Toeplitz Type Operators, Determining Functions, Principal Functions, and Trace Formulas

JOEL D. PINCUS

Department of Mathematics, State University of New York, Stony Brook, New York 11794

AND

DAOXING XIA

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37235

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We introduce and solve an operator valued Riemann–Hilbert problem defined by symbols associated with weak contractions $T$ and operators $A$ for which $TAT^* = A$. The solution of this problem leads to a connection between such objects as the eigenvalue distribution of $|\Theta_T(e^{it})|$ and the principal function of the hyponormal operator $H = (s\text{-lim}_{n \to \infty} T^n T^*)^{1/2} T^*$.

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1. INTRODUCTION

For a weak contraction $T$ the eigenvalue distribution of $|\Theta_T(e^{it})|$ is shown to give information about the $C^*$ algebra generated by $T$ and $A_T = s\text{-lim}_{n \to \infty} T^n T^*$. 

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If $N_T(\rho e^{it})$ is the number of eigenvalues of $\Theta_T(e^{it})\Theta_T(e^{it})$ in the interval $[0, \rho]$, then $g_T(\rho e^{it}) = -N_T(\rho^2 e^{it})$ is the principal function of the hyponormal operator $H = A^{1/2} T^*.$

This function gives us, among other things, relations such as

$$\begin{align*}
\text{Trace}\left[ f(T^*, T, A_T), h(T^*, T, A_T) \right] &= \frac{1}{\pi} \int \int \frac{\partial(f(e^{it}, e^{-it}, \rho^2), h(e^{it}, e^{-it}, \rho^2))}{\partial(\rho, \theta)} g_T(\rho e^{it}) \rho \, d\rho \, d\theta 
\end{align*}$$

where $f(T^*, T, A_T)$ is an operator associated with the complex valued function $f(e^{it}, e^{it}, x)$ on $S^1 \times S^1 \times \mathbb{R}$ by means of a suitable functional calculus.

Since the Fredholm index of $f(T^*, T, A_T)$ is determined by the principal current, a result such as the foregoing establishes a relation between the extension class of $\sigma_{\text{ess}}(H)$ determined by the $C^*$ algebra generated by $\{T, A_T\}$ and the characteristic operator $\Theta_T$. Indeed since $\sigma_{\text{ess}}(H)$, and thus the joint essential spectrum of $\{T, A_T\}$, is often determined completely by the essential discontinuity set of $g_T$ (for example, when $H^*H - HH^*$ has one dimensional range) even questions such as which operators $f(T^*, T, A_T)$ are Fredholm are seen now to be related to the eigenvalue distribution of $|\Theta_T(e^{it})|$. It is expected that the spectral density of this operator will play a similar role even in cases where $T$ is not a weak contraction. (Note that we have defined the principal function for an arbitrary hyponormal operator [10].)

The indicated results are obtained by a procedure which associates a Riemann–Hilbert problem to $T$ and any bounded operator $A$ for which $TAT^* = A$. Such operators are called $T^*$-Toeplitz.

The solution of this Riemann–Hilbert problem can be looked upon as a moment generating function for operator valued moments of the form $\{DA^*T^*kD; j, k = 1, 2, \ldots\}$, where $D = (I - T^*T)^{1/2}$ and the operator pair $\{T, A\}$ can be thought of as a “realization” of these moments.

The utility of these determining functions comes from their connection with symbols of a type introduced earlier [9, 3, 4]. The appropriate objects are

$$A_\pm = \lim_{n \to \infty} \tilde{U}^n \tilde{A} \tilde{U}^{-n},$$

where $\tilde{U}$ is the residual unitary part of the minimal isometric dilation of $T$ and $\tilde{A}$ is a certain (inesential) modification of $A$.

The Riemann–Hilbert Problem

One of our symbols is shown to coincide with an object recently studied by B. Sz.-Nagy and C. Foias [20, 21]. The other symbol, $A_-$, is shown to
have the form \([|\Theta|] A_+ [|\Theta|]\), where \([|\Theta|]\) specifies a certain action of the characteristic operator function of \(T\).

Forming the Riemann–Hilbert barrier in terms of the two symbols \(A_\pm\) as in \([9, 3, 4]\) and using the foregoing analysis leads us to the expression of the barrier in the form

\[
\xi(e^{it}, \mu) = I - \Theta(e^{it}) A_+(e^{it})(A_+(e^{it}) - \mu I)^{-1} \Theta(e^{it}),
\]

where \(I - \Theta(e^{it}) \Theta(e^{it}) = \Delta(e^{it})^2\) and \(\mu \in \rho(A)\).

