A CLASS OF SUBNORMAL OPERATORS WITH FINITE RANK SELF-COMMUTATORS

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In this paper, the family of pure subnormal operators $S$ with m.n.e. $N$ and finite rank self-commutators satisfying $\sigma(S) = \text{closed unit disk}$ and $\sigma(N) = \text{unit circle}$ is studied. A necessary and sufficient condition for a pair of matrices $\{A, C\}$ to be a complete unitary invariance of $S$ in this family is given. The concrete model for some sub-family of this family is also given in this paper.

1. Let $\mathcal{F}$ be the family of all pure subnormal operators $S$ with minimal normal extensions (m.n.e.) $N$ and finite rank self-commutator $[S^*, S]$ satisfying $\sigma(S) = D$ and $\sigma(N) = T \cup \{a_1, \ldots, a_m\}$, where $D$ is the unit disk $\{u \in \mathbb{C} : |u| < 1\}$, $T = \{u \in \mathbb{C} : |u| = 1\}$, and $a_1, \ldots, a_m$ are points in $D$. For a subnormal operator $S$ with finite rank self-commutator, let

$$M \overset{\text{def}}{=} \text{range of } [S^*, S],$$

where $M$ is a finite dimensional Hilbert space. As in [5], let

$$C_S \overset{\text{def}}{=} [S^*, S]|_M \quad \text{and} \quad \Lambda_S \overset{\text{def}}{=} \left( S^*|_M \right)^*.$$

Then the pair of matrices (operators on $M$) $\{\Lambda_S, C_S\}$ is a complete unitary invariance of $S$ if $S$ is also pure. For the rest of this paper we will denote $C_S$ and $\Lambda_S$ by $C$ and $\Lambda$ respectively.

The main tool used to investigate this subject is the analytic model theory introduced in [5]. In Theorem 1, we give an explicit form of the self-adjoint measure $\epsilon(\cdot)$ on $T$ and the mosaic $\mu(\cdot)$ in terms of $C$ and $\Lambda$ for $S \in \mathcal{F}$. We also establish some relationships between $C$ and $\Lambda$ for $S \in \mathcal{F}$. By means of the work in Theorem 1, in Theorem 2 we establish a necessary and sufficient condition for a pair of operators $\{\Lambda, C\}$ on $M$ to be the complete unitary invariant $\{\Lambda_S, C_S\}$ for some $S \in \mathcal{F}$.

Morrel's Theorem says that a pure subnormal operator with rank one self-commutator must be a linear combination of a unilateral shift and the identity. Connecting this with Theorem 3, we prove that if $S \in \mathcal{F}$ with m.n.e. $N$ satisfying $\sigma(N) = T$, then $S$ must be unitarily equivalent to a unilateral shift of multiplicity $m$ where $m = \text{rank } [S^*, S]$. In Theorem 4, we prove that if $S \in \mathcal{F}$ with m.n.e. $N$ satisfying $\text{rank } [S^*, S] = 2$, then either $\sigma(N) = T$ or $\sigma(N) = T \cup \{a\}$ where $|a| < 1$. In the second case, $S$ is unitarily equivalent...
to the multiplication operator multiplied by the independent variable of a function on the Hardy space $H^2(T)$ with a one point modification of the Hardy space inner product. In Theorem 5, we give the exact estimate of the cardinal number of $\sigma_p(N)$ where $N$ is the m.n.e. of $S$ by the rank of $[S^*, S]$.

All of the work in this paper may be easily generalized to the case where $\sigma(S)$ is any closed disk.

2. Let us begin by calculating the mosaic in [5] given by

$$
\mu(z) \overset{\text{def}}{=} \int_{\sigma(N)} \frac{uI - \Lambda}{u - z} e(du),
$$

for $|z| < 1$ and showing that it is a rational function. Denote $e_j = e(\{a_j\})$. We have

$$
\mu(z) = \int_T \frac{(I - \Lambda\overline{u})e(du)}{1 - z\overline{u}} + \sum_{i=1}^m \frac{a_iI - \Lambda}{a_i - z} e_i,
$$

$$
\int_{\sigma(N)} \frac{(I - \Lambda\overline{u})e(du)}{1 - z\overline{u}} - \sum_{i=1}^m \frac{I - \overline{a_i}\Lambda}{1 - \overline{a_i}z} e_i + \sum_{i=1}^m \frac{a_iI - \Lambda}{a_i - z} e_i, \quad |z| < 1
$$

since $|u| = 1$ for $u \in T$. But,

$$
\int_{\sigma(N)} f(\overline{u})e(du) = f(\Lambda^*)
$$

for any analytic function $f$ on a disk $\{u \in \mathbb{C} : |u| < 1 + \epsilon\}, \epsilon > 0$. Therefore,

$$
\mu(z) = (I - \Lambda\Lambda^*)(I - z\Lambda^*)^{-1} + \sum_{i=1}^m \frac{\overline{a_i}\Lambda - I}{1 - \overline{a_i}z} e_i + \sum_{i=1}^m \frac{a_iI - \Lambda}{a_i - z} e_i. \quad (1)
$$

