ON THE SEMI-HYPONORMAL n-TUPLE OF OPERATORS

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The theory of singular integral model and trace formula is extended to the context of hyponormal or semi-hyponormal n-tuple of operators. The spectrum of noncommutative n-tuple of operators is examined.

§1. INTRODUCTION

The theory of singular integral model and trace formula of hyponormal operators [9], [10], [12], [16], [17], semi-hyponormal operators [18] or nearly normal operators [2], [12] is now well known; and the theme appears with many variations.

We wish to extend this theory to n-tuple of operators and to establish a theory corresponding to singular integral of multivariables. Certainly some mathematicians have extended the theory in some important cases (cf. [3], [4], [8]). But it seems that the case which we have examined in this paper perhaps is a direct one.

In §2, we give the definition of semi-hyponormal tuple of operators and its general polar symbols. Besides, a special class of singular integral operator in the space of vector-valued square integrable functions is introduced.

In §3, the singular integral model of the semi-hyponormal tuple of operators is established.

In §4, the spectrum of semi-hyponormal tuple of operators is defined. The relation between the spectrum of semi-hyponormal tuple of operators and the joint approximated point spectrum of its general polar symbols is found.

In §5, the trace formula of semi-hyponormal tuple is established under certain conditions. A small part of Pincus' theory of principal function [1], [2], [3], [4] is generalized to
the semi-hyponormal tuple case. Also for the semi-hyponormal tuple, an inequality similar to Putnam's inequality [14] is proved.

In §6, the definitions and theorems corresponding to the hyponormal case are introduced.

§2. DEFINITIONS AND SINGULAR INTEGRAL OPERATORS

In this paper, $\mathcal{H}$ is a separable complex Hilbert space, $\mathcal{L}(\mathcal{H})$ is the algebra of all linear bounded operators in $\mathcal{H}$, $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_n)$ is a commuting $n$-tuple of unitary operators in $\mathcal{H}$, $(\mathcal{U}')$ being the set of all operators in $\mathcal{L}(\mathcal{H})$ which commute $\mathcal{U}$. Let $Q_j$ be the mapping in $\mathcal{L}(\mathcal{H})$ defined by

$$Q_j T = T - \mathcal{U}_j T \mathcal{U}_j^{-1}$$

for all $T \in \mathcal{L}(\mathcal{H})$ and $j = 1, 2, \ldots, n$. It is evident that

$$Q_j Q_k T = Q_k Q_j T.$$

If $A \in \mathcal{L}(\mathcal{H})$, $A \geq 0$ and

$$Q_j \cdots Q_{j_m} A \geq 0$$

for all $1 \leq j_1 < j_2 < \ldots < j_m \leq n$, then $A$ is said to be in the class $\text{SH}(\mathcal{U})$, and the $(n+1)$-tuple $(\mathcal{U}, A)$ is said to be semi-hyponormal. For $n = 1$, $(\mathcal{U}_1 A)$ is semi-hyponormal iff the operator $\mathcal{U}_1 A$ is semi-hyponormal [18]. For fixed $\mathcal{U}$, the linear combination of operators in $\text{SH}(\mathcal{U})$ with nonnegative coefficients is in $\text{SH}(\mathcal{U})$, i.e., $\text{SH}(\mathcal{U})$ is a cone in $\mathcal{L}(\mathcal{H})$. If $A \in \text{SH}(\mathcal{U})$ and $B \in \mathcal{U}'$, then

$$B^* A B \in \text{SH}(\mathcal{U}).$$

A set of operators $\{R_1, \ldots, R_N\}$ in $\text{SH}(\mathcal{U})$ is called a basis of $\text{SH}(\mathcal{U})$ if for every $A \in \text{SH}(\mathcal{U})$ there exist operators $B_1, \ldots, B_N$ in $\mathcal{U}'$ such that

$$A = \sum_{j=1}^{N} B_j^* R_j B_j.$$

Let $\mathcal{J}_j^\pm$ be the set of all operators $T \in \mathcal{L}(\mathcal{H})$, for which

$$\mathcal{J}_j^\pm T = \text{st - lim}_{n \to \infty} \mathcal{U}_j^{-n} T \mathcal{U}_j^n$$

exists. If $T \in \mathcal{J}_j^+ \cap \mathcal{J}_j^-$, then the operators $\mathcal{J}_j^\pm T$ are called the polar symbols [18] of $T$ with respect to $\mathcal{U}_j$. It is evident
that \( T \in \mathcal{F}_{j}^{\pm} \) iff
\[
F_{j}^{-} T = \operatorname{st-lim}_{N \to \infty} \sum_{n=1}^{N} u_{j}^{-n}(Q_{j} T) u_{j}^{n}
\]
e [18], and \( T \in \mathcal{F}_{j}^{-} \) iff
\[
F_{j}^{-} T = \operatorname{st-lim}_{N \to \infty} \sum_{n=0}^{N} u_{j}^{n}(Q_{j} T) u_{j}^{-n}
\]exists. If \( T \in \mathcal{L}(\mathcal{H}) \) and \( Q_{j} T \geq 0 \), then \( T \in \mathcal{F}_{j}^{\pm} \), \( T \preceq \mathcal{F}_{j}^{+} T \) and
\[
T = \mathcal{F}_{j}^{\pm} T + F_{j}^{\pm} T.
\]Thus \( \mathcal{SH}(\mathcal{U}) \subset \mathcal{F}_{j}^{\pm} \), for \( j = 1, 2, \ldots, n \). For simplicity, hitherto \( F_{j}^{-} \) and \( \mathcal{F}_{j}^{-} \) are also denoted by \( F_{j} \) and \( \mathcal{F}_{j} \) respectively.

