ON THE SPECTRUM OF HYPONORMAL OR SEMI-HYPONORMAL OPERATORS

DAOXING XIA

§ 1

Let $\mathcal{H}$ be a complex separable Hilbert space, $\mathcal{L}(\mathcal{H})$ be the algebra of all linear bounded operators in $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called semi-hyponormal [10], [11], if

$$(T^* T)^{1/2} - (TT^*)^{1/2} \succeq 0;$$

and $T$ is called hyponormal, if

$$T^* T - TT^* \succeq 0.$$

If $T$ is semi-hyponormal, then there is an isometric operator $U$ such that $T = U(T^* T)^{1/2}$ [10]. Let $U^{[n]} = U^n$ and $U^{[-n]} = U^* U^n$ for $n = 1, 2, 3, \ldots$. By results in [10], the polar symbols

$$T^\pm = \lim_{n \to \mp \infty} U^{[n]} T U^{[-n]}$$

exist. The operator $T^+$ is normal and the operator $T^-$ is subnormal. However if $U$ is unitary then $T^-$ is also normal.

If $T = X + iY$ is hyponormal, $X$ and $Y$ are self-adjoint, then the symbols [2], [13],

$$T_\pm = \lim_{t \to \pm \infty} e^{iXt} T e^{-iXt}$$

exist and are normal.

We construct the operators

$$T_k = kT_+ + (1 - k) T_-, \quad T^{(k)} = kT^+ + (1 - k) T^-, \quad 0 \leq k \leq 1.$$
It is easy to verify that these operators are normal when the operator $U$ in the polar decomposition $T = U(T^{*}T)^{1/2}$ is unitary in the semi-hyponormal case.

In a previous paper [11], the author proved that if $T$ is in a special subclass of semi-hyponormal operators, then

$$(1) \quad \sigma(T) = \bigcup_{0 \leq k \leq 1} \sigma(T^{(k)}).$$

The aim of the present paper is to prove that (1) is true for all semi-hyponormal operators and

$$(2) \quad \sigma(T) = \bigcup_{0 \leq k \leq 1} \sigma(T^{k}),$$

if $T$ is hyponormal.

§ 2

We shall consider the singular integral model of a hyponormal operator.

**Lemma 1.** [8], [9], [6]. If $T = X + iY$ is completely non-normal hyponormal operator, $X$ and $Y$ are self-adjoint, $\mathcal{B}$ is the $\sigma$-algebra of all Borel sets in $\sigma(X)$, $m$ is the Lebesgue measure on $(\sigma(X), \mathcal{B})$ and $\Omega = (\sigma(X), \mathcal{B}, m)$, then there are an auxiliary complex separable Hilbert space $\mathcal{D}$, a strongly measurable projection-valued function $Q(\cdot)$ with $Q(x) \in \mathcal{L}(\mathcal{D})$, a uniformly bounded strongly measurable $\mathcal{L}(\mathcal{D})$-valued function $z(\cdot)$, $\beta(\cdot)$ on $\Omega = (\sigma(X), \mathcal{B}, m)$ satisfying

$$zQ = Qz = z, \quad \beta Q = Q\beta = \beta, \quad z = z^*, \quad \beta = \beta^*,$$

a unitary operator $W: \mathcal{H} \mapsto \tilde{\mathcal{H}}$, where $\mathcal{H}$ is the Hilbert space of all strongly measurable, square integrable $\mathcal{B}$-valued functions $f$ satisfying $Qf = f$, and an operator $\tilde{T}$ in $\tilde{\mathcal{H}}$,

$$(3) \quad \tilde{(Tf)}(x) = (x + i\beta(x))f(x) + iz(x)P(\beta f), \quad \text{for } f \in \mathcal{H},$$

where

$$P(g) = \text{st-lim} \frac{1}{2\pi i} \int_{0}^{\infty} \frac{g(t)}{t - (s + ie)} ds,$$

such that $T = \tilde{W}\tilde{T}W^{-1}$.