By means of Plemelj's formula and the fact that $\mu(z) = 0$ for $z \in \rho(S)$, we may prove that the boundary value of $\mu(z), z \in \mathbb{R}$, satisfies

$$
\mu(z) = 2\pi i(zI - \Lambda)\frac{e(dz)}{dz} \quad (2)
$$

for almost all $z \in T$. By (15) of [5], we have

$$
\left((zI - \Lambda^*)(zI - \Lambda) - C\right)\frac{e(dz)}{dz} = 0 \quad \text{for a.e. } z \in T \quad (3)
$$

and

$$
((\overline{a_i}I - \Lambda^*)(a_iI - \Lambda) - C)e_i = 0, \quad i = 1, \ldots, m. \quad (4)
$$

From (2) and (3) we have

$$
\left(\frac{1}{z}I - \Lambda^* - C(zI - \Lambda)^{-1}\right)\mu(z) = 0 \quad (5)
$$

for a.e. $z \in T$. Let us redefine $\mu(\cdot)$ on $|z| \geq 1$ by (1). Then $\mu(\cdot)$ is a rational function on $\mathbb{C}$ and (5) holds for all $z \in \mathbb{C}$ except at a finite number of poles of the function on the left-hand
side of (5). By multiplying \((zI - \Lambda)C^{-1}\) on the left and \((I - z\Lambda^*)\) on the right of formula (5) and using (1), we have

\[
((zI - \Lambda)C^{-1}(I - z\Lambda^*) - z) \left( I - \Lambda\Lambda^* + \sum_{i=1}^{m} \left( \frac{\alpha_i I - \Lambda}{1 - z\alpha_i} + \frac{\alpha_i I - \Lambda}{a_i - z} \right) e_i(I - z\Lambda^*) \right) = 0. \tag{6}
\]

**LEMMA 2.1** For \(a_j \in \sigma_f(N),\)

\[
\Lambda^* e_j = \overline{a_j} e_j. \tag{7}
\]

**PROOF** For \(a_j \neq 0,\) the residue at \(z = \frac{1}{a_j}\) of the left-hand side of (6) must be zero. It follows that

\[
((I - \alpha_j\Lambda)C^{-1}(\alpha_j I - \Lambda^*) - \alpha_j)(I - \alpha_j\Lambda)e_j(\alpha_j I - \ Lambda^*) = 0. \tag{8}
\]

Multiplying equation (8) from the left by \((I - \alpha_j\Lambda)^{-1}\) and from the right by \((a_jI - \Lambda)\) and using the facts that (4) holds and \(e_j = e_j^*,\) we have

\[
(C^{-1}(\alpha_j I - \Lambda^*)(I - \alpha_j\Lambda) - \alpha_j) e_j = 0. \tag{9}
\]

By using (4) again, we may show that the left-hand side of (9) equals \((1 - |a_j|^2)C^{-1}(\alpha_j I - \Lambda^*)e_j\) which proves (7).

If one of the \(a_j,\) say \(a_1,\) is zero, then multiplying the left-hand side of (6) by \(z^3\) and letting \(z \to \infty,\) we have

\[
C^{-1}\Lambda^* e_1 \Lambda^* = 0
\]

or \(\Lambda e_1 = 0.\) Thus, \(\Lambda^* \Lambda e_1 = 0.\) In this case, we will then use (4) to show that

\[
C e_1 = \Lambda^* \Lambda e_1,
\]

It then follows that \(e_1 \Lambda = 0\) or \(\Lambda^* e_1 = 0.\)

Define \(R = I - \sum_{i=1}^{m} e_i;\) and \(Q = R - \Lambda^* R \Lambda.\)

**LEMMA 2.2** Suppose \(S \in \mathcal{F}.\) Then

\[
\mu(z) = (zI - \Lambda)(I - z\Lambda^*)^{-1} Q(zI - \Lambda)^{-1}, \quad |z| < 1 \tag{10}
\]

and

\[
e^{-\theta} = \frac{1}{2\pi} \int_{\mathbb{R}} (e^{-i\theta} I - \Lambda^*)^{-1} Q(e^{i\theta} I - \Lambda)^{-1} d\theta, \quad \theta \in \mathbb{R}. \tag{11}
\]