Let \( \mathcal{B} \) be an auxiliary separable complex Hilbert space, \( T \) be the unit circle \( \{ z : |z| = 1 \} \), \( T^{n} = T_{1} \times \cdots \times T_{n} \), where each \( T_{j} \) is a copy of \( T \), \( \mathcal{B} \) be the \( \sigma \)-algebra of all Borel sets in \( T^{n} \), \( m_{j} \) be the normalized Haar measure in \( T_{j} \), i.e.,
\[
dm_{j}(e^{i\theta_{j}}) = \frac{1}{2\pi} d\theta_{j}, \quad e^{i\theta_{j}} \in T_{j},
\]
and \( \nu_{j} \) be a singular measure on \( T_{j} \). Let \( \mu = \mu_{1} \times \cdots \times \mu_{n} \) and \( m = m_{1} \times \cdots \times m_{n} \), then \( \mu = m+\nu \), where \( \nu \) is a singular measure. Let \( \Omega \) be the measure space \((T^{n}, \mathcal{B}, \mu)\), \( R(\cdot) \) a projection valued function which is defined on \( T^{n} \) and measurable with respect to \( \mathcal{B} \), and \( \hat{\mathcal{H}} = L^{2}(\Omega, \mathcal{B}, R) \) be the Hilbert space of all \( \mathcal{B} \)-valued measurable functions \( f \) satisfying
\[
\|f\|^{2} = \int_{T^{n}} \|f(z)\|_{\mathcal{B}}^{2} d\mu(z) < +\infty
\]
and \( R(z)f(z) = f(z) \) for all \( z \in T^{n} \). Let \( S^{n} \) be the family of all subsets of \( \{1, 2, \ldots, n\} \). For \( \eta \in \mathcal{S}^{n} - \{\{1, 2, \ldots, n\}, \emptyset\} \), let \( T_{j}^{\eta} = \bigotimes_{j \in \eta} T_{j} \), \( \mathcal{B}^{\eta} \) be the \( \sigma \)-algebra of all the Borel sets in \( T_{j}^{\eta} \), \( \mu^{\eta} = \bigotimes_{j \in \eta} \mu_{j} \), \( \Omega^{\eta} = (T_{j}^{\eta}, \mathcal{B}^{\eta}, \mu^{\eta}) \) and
\[
\mathcal{B}^{\eta} = L^{2}(\Omega_{\frac{1}{\eta}}, \mathcal{B})
\]
where \( \frac{1}{\eta} = \{1, 2, \ldots, n\} - \eta \). Then every function \( f \in \mathcal{H} \) may be considered as a \( \mathcal{B}^{\eta} \)-valued function \( f^{\eta} \) on \( T_{j}^{\eta} \) which is measurable with respect to \( \mathcal{B}^{\eta} \) and satisfying
\[
R^{\eta}(z^{\eta})f^{\eta}(z^{\eta}) = f^{\eta}(z^{\eta}), \quad z^{\eta} \in T_{j}^{\eta}
\]
where \( R^{\eta}(z^{\eta}), z^{\eta} \in T_{j}^{\eta} \) is the projection from \( \mathcal{B}^{\eta} \) to the
subspace of all functions \( g(z^5), z^5 \in T^5 \) satisfying
\[
R(z^\eta, z^5)g(z^5) = g(z^5), \quad \xi \in T^5.
\]
If \( \eta = \{1, 2, \ldots, n\} \) then \( \Omega^\eta, \Theta^\eta \) and \( R^\eta \) denote \( \Omega, \Theta \) and \( R \) respectively. Thus \( \hat{\eta} = L^2(\Omega^\eta, \Theta^\eta, R^\eta) \) for \( \eta \in S_n - \{\varnothing\} \). Since \( \mu^\eta = m^\eta + v^\eta \), where \( m^\eta = \sum_{j \in \eta} m_j \) and \( v^\eta \) is a singular measure in \( T^\eta \) which concentrates at a \( m^\eta \)-null set \( F^\eta \), we have
\[
L^2(\Omega^\eta, \Theta^\eta) = L^2((T^\eta, \Theta^\eta, m^\eta), \Theta^\eta) \otimes L^2((T^\eta, \Theta^\eta, v^\eta), \Theta^\eta). \quad (4)
\]
For \( \eta = (j_1, \ldots, j_m) \), let \( \vartheta_{\eta} \) be the projection from \( L^2((T^\eta, \Theta^\eta, m^\eta), \Theta^\eta) \) to the Hardy space \( H^2((T^\eta, \Theta^\eta, m^\eta), \Theta^\eta) \) of all functions
\[
st_{-\infty} \sum_{n=0}^N f_n z^n, \quad z^n \in T^n
\]
satisfying \( f_\eta \in \Theta^\eta \) and \( \sum_n \| f_n \|_{\Theta^\eta}^2 < +\infty \), i.e.,
\[
(\vartheta_{\eta} f)(z_{j_1}, \ldots, z_{j_m}) = \text{st- lim}_{r \to 0} \left( \frac{1}{2\pi} \int \frac{dv}{v - rz_{j}} \right) f(v_{1}, v_{2}, \ldots, v_{m}).
\]
From (4), the operator \( \vartheta_{\eta} \) extends to a projection, which is still denoted by \( \vartheta_{\eta} \), in \( L^2(\Omega^\eta, \Theta^\eta) \) by defining
\[
\vartheta_{\eta} f = 0
\]
for \( f \in L^2((T^\eta, \Theta^\eta, v^\eta), \Theta^\eta) \). Further, let \( \vartheta_{\varnothing} = I \).

Let \( \hat{\vartheta} = (\hat{\vartheta}_1, \ldots, \hat{\vartheta}_n) \) be the \( n \)-tuple of unitary operators in \( \hat{N} \) defined by
\[
(\hat{\vartheta}_j f)(z_1, \ldots, z_n) = z_j f(z_1, \ldots, z_n), \quad (5)
\]
and \( \mathcal{M}(R(\cdot)) \) be the set of all bounded \( \mathcal{L}(\mathcal{B}) \)-valued measurable functions \( \alpha(\cdot) \) on \( T^n \) satisfying
\[
\alpha(z) = R(z)\alpha(z).
\]
If \( \{\alpha_\eta, \eta \in S_n\} \) is a subset of \( \mathcal{M}(R(\cdot)) \), then it is easy to prove that the operator
\[
\hat{A} = \sum_{\eta \in S_n} \alpha_\eta^* \vartheta_{\eta} \alpha_\eta \quad (6)
\]
is in $SH(\mathcal{U})$. In the next section, we shall prove that every operator in $SH(\mathcal{U})$ must be unitarily equivalent to one of the form (6) and then it is called the singular integral model of the operator in $SH(\mathcal{U})$.

§3. SINGULAR INTEGRAL MODEL OF SEMI-HYPONORMAL (n+1)-TUPLE OF OPERATORS

The following lemma gives a decomposition of an operator in $SH(\mathcal{U})$.

**Lemma 1.** If $A \in SH(\mathcal{U})$, then

$$A = \sum_{\eta \in \mathcal{S}_n} \prod_{j \in \eta} F_j \prod_{j \notin \eta} \mathcal{J}_j A.$$  

**Proof.** From (3), it follows that

$$A = \prod_{j=1}^{n} (F_j + \mathcal{J}_j) A.$$  

Let $T = \mathcal{J}_1 \cdots \mathcal{J}_m A$. We have to prove

$$\mathcal{J}_k \mathcal{J}_j T = \mathcal{J}_j \mathcal{J}_k T.$$  

Since $\mathcal{U}_j$ commutes $\mathcal{U}_1, \cdots, \mathcal{U}_m$, it is obvious that

$$Q_k T = \mathcal{J}_j \cdots \mathcal{J}_m Q_k A \geq 0$$

and also $Q_j T \geq 0$. Hence $T \in \mathcal{F}_j \cap \mathcal{F}_k$ and $\mathcal{U}_k^n \mathcal{U}_k^{-n} \geq \mathcal{J}_k T$ for $n \geq 0$. Thus

$$\mathcal{U}_k^n (\mathcal{J}_j T) \mathcal{U}_k^{-n} = \mathcal{J}_j \mathcal{U}_k^n \mathcal{U}_k^{-n} \geq \mathcal{J}_j \mathcal{J}_k T.$$  

But $\{\mathcal{U}_k^n \mathcal{U}_k^{-n}, n=1,2,\ldots\}$ is a monotonic sequence, and its limit is $\mathcal{J}_j \mathcal{J}_k T$. From (10), it follows that

$$\mathcal{J}_j \mathcal{J}_k T \geq \mathcal{J}_j \mathcal{J}_k T$$

for any $j$ and $k$, which proves (9). Thus

$$\mathcal{J}_k F_j T = \mathcal{J}_k T - \mathcal{J}_k \mathcal{J}_j T = \mathcal{J}_k T - \mathcal{J}_j \mathcal{J}_k T = F_j \mathcal{J}_k T.$$  

Expanding the product in the right-hand side of (8), it equals the right-hand side of (7), since (11).

**Theorem 1.** If $\mathcal{U} = (\mathcal{U}_1, \cdots, \mathcal{U}_n)$ is a commuting $n$-tuple of unitary operators in the Hilbert space $\mathcal{H}$, and $A \in SH(\mathcal{U})$, then there exists a certain $\mathcal{H} = L^2(\Omega, \mathcal{B}, \mathcal{P}(\cdot))$, (cf. §2), and a
unitary operator $W$ from $\mathcal{K}$ onto $\hat{\mathcal{K}}$ such that

$$W\hat{u}_jW^{-1} = \hat{u}_j$$

and

$$WAW^{-1} = \sum_{\eta \in S_n} \alpha^*_{\eta} \eta \eta,$$

where $\alpha_\eta, \eta \in S_n$ are the multiplication operators of function $\alpha_\eta(\cdot) \in \mathcal{M}(R(\cdot))$.