In this case $(W^{-1}T_{h}Wf)(x) = T_{h}(x)f(x)$, where

$$T_{h}(x) = \beta(x) + kz(x)^{2}.$$

Without loss of generality, in the following we shall assume that $Q(\cdot) = I$, since we can use $\tilde{T} \otimes 0 \mid_{\mathcal{H}_{1}}$ instead of $\tilde{T}$, where $\mathcal{H}_{1}$ is the Hilbert space of all strongly measurable and square integrable $\mathcal{B}$-valued functions $f$ satisfying $(I - Q(\cdot))f(\cdot) = f(\cdot)$. 


§ 3

We shall consider the hyponormal case first.

**Theorem 1.** If $T$ is hyponormal, then (2) is true.

**Proof.** (i) Without loss of generality, we assume that $T$ is completely non-normal. By Lemma 1, we also assume that $\mathcal{H} = \mathcal{H}$, $T$ is the singular integral operator (3) and $Q(\cdot) \equiv I$. Let $E = \bigcup_{0 < k < 1} \sigma(T_k)$ and $M = \text{ess sup} \|z(x)\|$. If $\text{dist}(z_0, E) = 0$, then there are a sequence of numbers $k_n, 0 < k_n < 1, k_n \to k_0$, and a sequence of unit vectors $\{f_n\} \subset \mathcal{H}$ such that

$$\|(T_{k_n} - z_0) f_n\| \to 0.$$  

Since $\text{ess sup} \|T_k(x) - T_k'(x)\| \leq M^2|k - k'|$, we have $\|(T_{k_0} - z_0) f_n\| \to 0$. Thus $z_0 \in \sigma(T_{k_0}) \subset E$. Therefore, $E$ is closed.

(ii) Let $z_0 = x_0 + iy_0 \notin E$. We have to prove that there is a number $\delta$,

$$0 < \delta < K\eta^2/(3M^2(1 + 2K))$$

such that

$$\|z(x) (\beta(x) + \gamma(x)\beta/2 - y_0I)\| \leq (K + 1/2)^{-1}$$

for almost all $x \in [x_0 - \delta, x_0 + \delta]$, where $\eta = \text{dist}(z_0, E)$, $K = \eta/(6M^2)$.

For any $f \in \mathcal{H}$, $\|f\| = 1$ and $0 < k < 1$, we have

$$\int_{\sigma(x)} \|(T_k(x)^* - z_0) f(x)\|^2 \, dx = \|(T_k^* - z_0) f\|^2 \geq \eta^2.$$ 

Thus

$$\text{ess sup} \|T_k(x)^* - z_0\| h \geq \eta \|h\|$$

for any $h \in \mathcal{D}$ and $0 < k < 1$. Hence, there is a number $\delta$, satisfying (4), and a null set $F_\delta$ such that

$$\| (\beta(x) + k(z(x) - sI)^2 - y_0I) h \| \geq \frac{2}{3} \eta \|h\|$$

for all

$$x \in [x_0 - \delta, x_0 + \delta] - F_\delta, \quad -K < k < 1 + K, \quad 0 \leq s \leq \delta.$$ 

In this case, we have $(\beta(x) + k(z(x) + sI)^2 - y_0I)^{-1} \in \mathcal{L}(\mathcal{D})$. Hence the spectrum of the self-adjoint operator

$$(z(x) + sI)^{-1}(\beta(x) + (z(x) + sI)^2/2 - y_0I)(z(x) + sI)^{-1}$$
is contained in \((- \infty, -K - 1/2] \cup [K + 1/2, \infty)\) for \(0 < s < \delta\) and \(x \in [x_0 - \delta, x_0 + \delta] - F_{\delta}\). Hence

\[
\| (\alpha(x) + sI)(\beta(x) + (\alpha(x) + sI)^2/2 - y_0I)^{-1}(\alpha(x) + sI)\| \leq (K + 1/2)^{-1}
\]

for \(0 < s < \delta, x \in [x_0 - \delta, x_0 + \delta] - F_{\delta}\). Put \(s \to 0\), in (7), we obtain (5).