**PROOF** For \(|z| < 1,\) it follows from (7) that
\[ \sum_{i=1}^{m} \frac{a_i \Lambda - I}{1 - z a_i} e_i = - \sum_{i=1}^{m} e_i + (\Lambda - zI) \sum_{i=1}^{m} \frac{a_i (1 - z a_i)^{-1} e_i}{1 - z a_i} = \left( -I + (\Lambda - zI) \Lambda^* (I - z \Lambda^*)^{-1} \right) \sum_{i=1}^{m} a_i \] \tag{12}

and

\[ \sum_{i=1}^{m} \frac{a_i I - \Lambda}{a_i - z} e_i = \sum_{i=1}^{m} e_i + (zI - \Lambda) \sum_{i=1}^{m} e_i (a_i - z)^{-1} = \sum_{i=1}^{m} e_i + (zI - \Lambda) \sum_{i=1}^{m} e_i (\Lambda - zI)^{-1}. \tag{13} \]

From (1), (12), (13), and \((I - \Lambda \Lambda^*) = I - z \Lambda^* + (zI - \Lambda) \Lambda^*\), it follows that

\[ \mu(z) = I + (zI - \Lambda)(\Lambda^* (I - z \Lambda^*)^{-1})R + (R - I)(zI - \Lambda)^{-1} \]

which proves (10). From (2) and (10), we have (11).

**Lemma 2.3** For \(S \in \mathcal{F}\), \(Q\) is an orthogonal projection on \(M\),

\[ Q = C - [\Lambda^*, \Lambda], \tag{14} \]

\[ Q e_i = 0, \quad i = 1, \ldots, m, \tag{15} \]

and

\[ \Lambda Q = 0. \tag{16} \]

**Proof** From Theorem 3 in [5], \(\mu(z) = \mu(z)^2\). This implies that

\[ Q(I - z \Lambda^*)^{-1} Q = Q \]

by (10). Therefore, \(Q = Q^2\). On the other hand, it is obvious that \(Q = Q^*\) since \(e_i \geq 0\) for all \(i\). Thus, \(Q\) is an orthogonal projection.

By (11) and (7), it is easy to calculate that

\[ \int_{\sigma(N)} (\overline{u}I - \Lambda^*) \alpha(du)(uI - \Lambda) = Q. \tag{17} \]

By the equation (cf. Lemma 4.1 of [4]) \(\int_{\sigma(N)} \overline{u} \alpha(du) = \Lambda^*, \int_{\sigma(N)} \alpha(du) = I, \) and (3), it is easy to see that the integral in the left-hand side of (17) is equal to

\[ \int_{\sigma(N)} (\overline{u}I - \Lambda^*) \alpha(du) = \int_{\sigma(N)} ((\overline{u}I - \Lambda^*) (uI - \Lambda) + \Lambda \overline{u} - \Lambda^* \Lambda) \alpha(du) = C - [\Lambda^*, \Lambda]. \]
Thus, (17) is true.

We will next prove (15). By (4), (7), and (14), we have

\[ Ce_i = (\overline{\alpha}_i I - \Lambda^*)(a_i I - \Lambda)e_i \]
\[ = [\overline{\alpha}_i I - \Lambda^*, a_i I - \Lambda]e_i + (a_i I - \Lambda)(\overline{\alpha}_i I - \Lambda^*)e_i \]
\[ = [\Lambda^*, a]e_i \]
\[ = (C - Q)e_i \]

which proves (15).

From (5) and (10), we have

\[ \left( \frac{1}{z} I - \Lambda^* \right)(zI - \Lambda)(I - z\Lambda^*)^{-1}Q - C(I - z\Lambda^*)^{-1}Q = 0 \]  

for all \( |z| < 1 \) except a finite number of points. Therefore, the residue of the left-hand side of (18) at \( z = 0 \) is zero which implies (16).

From (14) and \( Q = R - \Lambda^*R\Lambda \), it follows that

\[ C = I - \Lambda\Lambda^* - \sum_{i=1}^m (1 - |a_i|^2)e_i. \]

COROLLARY 2.4 Let \( S \in \mathcal{F} \). Then \( Q \neq 0 \) and

\[ 0 \in \sigma_p(\Lambda). \]  

If \( 0 \in \sigma_p(N) \), then

\[ \Lambda e(\{0\}) \neq 0. \]  

PROOF If \( Q = 0 \), then (11) implies that \( e(\{T\}) = 0 \). This implies that for \( f \in L^2\{e\} \),

\[ ||f||^2 = \sum_{i=1}^m (e_i f(a_i), f(a_i)). \]

and \( S \) is a subnormal operator on a finite dimensional space, i.e. \( S \) is normal. This leads to a contradiction.