**PROOF.** For $\eta = (j_1, \ldots, j_m) \in S_n-[\emptyset]$, it is easy to show that

$$\prod_{j \in \eta} (\mathcal{H}_j^+ - \mathcal{H}_j) \prod_{j \in \eta} \mathcal{H}_j \geq 0.$$  

Take its positive square root $A_\eta$, i.e.,

$$A_\eta = \left[ \prod_{j \in \eta} (\mathcal{H}_j^+ - \mathcal{H}_j) \prod_{j \in \eta} \mathcal{H}_j \right]^{1/2}.$$  

It is evident that $A_\eta \in \mathcal{K}'$. Let $R_\eta$ be the projection from $\mathcal{K}$ to $\overline{A_\eta \mathcal{K}}$.

Since $\prod_{j \in \eta} (I - \mathcal{H}_j) \prod_{j \in \eta} \mathcal{H}_j \leq \prod_{j \in \eta} (\mathcal{H}_j^+ - \mathcal{H}_j) \prod_{j \in \eta} \mathcal{H}_j$, there is a contraction $B_\eta \in \mathcal{L}(\mathcal{K})$ commuting with $R_\eta$ such that

$$\prod_{j \in \eta} (I - \mathcal{H}_j) \prod_{j \in \eta} \mathcal{H}_j = A_\eta B_\eta A_\eta.$$  

Since $F_j \mathcal{H}_j = 0$, we have

$$\prod_{j \in \eta} F_j \prod_{j \in \eta} \mathcal{H}_j = \prod_{j \in \eta} (I - \mathcal{H}_j) \prod_{j \in \eta} \mathcal{H}_j = A_\eta (\prod_{j \in \eta} F_j B_\eta) A_\eta.$$  

Denote $V_\eta = (\prod_{j \in \eta} Q_j B_\eta)^{1/2}$. Let $\mathcal{K}_\eta$ be the smallest subspace of $\mathcal{K}$ which contains $R_\eta \mathcal{K}$ as a subspace and is reducible with respect to $\mathcal{K}$. Denote $M_\eta = \overline{V_\eta \mathcal{K}_\eta}$ and $\xi = \{1, 2, \ldots, n\} - \eta$. If $\xi \neq \emptyset$, since $M_\eta$ is reducible with respect to $\mathcal{K}_j$, $j \in \xi$, there exist a singular measure $\nu_\xi$ on $T^\xi$, a projection-valued measurable function $R_\xi(\cdot)$ on $T^\xi$ and a unitary operator $\mathcal{V}_\eta$ from $M_\eta$ onto $\mathcal{B}^\eta = L^2(\Omega^\xi, \mathcal{B}^\xi, R^\xi)$, where $\Omega^\xi = (T^\xi, \mathcal{B}^\xi, m^\xi + \nu_\xi)$, such that

$$\mathcal{V}_\eta \mathcal{H}_j \mathcal{V}_\eta^{-1} f(z) = z_j f(z), \ z \in T^\xi$$

for $j \in \xi$. If $\xi \neq \emptyset$, denote $\mu_\eta = m_\eta \times (m^\xi + \nu_\xi)$. If $\xi \neq \emptyset$, let $\mu_\eta = m$, $\mathcal{B}^\eta = M_\eta$ and $\mathcal{V}_\eta = I$.

Define an operator $W_\eta$ from $\mathcal{K}_\eta$ onto a subspace
\( \mathcal{N} \subset L^2(\Omega_\eta, \mu_\eta) \) as follows:
\[
W_{\eta} \cdot \mathcal{U}_{\eta}^{m_1 \ldots m_n x} = z_1 \cdots z_n \cdot \sum_{k_j=0}^{\infty} \prod_{j \in \eta} \mathcal{U}_j^{k_j x},
\]
for \( x \in R_\eta \). From (2), it follows that
\[
R_\eta = \prod_{j \in \eta} (\omega_j^+ - \omega_j^-) B_\eta = \prod_{j \in \eta} (\mathbb{P}_j^+ + \mathbb{P}_j^-) B_\eta.
\]
The operator \( W_\eta \) is unitary from \( \mathcal{N}_\eta \) to \( \hat{\mathcal{N}}_\eta \), since (15) and
\[
(W_{\eta} \mathcal{U}_{\eta}^{m_1 \ldots m_n x} , W_{\eta} \mathcal{U}_{\eta}^{m_1' \ldots m_n' y} )
= \sum_{k_j=0}^{\infty} (\mathcal{V}_j \prod_{j \in \eta} \mathcal{U}_j^{k_j x} , \mathcal{V}_j \prod_{j \in \eta} \mathcal{U}_j^{k_j y} )
= (\prod_{j \in \eta} \mathcal{U}_j^{m_j x} , \prod_{j \in \eta} \mathcal{U}_j^{m_j y} ) = (\prod_{j \in \eta} \mathcal{U}_j^{m_j x} , \prod_{j \in \eta} \mathcal{U}_j^{m_j y} ).
\]
Similarly,
\[
W_{\eta} (\prod_{j \in \eta} \mathbb{P}_j^+ B_\eta) W_{\eta}^{-1} = \varphi_{\eta} \big|_{\hat{\mathcal{N}}_{\eta}},
\]
and
\[
W_{\eta} \mathcal{U}_j W_{\eta}^{-1} = \hat{\mathcal{U}}_j \big|_{\hat{\mathcal{N}}_{\eta}},
\]
where \( \hat{\mathcal{U}}_j \big|_{\hat{\mathcal{N}}_{\eta}} \) is the corresponding \( \hat{\mathcal{U}}_j \) in \( \hat{\mathcal{N}}_{\eta} \). This means \( \hat{\mathcal{U}}_j \big|_{\hat{\mathcal{N}}_{\eta}} f(z) = z_j f(z) \), and \( \hat{\mathcal{N}}_{\eta} \) is invariant with respect to \( \hat{\mathcal{U}}_j \).

The operator \( W_{\eta} \) extends to a unitary operator, which is still denoted by \( W_{\eta} \), from \( \mathcal{N}_\eta \) to \( \hat{\mathcal{N}}_{\eta} = L^2(\Omega_\eta, \mu_\eta, \nu_\eta) \), where
\[
\Omega_\eta = (\mathbb{T}_\eta, \mathcal{B}_\eta, \mu_\eta, \nu_\eta), \quad \mu_\eta = m + \nu_\eta, \quad \nu_\eta \text{ is a singular measure such that}
\]
\[
W_{\eta} \mathcal{U}_j W_{\eta}^{-1} = \hat{\mathcal{U}}_j. \quad \tag{17}
\]
Since \( A_\eta \) commutes \( \mathcal{U}_j \), \( W_{\eta} A_{\eta} W_{\eta}^{-1} \) is a multiplication operator
\[
(W_{\eta} A_{\eta} W_{\eta}^{-1} f)(\cdot) = \beta_{\eta}(\cdot) f(\cdot) \quad \tag{18}
\]
where \( \beta_{\eta}(\cdot) \) is a bounded measurable \( \mathcal{L}(\mathbb{T}_\eta) \)-valued function satisfying \( \beta_{\eta}(\cdot) \geq 0 \). From (14), (16) and (18), it follows that
\[ W_{\eta}(\prod_{j \in \eta} \mathcal{F}_j \prod_{j \notin \eta} \mathcal{J}_j A)W_{\eta}^{-1} = \beta_{\eta} \Theta_{\eta} \theta_{\eta}. \]  

(19)

For \( \eta = \emptyset \), (18) still holds, where \( \Theta_{\emptyset} = I \).