(iii) Now we have to prove that \(z_0 \in \rho(T)\). We suppose on the contrary, \(z_0 \in \sigma(T)\). Let \(A = [x_0 - \delta, x_0 + \delta], \mathcal{H}_A = \{ f : f \in \mathcal{H}, f(x) = 0 \text{ for } x \in A \}, P_A\) be the projection \(\mathcal{H} \to \mathcal{H}_A\) and \(T_A = P_AT|_{\mathcal{H}_A}\). It is well known that

\[
\sigma(T_A) \supset \{ \lambda : \text{Re}(\lambda) \in (x_0 - \delta, x_0 + \delta), \lambda \in \sigma(T) \}.
\]

Hence \(z_0 \in \sigma(T_A)\). Since \(\sigma(T_A) = \{ \lambda : \lambda \in \sigma_z(T_A) \} \cup \{ z_0 \} \) \([10]\), where \(\sigma_z(A)\) is the approximate point spectrum of \(A\), there is a sequence of unit vectors \(\{ f_n \} \subset \mathcal{H}_A\) such that

\[
c_n = \| (T_A^* - z_0I)f_n \| \to 0.
\]

Since \(0 \leq P \leq I\), we have

\[
\left\| P(f) - \frac{1}{2} f \right\| \leq \frac{1}{2} \| f \|.
\]

On the other hand

\[
\sigma f_n + \alpha(T_{1/2}(x) - y_0I)^{-1} \alpha \left( P_A \alpha f_n - \frac{1}{2} \alpha f_n \right) =
\]

\[
= i\alpha(T_{1/2}(x) - y_0I)^{-1}[T_A^* - z_0I]f_n - (x - x_0)f_n,
\]

by means of (5)-(8), we have

\[
\| \alpha f_n \|(1 - (1 + 2K)^{-1}) \leq 3M(c_n + \delta)/(2\eta).
\]

Thus

\[
\| (T_A^* - z_0I)f_n \| \geq \| (T_0(x)^* - z_0I)f_n \| - \| \alpha P(z_0)f_n \| \geq
\]

\[
\geq \eta - 3M^2(c_n + \delta)(1 + 2K)/(4K\eta).
\]

Put \(\eta \to \infty\) in (9). We obtain \(\delta \geq K\eta^2/[3M^2(1 + 2K)]\). This contradicts (4). Thus \(z_0 \in \rho(T)\), i.e. \(\sigma(T) \subset E\).

(iv) Let \(z_0 \in \sigma(T_0)\). We have to prove \(z_0 \in \sigma(T)\). We suppose on the contrary that \(z_0 \in \rho(T)\), then there is a positive \(b\) such that

\[
\{ x + iy : |x - x_0| = b, |y - y_0| \leq b \} \subset \rho(T).
\]
Let
\[ L(a) = \operatorname{ess inf}_x \| (T_k(x) - z_0 I) a \|, \quad \text{for} \quad a \in \mathcal{D}. \]

Since
\[ \| (T_k - z_0 I) f \| \geq \inf_{|\alpha| = 1} L(a) \| f \|, \quad \text{for} \quad f \in \mathcal{H}, \]
and \( z_0 \in \sigma(T_k) \), we have \( \inf_{|\alpha| = 1} L(a) = 0 \), i.e. there is a sequence of unit \( \{a_n\} \subset \mathcal{D} \) such that \( L(a_n) \to 0 \). Let \( \{\eta_n\} \) be a sequence of positive numbers such that \( \eta_n \to 0 \) and \( L(a_n) < \eta_n \leq b \). There is a sequence of measurable sets \( \{E_n\} \) in the real line such that \( m(E_n) > 0 \) and
\[
\sup_{x \in E_n} \sqrt{\|x - x_0\|^2 + \|Y(x) a_n\|^2} \leq \eta_n,
\]
where \( Y(x) = \beta(x) + kx(x)^2 - y_0 I \). Evidently \( E_n \subset [x_0 - \eta_n, x_0 + \eta_n] \) and
\[
(11) \quad \sup_{x \in E_n} \| Y(ax) a_n \| \leq \eta_n.
\]

Since \( \alpha(x)a_n \) and \( \alpha(x)^2 a_n \) are strongly measurable vector-valued functions, \( \| \alpha(x)a_n \| \leq M \) and \( \| \alpha(x)^2 a_n \| \leq M^2 \), there is a measurable set \( F_n \subset E_n, m(F_n) > 0 \) and the vectors \( e_n, v_n \in \mathcal{D} \) with \( \|e_n\| \leq M, \|v_n\| \leq M^2 \) such that
\[
(12) \quad \sup_{x \in F_n} \| \alpha(x)a_n - e_n \| \leq \eta_n/(1 + M) \sup_{x \in F_n} \| \alpha(x)^2 a_n - v_n \| \leq \eta_n.
\]