Let \( \eta \) be any eigenvector of \( Q \) with respect to the eigenvalue 1. Then (16) implies that \( \Lambda\eta = 0 \) which implies (19).

If \( \Lambda e(\{0\}) = 0 \), then by (7), \( [\Lambda^*, \Lambda]e(\{0\}) = 0 \). From (14) and (15) it then follows that \( Ce(\{0\}) = 0 \) which contradicts the existence of \( C^{-1} \). Therefore, (20) holds.

LEMMA 2.5 If \( S \in \mathcal{F} \), then

\[ \sigma_p(\Lambda) \subseteq \{u : |u| < 1\}. \]  

PROOF From (11) and \( \int_{\sigma(N)} e(du) = I \), it follows that
\[
\sum_{n=0}^{\infty} \Lambda^n Q \Lambda^n + \sum_{i=1}^{m} e_i = I.
\]

This implies that
\[
\lim_{N \to \infty} \Lambda^N R \Lambda^N = 0.
\]

But this is equivalent to
\[
\lim_{N \to \infty} \Lambda^N \Lambda^N - \sum_{i=1}^{m} |a_i|^{2N} e_i = 0.
\]

However, \(|a_i| < 1, i = 1, \ldots, m\). Therefore, \(\lim_{N \to \infty} \Lambda^N \Lambda^N = 0\) which implies (21).

Define
\[
F_S(u) = Q \left( (uI - \Lambda)^{-1} + C^{-1} \sum_{a_i \in \sigma_p(N) \setminus \{0\}} \frac{1 - |a_i|^2}{a_i} \frac{a_iI - \Lambda}{u - a_i} e_i \right) - \sum_{a_i \in \sigma_p(N) \cap \{0\}} \frac{C^{-1} \Lambda a_i}{u^2}
\]
for \(u \in \rho(\Lambda)\).

**LEMMA 2.6** If \(S \in \mathcal{F}\), then \(F_S(u)\) is a meromorphic function with only one simple pole at \(u = 0\).

**PROOF** From (14) of [5], we know that
\[
\int_{\sigma_p(N)} \frac{uI - \Lambda}{u - z} c(du) = 0 \quad \text{for } |z| > 1.
\]

By (11), we know that
\[
\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{uI - \Lambda}{u - z} (I - u\Lambda^*)^{-1} Q(uI - \Lambda)^{-1} du + \sum_{i=1}^{m} \frac{a_iI - \Lambda}{a_i - z} e_i = 0 \quad \text{for } |z| > 1.
\]

By Lemma 2.1, \(\sigma_p(N) \subset \sigma_p(\Lambda)\). Therefore, we may assume that \(\sigma_p(\Lambda) = \{a_i : i = 1, \ldots, n\}\) where \(n \geq m\). Then there are \(c_{ji} \in \mathcal{L}(M \to M)\) such that
\[
(uI - \Lambda)^{-1} = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \frac{c_{ji}}{(u - a_i)^j}
\]
where \(k_i\) is the order of the pole \(a_i\) of \((uI - \Lambda)^{-1}\), i.e. \(c_{ki} \neq 0\). Substituting (24) for \((uI - \Lambda)^{-1}\) in (23) and applying some calculations, we may see that (23) reduces to
\[
\sum_{i=1}^{n} \sum_{j=1}^{k_i} \frac{L_{ji}}{(a_i - z)^j} = 0
\]
where

\[ L_{li} = \begin{cases} 
\sum_{p=1}^{k_i} D_{lpi} + (a_i I - \Lambda) e_i & \text{if } l = 1, 1 \leq i \leq m \\
\sum_{p=1}^{k_i} D_{lpi} & \text{otherwise}
\end{cases} \]

and \( D_{lpi} = (-1)^l[(\Lambda - a_i I)(I - a_i \Lambda^*)^{-p+l-1} \Lambda^* (p-l) - \theta_{p-l-1}(I - a_i \Lambda^*)^{-p+l} \Lambda^* a_i (p-l-1)] Q_{cpi} \)

such that \( \theta_{\nu} = 1 \) for \( \nu \geq 0 \) and \( \theta_{\nu} = 0 \) for \( \nu < 0 \). Thus,

\[ L_{li} = 0, \quad l = 1, \ldots, k_i \text{ and } i = 1, \ldots, n. \quad (25) \]