Hitherto, for simplicity, \( W_{\emptyset} \) and \( \Theta_{\emptyset} \) are denoted by \( W \) and \( \Theta \), respectively. Since \( W_{\eta}W_{\eta}^{-1} \) is a unitary operator from \( \hat{\mathcal{H}}_{\eta} \) to \( \hat{\mathcal{U}}_{\eta} \) and

\[ \frac{\hat{\mathcal{U}}_{\eta}}{W_{\eta}W_{\eta}^{-1}} = W_{\eta}W_{\eta}^{-1} \frac{\hat{\mathcal{U}}_{\eta}}{W_{\eta}}. \]  

(20)

There exists a measurable function \( Z_{\eta}(\cdot) \), whose value is unitary operators from \( R(\cdot) \) to \( R_{\eta}(\cdot) \) such that

\[ (W_{\eta}W_{\eta}^{-1}f)(\cdot) = Z_{\eta}(\cdot)f(\cdot). \]  

(21)

Combining (19) and (20), we have

\[ W(\prod_{j \in \eta} \mathcal{F}_j \prod_{j \notin \eta} \mathcal{J}_j A)W_{\eta}^{-1} = \alpha_{\eta}^* \varphi_{\eta} \alpha_{\eta} \]  

where \( \alpha_{\eta}(\cdot) = \beta_{\eta}(\cdot)Z_{\eta}(\cdot) \). Thus (12-13) follows from (17), (20) and (22). The theorem is proved.

From Theorem 1, the set of self-adjoint operators

\[ R(\cdot) \varphi_{\eta} R(\cdot), \ \eta \in S_n, \]  

is a basis of \( SH(\hat{\mathcal{H}}) \).

§4. SPECTRUM.

If \( A \in SH(\hat{\mathcal{H}}) \), then the operator

\[ A_k = \prod_{j=1}^{n} (k_j \mathcal{J}_j + (1-k_j) \mathcal{F}_j)A \]  

is called a general polar-symbol of \( A \) with respect to \( \mathcal{H} \) corresponding to \( k = (k_1, \ldots, k_n) \in [0,1]^n \). It is obvious that \( A_k \in \mathcal{H}' \). If \( A \) is the singular integral model, then \( A_k \) is a multiplication operator

\[ (A_k f)(\cdot) = \sum_{\eta \in S_n} \prod_{j \in \eta} k_j \alpha_{\eta}^*(\cdot) \alpha_{\eta}(\cdot)f(\cdot). \]

Hitherto, the \( \mathcal{L}(\varphi) \)-valued function \( \sum_{\eta \in S_n} \sum_{j \in \eta} k_j \alpha_{\eta}^*(\cdot) \alpha_{\eta}(\cdot) \) is denoted by \( A_k(\cdot) \).

Let \( \sigma_{j\alpha}(A_k) \) be the joint approximated point spectrum of the commuting \((n+1)\)-tuple \((\mathcal{H},A_k) \), i.e., the set of all points \( (z_1, z_2, \ldots, z_n, \rho) \in \mathbb{T}^n \times [0, \infty) \) for which there exists
a sequence of unit vectors $\{f_m\} \subset \mathcal{X}$ such that
\[
\lim_{m \to \infty} \|\mathcal{U}_j z_j I f_m\| = 0, \quad j = 1, 2, \ldots, n
\] (23)
and
\[
\lim_{m \to \infty} \|A_k \rho I f_m\| = 0.
\]
This $\sigma_{ja}(\mathcal{U}, A_k)$ is the Taylor spectrum [6],[15] of the $(n+1)$-tuple $(\mathcal{U}, A_k)$.

Since $\mathcal{U}$ is a commuting $n$-tuple of unitary operators, its joint spectrum $\sigma(\mathcal{U})$ is in $T^n$. Let $E(\cdot)$ be the spectral measure of the $n$-tuple $\mathcal{U}$. For $z = (z_1, \ldots, z_n) \in \sigma(\mathcal{U})$, the set of all products $\Delta = \gamma_1 \times \cdots \times \gamma_n$ of open arcs $\gamma_j \subset T$, containing $z_j$, $j = 1, 2, \ldots, n$ is denoted by $\Gamma(z)$. For $A \in \mathcal{SH}(\mathcal{U})$, the set
\[
\sigma(\mathcal{U}, A) = \{(z, \rho) : z \in \sigma(\mathcal{U}), \rho \in \bigcap_{\Delta \in \Gamma(z)} E(\Delta) A E(\Delta)\}
\]
is called the joint spectrum of the $(n+1)$-tuple $(\mathcal{U}, A)$. If $A \in \mathcal{U}'$, it is easy to see the joint spectrum of the $(n+1)$-tuple $(\mathcal{U}, A)$ defined here coincides with the usual one [15].

**Theorem 2.** If $A \in \mathcal{SH}(\mathcal{U})$, then
\[
\sigma(\mathcal{U}, A) = \bigcup_{k \in [0, 1]} \sigma_{ja}(\mathcal{U}, A_k).
\] (24)

**Proof.** Without loss of generality, we may assume that $A$ is the singular integral model (13) in Theorem 1 and $\mathcal{U}$ is the $n$-tuple of multiplication operators (5) in $\mathcal{X}$. The spectral measure $E(\cdot)$ of $\mathcal{U}$ is of the form
\[
(E(M)f)(\cdot) = l_M(\cdot)f(\cdot),
\]
where $M$ is the Borel set in $T^n$, $l_M(\cdot)$ is the characteristic function of the set $M$.

Take any $(z_1, \ldots, z_n, \rho) \in \sigma_{ja}(\mathcal{U}, A_k)$ for certain $k = (k_1, \ldots, k_n) \in [0, 1]^n$, then there exists a sequence of unit vectors $\{f_m\} \in \mathcal{X}$ such that (23) and
\[
\lim_{n \to \infty} \|\sum_{\eta \in \eta} \Pi_{j \in \eta} k_j^{\alpha_j} \alpha_{\eta}^{*} - \rho f_m\| = 0
\]
hold. For $\Delta \in \Gamma(z_1, \ldots, z_n)$, denote
\[
\alpha_{\eta\Delta}(\cdot) = \alpha_{\eta}(\cdot) l_{\Delta}(\cdot)
\]
and
\[ A_1 = \sum_{1 \leq n_1} \prod_{j \in \eta} k_j \alpha_{\eta_1}^* \alpha_{\eta_1} + \sum_{1 \leq n_1} \prod_{j \in \eta, j \neq 1} k_j \alpha_{\eta_1}^* \phi_1 \alpha_{\eta_1}. \]

Then \( \mathcal{U}_1 A_1 \) is a semi-hyponormal operator \([18],[19]\) and its polar-symbols with respect to \( \mathcal{U}_1 \) are
\[ \mathcal{P}^+(\mathcal{U}_1 A_1) = \mathcal{U}_1 \sum_{1 \leq n_1} \prod_{j \in \eta} k_j \alpha_{\eta_1}^* \alpha_{\eta_1} \]
and
\[ \mathcal{P}(\mathcal{U}_1 A_1) = \mathcal{U}_1 \sum_{1 \leq n_1} \prod_{j \in \eta, j \neq 1} k_j \alpha_{\eta_1}^* \alpha_{\eta_1}. \]

