cdot) \) and \( \eta \neq 0 \). By the same methods we may prove that there is a function \( \phi(\cdot) \) of the form (90) which is regular and univalent on \( |\zeta| < 1 \) and maps \( |\zeta| < 1 \) onto \( D_1 \cup R \). In this case,
\[ \phi(\psi(\zeta)) = \frac{\bar{\alpha} + \beta \zeta + \bar{\gamma} \zeta^2}{\zeta + \delta \zeta^2} \]  
(110)

and \( S \) is unitarily equivalent to \( \psi(U) \).

If \( D_2 \neq \phi \), then by means of (92) and (97), \( \phi(\cdot) \) may be analytically continued from \( D_1 \) to \( D_2 \) and becomes a double valued analytic function on \( D_2 \). However, by the same method, we still may prove that there exists a function \( \psi(\cdot) \) of the form (90) which is regular and maps \( |\zeta| \leq 1 \) onto a Riemann surface which covers \( \sigma(S) \) and has two sheets on \( D_2 \). In this case \( \phi \) still satisfies equation (110) and \( S \) is also
unitarily equivalent to $\psi(U)$. But it is no longer cyclic by Corollary 3. Theorem 7 is proved.

References


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