We will begin by showing that if \( a_i \neq 0 \), then for \( k_i > 1 \),

\[ Q_{cji} = 0 \quad (26) \]

for \( j = 2, \ldots, k_i \). Let us prove (26) by induction. First, in this case \( L_{ki} = D_{kik_i} = 0 \) which implies that

\[ (\Lambda - a_i I)(I - a_i \Lambda^*)^{-1} Q_{cki} = 0. \quad (27) \]

By the commutation relation, (14), and (16),

\[ [(\Lambda - a_i I), (I - a_i \Lambda^*)^{-1}] = -a_i(I - a_i \Lambda^*)^{-1}[\Lambda^*, I]I - a_i \Lambda^*)^{-1}
\]

\[ = -a_i(I - a_i \Lambda^*)^{-1} C(I - a_i \Lambda^*)^{-1} + a_i(I - a_i \Lambda^*)^{-1} Q. \quad (28) \]

Therefore, (27) implies that

\[ -a_i(I - a_i \Lambda^*)^{-1}\left(I + C(I - a_i \Lambda^*)^{-1} - Q\right)Q_{cki} = 0. \]

Using \( Q = Q^2 \), we can prove that (26) holds for \( j = k_i \). Now, suppose (26) holds for \( j = k_i + 1, \ldots, k_i \) where \( 1 < k_i \leq k_i - 1 \). We only need to show that (26) holds for \( j = k \). In this case, \( D_{lpi} = 0 \) for \( p > k \). Therefore, \( L_{ki} = D_{kki} = 0 \), i.e.

\[ (\Lambda - a_i I)(I - a_i \Lambda^*)^{-1} Q_{cki} = 0. \]

By the same argument used to show that (26) holds for \( j = k_i \), we have \( Q_{cki} = 0 \).

Next, we will assume that \( a_i \neq 0 \) and study \( Q_{cji} \). Suppose \( \theta_{\nu} \) is defined as before. Notice that

\[ L_{li} = D_{l1i} + \theta_{m-i}(a_i I - \Lambda) e_i, \]

since \( D_{lpi} = 0 \) for \( p > 1 \) due to \( Q_{cpi} = 0 \) for \( p > 1 \). Therefore, (25) implies that

\[ (a_i I - \Lambda)(I - a_i \Lambda^*)^{-1} Q_{c1i} + \theta_{m-i}(a_i I - \Lambda) e_i = 0. \quad (29) \]

By the same argument used previously, we may show that

\[ Q_{c1i} = 0 \quad \text{for } i > m. \quad (30) \]

For \( i \leq m \), by (16), (28), and (29), we have

\[ (I - a_i \Lambda^*)(a_i - \Lambda) e_i = -a_i C(I - a_i \Lambda^*)^{-1} Q_{c1i}. \quad (31) \]

However, by (14) and (15), we have
Thus, (31) implies that
\[ Qc_{i} = -(I - a_{i}A^{*})e_{i} - \frac{1}{a_{i}}(I - a_{i}A^{*})C^{-1}(a_{i}I - \Lambda)(1 - |a_{i}|^{2})e_{i} \]  \hspace{1cm} (32)

using (7). Multiplying by $Q$ from the left on both sides of (32) and using (16), we then have
\[ Qc_{i} = -\frac{1 - |a_{i}|^{2}}{a_{i}}QC^{-1}(a_{i}I - \Lambda)e_{i}, \quad i \leq m. \]  \hspace{1cm} (33)

Now consider the case where $a_{i} = 0$. Assume that $i > m$, i.e. $0 \notin \sigma_{p}(N)$. We have to show that (26) holds for $j = 2, \ldots, k_{i}$. We will prove this by mathematical induction. Suppose that $k_{i} \geq 2$. From
\[ D(k_{i-1})(k_{i-1})i + D(k_{i-1})k_{i} = L(k_{i-1})i = 0, \]
we have
\[ (\Lambda^{*} - I)Qc_{k_{i}} = 0. \]  \hspace{1cm} (34)

By (14), $(\Lambda\Lambda^{*} - I)Q = CQ$. Using this fact, $Q = Q^{2}$, and (16), we know that (34) implies $Qc_{k_{i}} = 0$. Thus, (26) holds for $j = k_{i}$. Now suppose that (34) is true for $j = l + 1, \ldots, k_{i}$ where $l > 1$. Then
\[ D(l-1)(l-1)i + D(l-1)i = L(l-1)i = 0. \]
By the same argument used to show that (26) holds for $j = k_{i}$, we may prove that $Qc_{l} = 0$. Hence, (26) holds for $j = 2, \ldots, k_{i}$.