Thus \( k_1 \mathcal{P}^+(\mathcal{U}_1 A_1) + (1-k_1) \mathcal{P}(\mathcal{U}_1 A_1) = \mathcal{U}_1 A_k 1_\Delta \) and
\[ z_1 \rho \in \sigma(\mathcal{U}_1 A_k 1_\Delta). \]
From \([19]\), we know that \( \sigma(k_1 \mathcal{P}^+(\mathcal{U}_1 A_1) + (1-k_1) \mathcal{P}(\mathcal{U}_1 A_1)) \subset \sigma(\mathcal{U}_1 A_1) \). Therefore
\[ z_1 \rho \in \sigma(\mathcal{U}_1 A_1). \]
From the projection property of the \( \sigma(\mathcal{U}_1 A_1) \), it follows that
\[ \rho \in \sigma(A_1). \]
Since \( \mathcal{U}_2 A_1 \) is normal there is a \( z' \in \sigma(\mathcal{U}_2) \) such that
\[ \rho z_2' \in \sigma(\mathcal{U}_2 A_2). \]
Define
\[ A_2 = \sum_{1 \leq n_2} \prod_{j \in \eta} k_j \alpha_{\eta_2}^* \alpha_{\eta_2} + \sum_{1 \leq n_2} \prod_{j \in \eta, j \neq 2} k_j \alpha_{\eta_2}^* \phi_2 \alpha_{\eta_2} \]
+ \[ \sum_{1 \leq n_2, 2 \leq n_2} \prod_{j \in \eta} k_j \alpha_{\eta_2}^* \phi_1 \alpha_{\eta_2} + \sum_{1 \leq n_2, 2 \leq n_2} \prod_{j \in \eta, j \neq 1} k_j \alpha_{\eta_2}^* \phi_1 \alpha_{\eta_2} \].
Then by calculation it is easy to show that
\[ k_2 \mathcal{P}^+ A_2 + (1-k_2) \mathcal{P} A_2 = \mathcal{U}_2 A_1. \]
Thus \( \rho z_2' \in \sigma(\mathcal{U}_2 A_2) \) and \( \rho \in \sigma(A_2) \). By this procedure, we can prove that
\[ \rho \in \sigma(\mathcal{E}(\Delta)\mathcal{A}(\Delta)). \]
Thus \( (z_1, \cdots, z_n, \rho) \in \sigma(\mathcal{U}, \mathcal{A}). \)

Next, if \((z_1, \cdots, z_n, \rho) \in \sigma(\mathcal{U}, \mathcal{A}), \) let \( A_\Delta = \mathcal{E}(\Delta)\mathcal{A}(\Delta), \) where \( \Delta \in \Gamma(z), \) then \( \rho \in \sigma(A_\Delta). \) Since \( \mathcal{U}_n A_\Delta \) is semi-
hyponormal, from [19], there is a \( k_n \in [0,1] \) such that

\[
\rho_{z_n} \in (\mathcal{U}_n A_n),
\]

where

\[
A_n = \sum_{n \in \mathbb{N}} \eta_n \eta_n^* \sum_{j \in \mathbb{N}} \eta_j \eta_j^* + k_n \sum_{n \in \mathbb{N}} \eta_n \eta_n^* \sum_{j \neq n} \eta_j \eta_j^*
\]

is a linear combination of the polar-symbols of \( \mathcal{U}_n A \) with coefficients \( k_n \) and \( (1-k_n) \). Thus \( \rho \in \sigma(A_n) \). By the same procedure, we can prove that there is a \( k^\Delta = (k_1, \ldots, k_n) \in [0,1]^n \)

such that

\[
\rho \in \sigma(E(\Delta)A_k^\Delta E(\Delta)).
\]

Let \( \{\Delta_m\} \) be a sequence in \( \Gamma(z) \) such that \( \bigcap_{m} \Delta_m = \{z\}, \]

\( \text{diam}(\Delta_m) \to 0 \) and \( k_m \to k^0 \in [0,1] \). There is a sequence of unit vectors \( \{f_m\} \subset E(\Delta_m)_\mathcal{E} \) such that

\[
\| (A_{\Delta_m} - \rho)f_m \| < \frac{1}{m}.
\]

Thus \( \| (A_{\Delta_m} - \rho)f_m \| \to 0 \) and \( \| (\mathcal{U} - z_j I)\rho_m \| \to 0 \). Hence

\[
(z_1, \ldots, z_n, \rho) \in \sigma_j (\mathcal{U}_n A_{k_0}^j) \text{ and } (24) \text{ is proved.}
\]

§5. TRACE FORMULA

Let \( \mathcal{J} = \{ k = (k_1, \ldots, k_n) : k_j = 0 \text{ or } 1, j = 1,2,\ldots,n \} \).

Denote \( \| (k_1, \ldots, k_n) \| = \sum_{j=1}^n k_j \). If \( E \) is a closed set in \( \mathcal{T} \), let \( M(E) \) be the set of all functions \( \varphi(\cdot) \) on \( E \) satisfying the following conditions, \( \varphi(z) \in \mathcal{T} \) for \( z \in E \) and \( \varphi(z) \) varies clockwise if \( z \) varies clockwise. Denote

\[
q_{\varphi_j}(\mathcal{T}) = \mathcal{T} - \varphi_j(\mathcal{U}_j) T \varphi_j(\mathcal{U}_j)^{-1}, \quad j = 1,2,\ldots,n,
\]

where \( \varphi_j \in M(\sigma(\mathcal{U}_j)) \).

THEOREM 3. If \( A \in \text{SH}(\mathcal{U}), \varphi_j \in M(\sigma(\mathcal{U}_j)), j = 1,2,\ldots,n, \)

\( \mathcal{G}(\cdot) \geq 0 \) is a continuous function on \( \sigma(A) \) such that

\[
\mathcal{G}(A) \in \text{SH}(\varphi_1(\mathcal{U}_1), \ldots, \varphi_n(\mathcal{U}_n)) \text{ and }
\]

\[
\int_{\mathcal{T}^n} \int_{\mathcal{J}} \mathcal{G}(A_{k}(z)) \prod_{k \in \mathcal{J}} (\mathcal{T} - \varphi_j(\mathcal{U}_j)) \sum_{k \in \mathcal{J}} (-1)^{n-1} k \mathcal{G}(A_{k}(z)) \prod_{k \in \mathcal{J}} \mathcal{T} - \varphi_j(\mathcal{U}_j) \prod_{j=1}^n \mathcal{G}(\varphi_j(\mathcal{U}_j)) \prod_{k \in \mathcal{J}} \mathcal{T} - \varphi_j(\mathcal{U}_j)
\]

\[
< +\infty.
\]

Then

\[
\text{tr}(q_{\varphi_1}(\mathcal{T}) \cdots q_{\varphi_n}(\mathcal{T}) \mathcal{G}(A)) = \\
\int_{\mathcal{T}^n} \int_{\mathcal{J}} \mathcal{G}(A_{k}(z)) \prod_{k \in \mathcal{J}} (\mathcal{T} - \varphi_j(\mathcal{U}_j)) \sum_{k \in \mathcal{J}} (-1)^{n-1} k \mathcal{G}(A_{k}(z)) \prod_{k \in \mathcal{J}} \mathcal{T} - \varphi_j(\mathcal{U}_j) \prod_{j=1}^n \mathcal{G}(\varphi_j(\mathcal{U}_j)) \prod_{k \in \mathcal{J}} \mathcal{T} - \varphi_j(\mathcal{U}_j). \quad (25)
\]
Moreover, if in the singular integral model,
\[ \int_{\mathcal{S}} (a_\eta a_\eta) dm(z) < +\infty, \text{ for all } \eta \in S_n - \{\varnothing\}, \]
then there is an integrable function \( G(z, \rho) \) on \( \sigma(\mathcal{U}, \mathcal{A}) \) such that
\[
\text{tr}(Q_1 \cdots Q_n \mathcal{S}(\mathcal{A})) = \int_{\mathcal{S}} \mathcal{S}'(\rho) \prod_{j=1}^{n} \varphi_j(z_j) z_j G(z, \rho) dm(z) d\rho.
\]