We may suppose that \( \lim_{n \to \infty} \| e_n \| = a, \lim_{n \to \infty} \| v_n \| = a' \) exist. It is obvious that
\[
(13) \quad a^2 \leq a' \quad \text{and} \quad a' \leq Ma.
\]

Since \( F_n \subset [x_0 - \eta_n, x_0 + \eta_n] \) and \( m(F_n) > 0 \) there is an interval
\[
\Delta_n = [x_n - q_n, x_n + q_n] \subset [x_0 - \eta_n, x_n + \eta_n]
\]
such that
\[
(14) \quad m(\Delta_n - F_n) < m(\Delta_n) \eta_n^2.
\]

From (13) and (14), it is easily to verify that
\[
(15) \quad \frac{1}{m(\Delta_n)} \int_{\Delta_n} \| \alpha(x)a_n - e_n \|^2 \, dx \leq (1 + 4M^2) \eta_n^2
\]
and
\[
(16) \quad \frac{1}{m(\Delta_n)} \int_{\Delta_n} \| \alpha(x)^2 a_n - v_n \|^2 \, dx \leq (1 + 4M^4) \eta_n^2.
\]
We now construct an operator $T_n$ in $H_{\Delta_n}$

$$(T_n f) (x) = \left( \frac{x - x_n}{q_n} + i \frac{\beta(x) - y_0}{b} \right) f(x) + i \frac{\alpha(x)}{b} P(\alpha f).$$

By a spectral mapping theorem [4], [13]

$$\sigma(T_n) = \left\{ \left( \frac{x - x_n}{q_n} + i \frac{y - y_0}{b} \right) \left| \begin{array}{c} x + iy \in \sigma(T), \ x \in \Delta_n \end{array} \right\} \right\} \cup \left\{ (\pm 1 + iy) \right\} \quad -\infty < y < \infty \},$$

Thus

$$\{ x + iy \ | \ |x - x_0| < 1, \ |y - y_n| < 1 \} \subset \rho(T_n).$$

From (17), it is obvious

$$(18) \quad \|T_n^* f\| \geq \text{dist}(0, \sigma(T_n)) \|f\| \geq \|f\|.$$ 

Let $\gamma = [-1,1] \mathcal{B}$, be the $\sigma$-algebra of all Borel sets in $\gamma$, $\Omega_t = (\gamma, \mathcal{B}, m)$, $\mathcal{F}$ be the family of all functions in $L^2(\Omega_t)$ satisfying

$$\text{ess sup} \ |h(t)| < \infty, \ \text{ess sup} \ |P(h)| < \infty.$$ 

Evidently, $\mathcal{F}$ is dense in $L^2(\Omega_t)$.

If $f_n(x) = a_n h((x - x_n)/q_n) q_n^{-1/2}$, where $h \in \mathcal{F}$ and $\|h\| = 1$, then $\|f_n\| = 1$.

From (11), (15) and (16), we obtain

$$(19) \quad \lim_{n \to \infty} \|T_n^* f_n\|^2 = \begin{cases} \|th\|^2 & a = a' = 0 \\ \left( \frac{a^2 t}{a'} h - \frac{ia'}{b} (P(h) - k h) \right) \|th\|^2 + \left( 1 - \frac{a^4}{a'^2} \right) \|th\|, & a' > 0. \end{cases}$$

If $a' > 0$, the spectrum of the operator

$$T': h \mapsto \frac{a^2 t + ia^2 k/b}{a'} h - \frac{ia'}{b} P(h)$$

in $L^2(\Omega_t)$ is $\left\{ \frac{a^2 t + ia^2 k/b}{a'} - \frac{ia'}{b} t \in [-1,1], \ y \in [0,1] \right\}$ which contains 0, then

we can choose a sequence $\{h_n\} \subset \mathcal{F}$ such that $\|h_n\| = 1$ and

$$\lim_{n \to \infty} \|T' h_n\| = 0.$$
From (18) and (19), we have $1 \leq 1 - a^4/a^2$; this contradicts to (13).