Suppose that $a_{i} = 0$ for some $i \in \{1, \ldots, m\}$. Without loss of generality, assume $i = 1$. Then by the same method as before, we may prove that (26) holds for $j = 3, \ldots, k_{1}$. Now we want to determine $Qc_{21}$ if $k_{1} \geq 2$. In this case,
\[ L_{11} = D_{111} + D_{121} - \Lambda e_{1} = 0. \]

Therefore,
\[ (\Lambda^{*} - I)Qc_{21} + \Lambda e_{1} = 0 \]
which implies
\[ Qc_{21} = C^{-1}\Lambda e_{1}. \]

Now we have to prove that $k_{1} > 1$ if $a_{1} = 0 \notin \sigma_{p}(N)$. Suppose on contrary that $k_{1} = 1$. Then from (25)
\[ -\Lambda Qc_{11} - \Lambda e_{1} = D_{111} - \Lambda e_{1} = L_{11} = 0. \]
which implies
\[ \Lambda e_1 = 0 \quad (35) \]
which contradicts (20).

From these lemmas we have the following theorems.

**THEOREM 1** Suppose \( S \in \mathcal{F} \) with m.n.e. \( N \). Let 
\[ \sigma_p(N) = \{a_i : i = 1, \ldots, m\} \] and \( e_i = e(\{a_i\}) \). Then

(i) \( C > 0 \), \( ||A|| \leq 1 \), \( 0 \in \sigma_p(\Lambda) \), and \( \{a_1, \ldots, a_m\} \subset \sigma_p(\Lambda) \subset \{u \in \mathbb{C} : |u| < 1\} \)

(ii) \( Q \overset{\text{def}}{=} C - [\Lambda^*, \Lambda] \) is a non-zero orthogonal projection, \( Qe_i = 0 \), \( i = 1, \ldots, m \),
\[
Q = I - \Lambda^*\Lambda - \sum_{i=1}^{m} (1 - |a_i|^2)e_i, \text{ and } \Lambda Q = 0
\]

(iii) \( e_i \geq 0 \), \( e_i \neq 0 \), \( 0 \leq I - \sum_{i=1}^{m} e_i \), and \( \Lambda^*e_i = \overline{a_i}e_i \) for \( i = 1, \ldots, m \)

(iv) the function
\[
F_s(u) = Q \left( (uI - \Lambda)^{-1} + C^{-1} \sum_{a_i \in \sigma_p(N) \setminus \{0\}} \frac{1 - |a_i|^2}{a_i} \frac{a_i I - \Lambda}{u - a_i} e_i \right) - \sum_{a_i \in \sigma_p(N) \cap \{0\}} \frac{C^{-1}\Lambda a_i}{u^2}
\]
is a meromorphic function with only one simple pole at \( u = 0 \).

Also, \( e(\text{de}^{i\theta}) = \frac{1}{2\pi i} (e^{-i\theta}I - \Lambda^*)^{-1} Q(e^{i\theta}I - \Lambda)^{-1} \text{d}\theta \) and the mosaic of \( S \) is
\[
\mu(z) = (zI - \Lambda)(J - z\Lambda^*)^{-1}Q(zI - \Lambda)^{-1}.
\]

**THEOREM 2** Let \( M \) be a finite dimensional Hilbert space. Let \( C \) and \( \Lambda \) be two operators in \( \mathcal{L}(M \rightarrow M) \). Then there exists a subnormal operator \( S \in \mathcal{F} \) satisfying \( CS = C \) and \( LS = \Lambda \) if and only if there exist \( \{a_i \in \mathbb{C} : |a_i| < 1, i = 1, \ldots, m\} \) and \( \{e_i : e_i \geq 0, e_i \neq 0, i = 1, \ldots, m\} \subset \mathcal{L}(M \rightarrow M) \) such that (i) - (iv) of Theorem 1 hold.