PROOF. We consider the singular integral model of \( \mathcal{A} \). For simplicity we prove this theorem in the case of \( n = 2 \) only, and for the general case it can be proved similarly. In this case,
\[
A = a_0^2 a_0 + a_1^2 a_1 + a_2^2 a_2 + a_3^2 a_3,
\]
where \( a_j, j = 0, 1, 2, 3 \) are \( \mathcal{L}(\mathcal{B}) \)-valued bounded measurable functions. Without loss of generality, we may suppose \( \nu(\cdot) = 0 \), \( R(\cdot) = I \) and \( \varphi_j(z_j) = z_j \). Let \( \mathcal{B} = L^2(T, \mathcal{B}^1) \), then \( \mathcal{X} = L^2(T, \mathcal{B}^1) \). The operator \( \mathcal{U}_1 \mathcal{S}(\mathcal{A}) \) is semi-hyponormal in \( L^2(T, \mathcal{B}^1) \). For \( e \in \mathcal{B}^1 \), \( e \neq 0 \), let \( P_e \) be the operator
\[
(P_e f)(z_1) = (f(z_1), e / \|e\|_\mathcal{B}^1) e, \text{ for } f \in L^2(T, \mathcal{B}^1).
\]
Let \( \mathcal{C}_1(\mathcal{U}_1) \) be the class of all operators \( B \in \mathcal{L}(\mathcal{X}) \), for which the series
\[
\Sigma(B f_n \otimes a, f_n \otimes b)
\]
converges whenever \( \{f_n\} \) is an orthonormal basis in \( L^2(T) \) and \( a, b \in \mathcal{B}^1 \). If \( B \in \mathcal{C}_1(\mathcal{U}_1) \), then (27) is independent on the choice of \( \{f_n\} \), and there is an operator in \( \mathcal{L}(\mathcal{B}^1) \), which is denoted by \( \text{tr}_{\mathcal{U}_1}(B) \), such that (27) equals \( \text{tr}_{\mathcal{U}_1}(B)a, b \) for every \( a, b \in \mathcal{B}^1 \). An operator \( B \in \mathcal{L}(\mathcal{X}) \) belongs to \( \mathcal{C}_1(\mathcal{U}_1) \) iff \( P_a B P_a \) is in the trace class for every \( a, b \in \mathcal{B}^1 \). If \( B \in \mathcal{C}_1(\mathcal{U}_1) \), then
\[
\text{tr}_{\mathcal{U}_1}(B) = \text{tr}(P_a B P_a), \text{ for all } a, b \in \mathcal{B}^1.
\]
If we consider \( \mathcal{U}_2 \) as an operator in \( \mathcal{B}^1 \), then
\[
Q_2 B \in \mathcal{C}_1(\mathcal{U}_1) \text{ and } Q_1 \text{tr}_{\mathcal{U}_1}(Q_2 B) = Q_2 \text{tr}_{\mathcal{U}_1}(B)
\]
for \( B \in \mathcal{C}_1(\mathcal{U}_1) \). Since \( P_a \) commutes \( \mathcal{U}_1 \), \( \mathcal{U}_1 P_a \mathcal{S}(\mathcal{A}) P_a \) is semi-hyponormal and the difference of the polar symbols of \( P_a \mathcal{S}(\mathcal{A}) P_a \)
is $P_a(\xi(A_1)-\xi(A_0))P_a$, where $A_0 = a_0^*a_0 + a_1^*a_1$ and

$$A_1 = a_0^*a_0 + a_1^*a_1 + a_2^*a_2 + a_3^*a_3.$$

By the trace formula [13] of semi-hyponormal operators, it is easy to see that $P_a(Q_1\xi(A))P_a = Q_1(P_a\xi(A)P_a)$ is in the trace class and

$$\text{tr}(P_a(Q_1\xi(A))P_a) = \int_T ((\xi(A_1(z_{1}))-\xi(A_0(z_{1}))))a_{1}a_{1}dm(z_{1}).$$

Thus $Q_1\xi(A) \in C_1(\mathcal{U}_1)$,

$$\text{tr}_{\mathcal{U}_1} (Q_1\xi(A)) = \int_T (\xi(A_1(z_{1}))-\xi(A_0(z_{1})))dm(z_{1})$$

and then

$$Q_1Q_2\xi(A) = Q_2Q_1\xi(A) \in C_1(\mathcal{U}_1),$$

$$\text{tr}_{\mathcal{U}_1} (Q_1Q_2\xi(A)) = \int_T (Q_2\xi(A_1(z_{1}))-Q_2\xi(A_0(z_{1})))dm(z_{1}). \quad (28)$$

But $\mathcal{U}_2Q_2\xi(A_1(z_{1}))$ and $\mathcal{U}_2Q_2\xi(A_0(z_{1}))$ are semi-hyponormal operators in $\mathcal{B}$. Therefore, from [13], it follows that

$$\text{tr}_{\mathcal{B}_1} (Q_2\xi(A_1(z_{1}))) = \int_T \text{tr}_B \xi(A_1(z_{1},z_{2}))-\xi(A_1(z_{1},z_{2}))\text{dm}(z_{2}) \quad (29)$$

$$\text{tr}_{\mathcal{B}_1} (Q_2\xi(A_0(z_{1}))) = \int_T \text{tr}_B \xi(A_0(z_{1},z_{2}))-\xi(A_0(z_{1},z_{2}))\text{dm}(z_{2}). \quad (30)$$

From (28), (29) and (30) it follows that

$$\text{tr}_{\mathcal{B}_1} \text{tr}_{\mathcal{U}_1} (Q_1Q_2\xi(A))$$

$$= \int_T \int_T (\prod_{k \in J} (-1)^k |k| \xi(A_k(z_{1},z_{2})))\text{dm}(z_{1})\text{dm}(z_{2}). \quad (31)$$

Since $Q_1Q_2\xi(A) \geq 0$, the fact $\text{tr}_{\mathcal{B}_1} \text{tr}_{\mathcal{U}_1} (Q_1Q_2\xi(A)) < +\infty$ implies that $Q_1Q_2\xi(A)$ is in the trace class and

$$\text{tr}(Q_1Q_2\xi(A)) = \text{tr}_{\mathcal{B}_1} \text{tr}_{\mathcal{U}_1} (Q_1Q_2\xi(A)). \quad (32)$$

By (31) and (32), (25) is proved.

From the theory of nearly normal operators [2],[13], we know that if $\alpha \in \mathcal{L}(\mathcal{B})$, $\text{tr}_{\mathcal{B}}(\alpha^*\alpha) < +\infty$, $\beta \in \mathcal{L}(\mathcal{B})$, $\beta \geq 0$, and

$$E = \bigcup_{0 \leq k \leq l} \sigma(\beta+k\alpha^*\alpha)$$

then there is a nonnegative integrable function on $E$ such that
\[ \text{tr}_B[\xi(\beta + \gamma + \lambda) - \xi(\beta)] = \int_E \xi'(\rho)G(\rho)\,d\rho \]

for any monotonic continuous function \( \xi \) on \( E \) such that \( \xi(\beta + \gamma + \lambda) - \xi(\beta) \geq 0 \).