If $a' = 0$, then $a = 0$ by (13). In this case, (18) and (19) implies

$$\|th\| \geq \|h\|, \quad \text{for } h \in \mathcal{F}.$$  

But it is impossible. Hence $E \subset \sigma(T)$ and (2) is proved.

§ 4

Let us consider the class $H_1$ of all hyponormal operators $X + iY$ with non-negative imaginary parts $Y$ and the class $S_1$ of all semi-hyponormal operator $T = U(T^*T)^{1/2}$ with unitary $U$ satisfying $1 \notin \sigma(U)$.

The mapping

$$L: X + iY \mapsto (X + iI)(X - iI)^{-1}Y$$

is bijective from $H_1$ to $S_1$. The mapping $x + iy \mapsto (x + i)(x - i)^{-1}y$ from the upper half-plane to the complex plane is also denoted by $L$.

**Lemma 2.** [3, 12] Let $R$ be a set in the complex plane, $T(t)$ be an operator-valued function of $t \in [0,1]$ which is continuous with respect to the operator norm, $\{\tau_t, t \in [0,1]\}$ be a family of topological mappings from $R$ to itself such that $\tau_0(z)$ is a continuous function of $t \in [0,1]$ for every $z \in R$. If $\tau_0$ is the identity mapping and

$$\sigma(T(t)) \cap R = \tau_0(\sigma(T(0)) \cap R) \quad \text{for } t \in [0, 1],$$

then

$$\sigma(T(t)) \cap R = \tau_0(\sigma(T(0)) \cap R) \quad \text{for } t \in [0,1].$$

**Theorem 2.** If $T \in H_1$, then

$$L(\sigma(T)) = \sigma(L(T)).$$

**Proof.** Let $R = \{z \mid \text{Im}(z) > 0\}, \quad \phi_t(x) = (1 - itx)/(tx - i), \quad \psi_t(y) = (1 - t^2)/2 + ty,$

$$\tau_t(x + iy) = \begin{cases} \left((\phi_t(x))\psi_t(y) - i/2\right)/t, & 0 < t \leq 1, \\ x + iy, & t = 0, \end{cases}$$

and

$$T(t) = \begin{cases} \left((\phi_t(X))\psi_t(Y) - i/2\right)/t, & 0 < t \leq 1, \\ X + iY, & t = 0, \end{cases}$$

where $X, Y$ are given, $\phi_t$ and $\psi_t$ are analytic in $R$.\]
where $X + iY = T$, $X$ and $Y$ are self-adjoint. In this case

$$\tau_1(x + iy) = -iL(x + iy) - i/2, \quad T(1) = -iL(X + iY) - i/2.$$  

It is easy to verify that $\tau_1$ and $T(t)$ satisfy all the assumptions of Lemma 2 except, (20). Now we have to verify (20).

It is obvious that

$$\|T(t)f - \tau_1(x_0 + iy_0)f\|^2 = \|\psi_\epsilon(Y) - \psi_\epsilon(y_0)f/t\|^2 + 2\psi_\epsilon(y_0)\text{Re}((\psi_\epsilon(Y) - \phi_\epsilon(x_0)\psi_\epsilon(Y)\phi_\epsilon(X)^*)f, f)/t^2.$$  

Since $\text{Re}((1 - \phi_\epsilon(x_0)\phi_\epsilon(X)^*)f, f) \geq 0$ and $\text{Re}((Y - \phi_\epsilon(x_0)Y\phi_\epsilon(X)^*)f, f) \geq 0$ we have

$$\|T(t)f - \tau_1(x_0 + iy_0)f\|^2 \geq \|(Y - y_0I)f\|^2.$$  

If $x_0 + iy_0 \in R$ and $\tau_1(x_0 + iy_0) \in \sigma_q(T(t))$, then there is a sequence of unit vectors $\{f_n\}$ such that $\|T(t)f_n - \tau_1(x_0 + iy_0)f_n\| \to 0$. From (23), we see that