**PROOF** Theorem 1 implies the “only if” part of this theorem. Now we have to prove the “if” part.

Define a \( \mathcal{L}(M \rightarrow M) \)-valued measure on \( \gamma = \mathbb{T} \cup \{a_1, \ldots, a_m\} \) by (11) where \( e(\{a_i\}) = e_i \) for all \( i \). Then it is obvious that \( e(F) \geq 0 \) for any Borel set \( F \subset \gamma \). It is also easy to see that
\[
\int_{\gamma} e(\text{du}) = \sum_{n=0}^{\infty} \Lambda^*Q\Lambda^n + \sum_{i=1}^{m} e_i.
\]
But \( Q = R - \Lambda^*RA \) where \( R = I - \sum_{i=1}^{m} e_i \). Therefore,
\[
\sum_{n=0}^{\infty} \Lambda^*Q\Lambda^n = R - \lim_{N \to \infty} \Lambda^NRA^N = R
\]
since \( \sigma_p(\Lambda) \subset \{ u : |u| < 1 \} \). Hence,

\[
\int_U e(du) = I.
\]

From methods in the proof of Lemma 2.6 we may prove that

\[
\int_U \frac{uI - \Lambda}{u - z} e(du) = 0 \quad \text{for } |z| < 1.
\]

For \( z \in T \), we have

\[
Q = C - \Lambda^* \Lambda \text{ is a projection and } \Lambda Q = 0.
\]

Therefore, (3) holds. Similarly, we may prove (4). Thus,

\[
\int_F ((uI - \Lambda^*)(uI - \Lambda) - C)e(du) = 0
\]

for any Borel subset \( F \subset T \).

By Theorem 2 of [5], we have a pure subnormal operator \( S \) with m.n.e. \( N \) such that \( \Lambda_S = \Lambda, C_S = C, \) and \( \sigma(N) = \gamma \). Thus, \( S \in \mathcal{F} \).

3. Some applications of Theorems 1 and 2 are included in this section.

**THEOREM 3** Let \( S \) be a pure subnormal operator on a Hilbert space with m.n.e. \( N \). If \( m = \text{rank } [S^*, S] < +\infty \) and \( \sigma(N) \) is a circle, \( \{ u \in \mathbb{C} : |u - \lambda| = c \} \), then \( S = cS_m + \lambda I \)

where \( S_m \) is a unilateral shift with multiplicity \( m \).

**PROOF** Without loss of generality, we may assume that \( c = 1 \) and \( \lambda = 0 \). Then \( \sigma(N) = T \) and \( S \in \mathcal{F} \). Thus, \( R = I, Q = I - \Lambda^* \Lambda, C = I - \Lambda \Lambda^* \), and

\[
\mu(z) = C(I - z\Lambda^*)^{-1}
\]

by (1). From \( \mu(z) = \mu(z)^2 \), we have \( C(I - z\Lambda^*)^{-1} = I \). Hence, \( C = I \) and \( \Lambda = 0 \). Therefore, \( Q = I \) and

\[
e(d\theta) = \frac{1}{2\pi} d\theta.
\]

Thus, \( S = S_m \).

Let \( \mathcal{H}_{a,k}, 1 < |a| < 1 \) and \( k > 0 \), be the space of all analytic functions in \( H^2(T) \) with inner product

\[
(f, g) = \int_{\mathbb{T}} f(z)\overline{g(z)}\,dz + k f(a)\overline{g(a)}.
\]

Let \( M_z \) be the multiplication operator on \( \mathcal{H}_{a,k} \) defined by

\[
(M_z f)(z) = (z)f(z) \quad f \in \mathcal{H}_{a,k}.
\]
THEOREM 4  Let $S \in \mathcal{F}$ with m.n.e. $N$. Suppose rank $[S^*, S] = 2$. Then either
$
\sigma_p(N) = \emptyset$ and $S$ is a unilateral shift with multiplicity two or $\sigma_p(N) = \{a\}$, $|a| < 1$, and $S$

is unitarily equivalent to the operator $M_z$ on $\mathcal{H}_{a,k}$ with some $k > 0$.

PROOF If is easy to see that $\sigma_p(N)$ contains at most two points since
$\dim M = 2$ where $M = \text{range of } [S^*, S]$. We have to show that $\sigma_p(N)$ contains at most one
point. If $0 \in \sigma_p(N)$, then $0$ is a pole of order two of $(uI - \Lambda)^{-1}$. Therefore, $\sigma_p(N) = \{0\}$. If
$0 \notin \sigma_p(N)$ but $0 \in \sigma_p(\Lambda)$, then $\sigma_p(N) \subset \sigma_p(\Lambda)\{0\}$ contains at most one point. Thus, $\sigma_p(N)$
contains at most one point.