Thus for \( k_1 = 0,1 \) and \( z \in \sigma(\mathcal{U}) \) there are functions \( G_{k_1}(z,\rho) \geq 0 \) of \( \rho \) in \( E_{k_1}(z) = \bigcup_{0 \leq k_2 \leq 1} \sigma(\alpha^*_0 + k_1 \alpha^*_1 + k_2 \alpha^*_2 + k_1^2 \alpha^*_3) \) such that

\[ \text{tr}[\xi(\alpha^*_0 + k_1 \alpha^*_1 + k_2 \alpha^*_2 + k_1^2 \alpha^*_3) - \xi(\alpha^*_0 + k_1 \alpha^*_1)] = \int_{E_{k_1}}(z) \xi'(\rho)G_{k_1}(z,\rho)\,d\rho. \quad (33) \]

From (24), \( E_{k_1}(z) \subseteq \{ \rho : (z,\rho) \in \sigma(\mathcal{U},A) \} \). Let

\[ G(z,\rho) = G_0(z,\rho) + G_1(z,\rho), \quad (34) \]

then (26) follows from (25), (33) and (34) which proves theorem 3.

The function \( G(z,\rho) \) is a generalization of Pincus principal function (cf. [2],[13]).

Let \( B \) be a self-adjoint operator in \( L^2(\mathbb{T}^n,\mathcal{B}) \). If the series

\[ \sum_{k,l} (Bf_k \otimes e_l, f_k \otimes e_l) \]

converges for any orthonormal basis \( \{e_j\} \) in \( \mathcal{B} \) and orthonormal basis \( \{f_k\} \) in \( L^2(\mathbb{T}) \) and its sum does not depend on the proper choice of the basis \( \{e_j\} \) and the basis \( \{f_k\} \) in \( L^2(\mathbb{T}) \), then this sum is defined to be the generalized trace of \( B \) and is denoted by \( \text{gtr}(B) \). It is obvious that if \( B \) is in the trace class then \( \text{gtr}(B) = \text{tr}(B) \). The trace norm of an operator \( L \in \mathcal{L}(\mathcal{B}) \) is denoted by \( \|L\|_1 \).

By the same method in the proof of Theorem 3, we can prove the following theorem, in which the operator \( A \) is the singular integral model.

**THEOREM 4.** If \( A \in \text{SH}(\mathcal{U}) \), \( \varphi_j(\cdot) \) and its derivative are continuous function in \( \mathbb{T} \), \( |\varphi_j(\cdot)| = 1, j = 1,2,\ldots,n \), \( \xi(\cdot) \) is a continuous function on \( \sigma(A) \) and

\[
\int_{\mathbb{T}^n} \left| \varphi_j(z_j) \right| \| \sum_{k \in \mathcal{G}} (-1)^{k_j} \xi(A_k(z)) \|_1 \,dm(z) < +\infty.
\]
Then
\[
gtr(Q_{\varphi_1} \cdots Q_{\varphi_n} \xi(A)) = \sum_{k \in \mathcal{F}} (-1)^{n-|k|} \text{tr}_g \left( \sum_{j=1}^n \frac{\partial'(z_j)z_j}{\partial_j(z_j)} \right) g(A_k(z)) \text{d}m(z).
\]
Moreover, if \( \int \text{tr}_g (\alpha_\eta^* \eta) \text{d}m < +\infty \), for \( \eta \in S_n - \{\emptyset\} \) then there is an integrable function \( G(z, \rho) \) defined on \( \sigma(\mathcal{U}, A) \) such that
\[
gtr(Q_{\varphi_1} \cdots Q_{\varphi_n} \xi(A)) = \sum_{\sigma(\mathcal{U}, A)} \xi'(\rho) \prod_{j=1}^n \frac{\partial'_j(z_j)z_j}{\partial_j(z_j)} G(z, \rho) \text{d}m(z) \text{d}\rho
\]
provided that \( \xi'(\cdot) \) is continuous on an interval containing \( \sigma(A) \).

**THEOREM 5.** If \( A \in \text{SH}(\mathcal{U}) \), then
\[
\|Q_1 \cdots Q_n A\| \leq m(\sigma(\mathcal{U}, A)),
\]
where \( m \) is the product of the Haar measure on \( \mathbb{T}^n \) and the Lebesgue measure on \( \mathbb{R}^1 \).

**PROOF.** In the singular integral model
\[
(Q_1 \cdots Q_n A)(\cdot) = \alpha^* (\cdot) \int_{\mathbb{T}^n} \alpha(z) f(z) \text{d}m(z)
\]
where \( \alpha = \alpha_1, 2, \ldots, n \) in \((13)\). It is easy to show that
\[
\|Q_1 \cdots Q_n A\| \leq \int_{\mathbb{T}^n} \alpha(z) \alpha^*(z) \text{d}m(z).
\]
For fixed \( z' = (z_2, \ldots, z_n) \in \sigma(\mathcal{U}_2, \ldots, \mathcal{U}_n) \), consider the operator
\[
B(z') = \sum_{1 \in \eta} \alpha_\eta(\cdot, z')^* \alpha_\eta(\cdot, z') + \sum_{l \in \eta} \alpha_\eta(\cdot, z')^* \alpha_\eta(\cdot, z')
\]
In this case, \( \mathcal{U}_1 B(z') \) is semi-hyponormal. From \([13]\), it follows that
\[
\|Q_1 B(z')\| \leq \int_{\mathbb{T}^n} \alpha_1(z_1, z') \text{d}m(z_1).
\]
It is obvious that
\[
\int_{\mathbb{T}^n} \alpha_1(z_1, z')^* \alpha_1(z_1, z') \text{d}m(z_1) \| \leq \|Q_1 B(z')\|
\]
and
\[ \sigma(\mathcal{U}_1 B) = \bigcup_{0 \leq k_1 \leq 1} \sigma \left( \sum_{l \in \mathbb{N}} \alpha_{\eta}(\cdot, z') \alpha_{\eta}(\cdot, z') \right) \]
\[ + k_1 \sum_{l \in \mathbb{N}} \alpha_{\eta}(\cdot, z') \alpha_{\eta}(\cdot, z') \]
\[ = \left\{ \rho z_1 : (z_1, z', \rho) \in \bigcup_{k \in [0,1]} \sigma(A_k) \right\} . \]

Thus
\[ \left\| \int_{\mathcal{T}_n} \alpha(z)^* \alpha(z) \, dm(z) \right\| \leq \int_{\mathcal{T}_n-1} \left( \int (\mathcal{U}_1 B(z')) \, dp \, dm(z_1) \right) \, dm(z') \]
\[ \leq m(\sigma(\mathcal{U}_1 A)) . \quad (37) \]

From (36) and (37), it follows (35).

The inequality (35) is a generalization of the corresponding one for semi-hyponormal operator (cf. \cite{13},\cite{14},\cite{18}).

§6. HYPERSONAL TUPLE.

Now we consider the hyponormal case, give the definitions and results here, and omit the proofs.

Let \( X = (X_1, \ldots, X_n) \) be the commuting tuple of bounded self-adjoint operators in \( \mathcal{K} \). Denote
\[ D_1 T = i[ X_1, T ] , \quad T \in \mathcal{L}(\mathcal{K}) , \]
where \([A,B]\) denotes the commutator \( AB-BA \). It is evident that
\[ D_1 D_2 T = D_2 D_1 T . \]

If \( Y \in \mathcal{L}(\mathcal{K}) \) and
\[ D_1 \cdots D_m Y \geq 0 \]
for all \( 1 \leq j_1 < \cdots < j_m \leq n \), then \( Y \) is said to be in the class \( \text{HN}(X) \) and the \((n+1)\)-tuple \( (X,Y) \) is said to be hyponormal. If \( Y \in \text{HN}(X) \) then \( X_1 + i Y, j = 1,2,\ldots,n \) are hyponormal operators.