$$\lim_{n \to \infty} \|(Y - y_0)f_n\| = 0.$$  

Then $\|\psi_\epsilon(Y) - \psi_\epsilon(y_0)f_n\| \to 0$ and $\|\phi_\epsilon(X) - \phi_\epsilon(x_0)f_n\| \to 0$. Hence $\|(Y - x_0)f_n\| \to 0$. Thus (20) holds. From (21) we have

$$\sigma(L(T)) \cap R = L(\sigma(T) \cap R).$$  

On the other hand, it is well-known [7] that if a real $x_0 \in \sigma(T)$ then $0 \in \sigma(Y)$ and $0 \in \sigma(L(T))$. Similarly if $0 \in \sigma(L(T))$ then $0 \in \sigma(Y)$ and there is a real $x_0$ such that $x_0 \in \sigma(T)$. Let $R_1$ be the real line, then

$$\sigma(L(T)) \cap R_1 = L(\sigma(T) \cap R_1).$$  

(24) and (25) imply (22).

**Lemma 3.** [13]. If $T = X + iY \in H_1$, then

$$L(T_{\pm}) = (L(T))^\pm.$$  

§ 5

**Theorem 3.** If $T$ is semi-hyponormal then (1) is true.

**Proof.** First we assume that $T = U(T^*T)^{1/2}$ where $U$ is unitary. From Theorem 1, Theorem 2 and Lemma 3, it is easy to prove that (1) is true for $T \in S_2$. Now we consider the general case, $T \notin S_2$.  

Let $\gamma$ be any open arc in the unit circle $C_1 = \{z \mid |z| = 1\}$ satisfying $\gamma \neq C_1$. For simplicity, we suppose that $1 \notin \gamma$. Let

$$U = \int_{C_1} \lambda \mathcal{E}(d\lambda)$$

be the spectral decomposition of the unitary operator $U$, $\mathcal{H}_\gamma = E(\gamma)\mathcal{H}$, $T(\gamma) = E(\gamma)T|_{\mathcal{H}_\gamma}$ and

$$\mathcal{D}_\gamma = \{z \mid z \neq 0, \ z/|z| \in \gamma\}.$$ 

By [10], we have

(27) \hspace{1cm} \sigma(T) \cap \mathcal{D}_\gamma = \sigma(T_\gamma) \cap \mathcal{D}_\gamma.

It is easily to verify that

(28) \hspace{1cm} (T_\gamma)^\pm = E(\gamma)T^\pm|_{\mathcal{H}_\gamma}

and

(29) \hspace{1cm} \sigma(T^{(k)}) = \sigma(T^{(k)}) \cap \mathcal{D}_\gamma.

Since $T_\gamma \in S_1$, we have

(30) \hspace{1cm} \sigma(T_\gamma) = \bigcup_{0 < k < 1} \sigma(T^{(k)})\cdot$

From (1), (27), (29) and (30) we have

$$\sigma(T) \cap \mathcal{D}_\gamma = \bigcup_{0 < k < 1} (\sigma(T^{(k)}) \cap \mathcal{D}_\gamma).$$

Since $\gamma$ is arbitrary, we have

$$\sigma(T) - \{0\} = \bigcup_{0 < k < 1} (\sigma(T^{(k)}) - \{0\}).$$

But it is obvious that $\sigma(T) \cap \{0\} = (\bigcup_{0 < k < 1} \sigma(T^{(k)})) \cap \{0\}$, thus (1) is true when $U$ it unitary.

If the operator $U$ in the polar decompositon $T = U(T^*T)^{1/2}$ is not unitary, by a technique used in [10], we extend $T$ to be an operator $\tilde{T}$ in a larger space such that the corresponding operator $U$ becomes unitary and then $\tilde{T}$ satisfy (1). From this, we can easily verify that $T$ aslo satisfies (1).
Acknowledgments

The author wishes to thank Professors J. D. Pincus and R. G. Douglas for their kind hospitality and useful discussions.

This paper was finished when the author was visiting the Department of Mathematics, State University of New York, Stony Brook, U.S.A.

The author acknowledges partial support of this research from the U.S. National Science Foundation.

REFERENCES

13. Zhang, Yenlan, On the hyponormal operators, J. Fudan University, 18(1979), 76—82.

Daoxing Xia
Research Institute of Mathematics,
Fudan University,
Shanghai,
China.

Received April 4, 1980.