If $\sigma_p(N) = \emptyset$, then Theorem 4 follows from Theorem 3. Now we only have
to consider the case where $\sigma_p(N) = \{a\}$. We will consider the analytic model of $S$. The
projection $Q \neq I$ since $Qe(\{a\}) = 0$. Also, the projection $Q \neq 0$. Therefore, rank $Q = 1$. We
may then choose an orthonormal basis for $M$ such that
$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

From $\Lambda Q = 0$, it follows that
$$\Lambda = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}.$$ 

From $C = [\Lambda^*, \Lambda] + Q$ and $C > 0$, we have $b \neq 0$ where $|a|^2 + |b|^2 < 1$. From $Qe(\{a\}) = 0$, we
know that
$$e(\{a\}) = \begin{pmatrix} e_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Using the fact that $m(z) = \int_{\sigma(N)} \frac{uI - \Lambda}{u - z} e(du) = 0$ for $|z| > 1$ and (11), we can show that
$$e_{11} = \frac{1 - |b|^2 - |a|^2}{1 - |a|^2}. \quad \text{For }$$ 

let
$$\hat{f}(u) = \frac{1}{1 - au} (bf_1(u) + (u - a)f_2(u)).$$ 

It is easy to calculate that
$$\int_{\sigma(N)} (e(du)f(u), f(u)) = \frac{1}{2\pi} \int_0^\pi |\hat{f}(e^{i\theta})|^2 d\theta + k|\hat{f}(a)|^2$$

where $k = \frac{(1 - |a|^2)^2(1 - |b|^2)}{|b|^2}$.

Let $W$ be the operator such that $Wf = \hat{f}$. Then $W$ is a
unitary operator from the Hilbert space $\mathcal{H}$ defined in Theorem 1 of [5] to $\mathcal{H}_{a,k}$. It is easy to see that $WSW^{-1} = M_z$ and so the theorem is proven.
Let $|A|$ denote the cardinal number of the set $A$.

**Theorem 5** Let $\mathcal{F}_m$ be the sub-family of operators $S$ in $\mathcal{F}$ satisfying
\[
\text{rank } [S^*, S] = m, \ 0 < m < +\infty.
\]
For an operator $S \in \mathcal{F}_m$, let $N_S$ be the m.n.e. of $S$. Then
\[
\max \{|\sigma_p(N_S)| : S \in \mathcal{F}_m\} = m - 1.
\]

Remark: In the case of $m = 1$, Theorem 5 is a corollary of Morrel's Theorem in [2]. In the case of $m = 2$, Theorem 5 is a corollary of Theorem 4.

**Proof** We will assume that $m > 2$. For $S \in \mathcal{F}_m$, it is obvious that $|\sigma_p(\Lambda)| \leq m$. By Corollary 2.4, $0 \in \sigma_p(\Lambda)$. Let us consider the following cases.

Case 1: If the order of the pole 0 of $(uI - \Lambda)^{-1}$ is at least 2, then $|\sigma_p(\Lambda)| \leq m - 1$. But, $\sigma_p(N_S) \subset \sigma_p(\Lambda)$. Therefore, $|\sigma_p(N_S)| \leq m - 1$.

Case 2: If 0 is a simple pole of $(uI - \Lambda)^{-1}$, then $0 \notin \sigma_p(N_S)$. Thus, $\sigma_p(N_S) \subset \sigma_p(\Lambda) \setminus \{0\}$. But, $|\sigma_p(\Lambda) \setminus \{0\}| \leq m - 1$. Therefore, we still have
\[
|\sigma_p(N_S)| \leq m - 1 \quad \text{for } S \in \mathcal{F}_m. \tag{36}
\]

Next, we will construct an example of $N_S$, $S \in \mathcal{F}_m$, such that $|\sigma_p(N_S)| = m - 1$.

Let $\{a_1, \ldots, a_{m-1}\}$ be different points in the open unit disk and $a_i \neq 0$, $i = 1, \ldots, m - 1$. Let $\mathcal{H} = H^2(\mathbb{T})$ with a different inner product defined by
\[
(f, g) = (f, g)_{H^2(\mathbb{T})} + \sum_{i=1}^{m-1} f(a_i)\overline{g(a_i)}.
\]
Define
\[
(Sf)(z) = zf(z), \quad f \in \mathcal{H}.
\]
By a direct calculation, it is not difficult to show that
\[
\text{rank } [S^*, S] \leq m
\]
and $\sigma_p(N_S) = \{a_1, \ldots, a_{m-1}\}$. But, $\text{rank } [S^*, S] \geq |\sigma_p(N_S)| + 1$ by (36). Therefore, $S \in \mathcal{F}_m$ and $|\sigma_p(N_S)| = m - 1$.

**References**


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