If \( T \in \mathcal{L}(\mathcal{K}) \) and
\[ \text{st-lim} \quad e^{itX_1} - e^{-itX_1} \quad t \to +\infty \]
exist then these operators are denoted by \( S_j^\pm T \) and are called the symbols \( [1],[5] \), respectively. If \( Y \in \text{SH}(X) \), then for every \( k = (k_1, \ldots, k_n) \in [0,1]^n \) there exists
\[ Y_k = \sum_{j=1}^n (k_j S_j^+ + (1-k_j) S_j^-) Y \]
which is called the general symbol [19] of $Y$ with respect to $X$
and corresponding to $k$, and is denoted by $Y_k$.

For $\eta \in S_n\setminus \{\emptyset\}$, let $R^n$ be the Cartesian product of
real lines $R^1_j$, $j \in \eta$, $L^2(R^n,\mathcal{B})$ be the space of all measurable
$\mathcal{B}$-valued square integrable functions on $R^n$. Denote the projection
from $L^2(R^n,\mathcal{B})$ to the Hardy space by $P_{\eta}$, i.e., for
$f \in L^2(R^n,\mathcal{B})$,
\[(P_{\eta}f)(x) = \text{st-lim}_{\varepsilon \to 0^+} \prod_{j \in \eta} \left( \frac{1}{2\pi i} \int_{R^1_j} \frac{1}{x_j - (s_j + i\varepsilon)} \right) f(s) ds,
\]
where $s \in R^n_j$ and $ds$ is the Lebesgue measure in $R^n$.

Let $M$ be a bounded closed set in $R^n = \{ x = (x_1, \ldots, x_n) : x_j \in R^1 \}$, $\mathcal{B}$ be the $\sigma$-algebra of all Borel sets in $M$, $m$ be
the Lebesgue measure in $R^n$, $\nu$ be a singular measure in $M$, $\mu = m + \nu$, $\Omega = (M, \mathcal{B}, \mu)$, $\mathcal{H}$ be an auxiliary Hilbert space, $R(\cdot)$
be a projection-valued measurable function in $\Omega$ and $\hat{\mathcal{H}}$ be the
Hilbert space of all $\mathcal{B}$-value measurable function in $\Omega$ satisfying
\[
\|f\|^2 = \int_M \|f(x)\|^2_\mathcal{B} d\mu(x) < +\infty
\]
and
\[R(\cdot)f(\cdot) = f(\cdot).
\]

THEOREM 6. Let $X = (X_1, \ldots, X_n)$ be an abelian set of
bounded self-adjoint operators in $\mathcal{H}$, $Y \in \text{HN}(X)$. Then there
exist a space $\hat{\mathcal{H}}$, in which $M = \sigma(X)$, a set of bounded measurable
$L(\mathcal{B})$-valued functions $\{\alpha_{\eta}(\cdot) : \eta \in S_n\}$ satisfying
$\alpha_{\eta}(\cdot)R(\cdot) = \alpha_{\eta}(\cdot)$ and a unitary operator $W$ from $\mathcal{H}$ onto $\hat{\mathcal{H}}$
such that
\[(WX_j W^{-1} f)(x) = x_j f(x),
\]
and
\[WYW^{-1} = \sum_{\eta \in S_n} \alpha_{\eta}^* \eta_{\eta} \eta_{\eta}.
\]

The operator in (38) is called the singular integral
model of $Y$.

Let $E(\cdot)$ be the spectral measure of the $n$-tuple $X$.
For $x = (x_1, \ldots, x_n) \in \sigma(X)$, let $\Gamma(x)$ be all cubics in $R^n$
with center $x$. If $Y \in \text{HN}(X)$, then
The joint spectrum of the \((n+1)\)-tuple \((X,A)\).

**THEOREM 7.** If \(Y \in HN(X)\), then
\[
\sigma(X,Y) = \bigcup_{k \in [0,1]} \sigma_{ja}(X,Y_k),
\]
where \(Y_k\) is the general symbol of \(Y\) with respect to \(X\) and corresponding to \(k\).

If \(E\) is a bounded closed set in \(\mathbb{R}^l\), then the family of all continuous monotonic increasing functions on \(E\) is denoted by \(M(E)\). Let \(\varphi_j \in M(\sigma(X_j))\) and
\[
D_{\varphi_j} T = i[\varphi_j(X), T].
\]

In the following theorems, we consider the singular integral model only. In this case, the general symbol \(Y_k\) is a multiplication operator, i.e.,
\[
(Y_k f)(\cdot) = Y_k(\cdot) f(\cdot)
\]
where \(Y_k(\cdot) = \sum \prod_{j \in \eta} \alpha_{k_j}(\cdot) \alpha_{\eta}(\cdot)\), for \(k = (k_1, \ldots, k_j)\).

**THEOREM 8.** If \(Y \in HN(X)\), \(\varphi_j \in M(\sigma(X_j))\), \(j = 1, 2, \ldots, n\), \(\xi(\cdot) \geq 0\) is a continuous function on \(\sigma(Y)\) such that
\[
\xi(Y) \in HN(\varphi_1(X_1), \ldots, \varphi_n(X_n))
\]
and
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} tr\left(\sum_{k \in \xi} (-1)^{n-|k|} \xi(Y_k(x))\right) d\varphi_1(x_1) \cdots d\varphi_n(x_n) < +\infty.
\]

Then
\[
tr(D_{\varphi_1} \cdots D_{\varphi_n} \xi(Y)) = \frac{1}{n^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} tr\left(\sum_{k \in \xi} (-1)^{n-|k|} \xi(Y_k(x))\right) d\varphi_1(x_1) \cdots d\varphi_n(x_n).
\]

Moreover, if \(\int tr_2(\alpha_{k_j}^* \alpha_{k_j}) dx < +\infty\) for all \(k \in S_n - \{\varnothing\}\), then there is an integrable function \(G(x,y)\) on \(\sigma(X,Y)\) such that
\[
tr(D_{\varphi_1} \cdots D_{\varphi_n} \xi(Y)) = \frac{1}{n^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \xi'(y) \prod_{j=1}^{n} \varphi_j'(x_j) G(x,y) dx dy.
\]

The function \(G(x,y)\) is a generalization of Pincus.
principal function.

We can define similarly the generalized trace in this case.

THEOREM 9. If $Y \in HN(X)$, $\varphi_j(\cdot)$ and its derivative are continuous functions in an interval containing $\sigma(X_j)$, $\xi(\cdot)$ is a continuous function on $\sigma(Y)$ and

$$\int_{\mathbb{R}^n} \prod_{j=1}^n |\varphi_j'(x_j)| \sum_{k \in \mathcal{S}} (-1)^{|k|} |\xi(Z_k(x))| \, dx < +\infty.$$  

Then

$$\text{gtr}(\prod_{j=1}^n \varphi_j(x_j)) = \int_{\mathbb{R}^n} \prod_{j=1}^n \varphi_j'(x_j) \text{tr}_B \sum_{k \in \mathcal{S}} (-1)^{|k|} \xi(Z_k(x)) \, dx.$$  

Moreover, if $\int \text{tr}_B (a_k^* a_k) \, dx < +\infty$ for $\eta \in S_n - \{\emptyset\}$, then there is an integrable function $G(x,y)$ on $\sigma(X,Y)$ such that

$$\text{gtr}(\prod_{j=1}^n \varphi_j(x_j)) = \frac{1}{(2\pi)^n} \int_{\sigma(X,Y)} \int_{\mathbb{R}^n} \varphi_j'(x) G(x,y) \, dx \, dy,$$

provided that $\xi'(\cdot)$ is continuous in an interval containing $\sigma(Y)$.

THEOREM 10. If $Y \in SH(X)$, then

$$\|D_1 \cdots D_n Y\| \leq \frac{1}{(2\pi)^n} m(\sigma(X,Y)).$$

This is a generalization of Putnam's inequality of a hyponormal operator.

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