We then show that, in a suitable sense, and this is our main result,

\[
\lim_{\rho \to 1^+} \Theta(\rho e^{it}, \mu) \xi(e^{it}, \mu) = \lim_{\rho \to 1^+} \Theta(\rho e^{it}, \mu),
\]

where \(\Theta(\lambda, \mu) = (-T + \lambda D_*(A - \mu I)^{-1} A(I - \lambda T^*)^{-1} D)|_{\gamma^{\prime}}\) for \(|\lambda| < 1\), \(\mu \in \rho(A)\) and

\[
\Theta(\lambda, \mu) = -TD_\star \left( \frac{1}{\lambda}, \mu \right)^*.
\]

for \(|\lambda| > 1\), \(\mu \in \rho(A)\) with

\[
D_\star(\lambda, \mu) = (I - D(A^* - \mu I)^{-1} A^*(I - \lambda T^*)^{-1} D)|_{\gamma^{\prime}}.
\]

This solution of the Riemann–Hilbert problem is exploited in the remainder of the paper in two main cases:

(1) First we treat the case whose description we gave above where \(A = A_T\) (and \(A_+ \equiv I\)).

In this case we also obtain such relations as

\[
\det D_\star(\lambda, \mu) = \exp \frac{1}{\pi} \int \frac{g_T(\rho e^{it}) \rho \, d\rho \, dt}{(\rho^2 - \mu)(1 - \lambda e^{it})}
\]

and we identify \(g_T(\rho e^{it})\) as the principal function of the hyponormal operator \(H \equiv A_T^{1/2} T^*\).

Further developments, which extend significantly results of Sachnovich and Adamjan-Pavlov for phase shift formulae for dissipative operators, are described in Theorems 5.5 and 6.5.

(2) Our second application concerns contraction operators \(T\) so that \(H^* = TH\), where \(H\) is a hyponormal operator with \([H^*, H] \in \mathcal{L}_1\). We note that \(H\) is \(T^*-\)Toeplitz, and find that

\[
\det \xi(\rho^{it}, \mu) = \exp \int_0^{\infty} \left( \frac{g(re^{it/2})}{r - \mu e^{it/2}} + \frac{g(-re^{-it/2})}{r + \mu e^{-it/2}} \right) \, dr,
\]
where $g$ is the principal function of the given $H$. Indeed, we show there is a positive operator $B(\cdot)$ so that $B(\cdot)$ is a complete unitary invariant for $H$.

$$\xi(e^{it}, ze^{-it^2}) = \exp - \int_0^\infty \frac{B(\rho e^{-it^2})}{\rho - z} \, d\rho,$$

and trace $B(\xi) = -g(\xi)$. (We show that this $B$ is different from the mosaic of $H$ introduced previously by Carey and Pincus in [4].)

In Section 10 we consider the implications in the case where $[H^*, H]$ has one dimensional range. We show that $T = H^*H^{-1}$ is completely non-unitary iff the intersection of the ray $(\rho e^{it})$ with $\sigma(H)$ is either a $\rho$ interval for almost all $t$ or is empty.

We also show that even if $H$ is not invertible there is a contraction $T$ so that $H^* = TH$ and the spectral multiplicity $m_\nu(\xi)$ of the minimal unitary dilation $\mathcal{U}$ of $T$ is geometrically determined by $\sigma(H)$; namely, if $-g(\rho e^{-it^2})$ is $L^1$ equivalent to the characteristic function of $n$ disjoint intervals then $m_\nu(\xi) = n$; otherwise, $m_\nu(\xi) = \infty$.

## 2. Preliminary Remarks

In this paper, $\mathcal{L}(\mathcal{H})$ denotes the algebra of all operators on Hilbert space $\mathcal{H}$, $\mathcal{L}_1(\mathcal{H})$ the trace ideal and $\mathcal{L}_2(\mathcal{H})$ the Hilbert–Schmidt class, and the letter $H$ always denotes a hyponormal operator, i.e., a linear bounded operator in $\mathcal{L}(\mathcal{H})$ satisfying $[H^*, H] \geq 0$, where $[H^*, H]$ denotes the self-commutator $H^*H - HH^*$ of $H$. For the hyponormal operator $H$, there is a largest subspace $\mathcal{K}_n$ of $\mathcal{H}$ such that $H$ reduces $H$ and $H|_{\mathcal{K}_n}$ is normal. The operator $H|_{\mathcal{K}_n}$ is called the completely non-normal part of $H$. Since we are only interested in the non-normal part, we may suppose that $\mathcal{K}_n = \{0\}$, in this case $H$ is called completely hyponormal.

Let $T$ be a contraction on $\mathcal{H}$, i.e., $T \in \mathcal{L}(\mathcal{H})$ and $\|T\| \leq 1$. Denote

$$D = (I - T^*T)^{1/2}, \quad D_* = (I - TT^*)^{1/2},$$

which are called the defect operators of $T$. The subspaces

$$\mathcal{D} = \overline{D\mathcal{H}}, \quad \mathcal{D}_* = \overline{D_*\mathcal{H}}$$

are called the defect subspaces of the contraction $T$. Some notation concerning the contraction will be adapted here from [19, 23] without any explanation. The Hilbert space $\mathcal{H}$ may be decomposed into

$$\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_u.$$
such that $\mathcal{H}$ reduces $T$, $U_u = T|_{\mathcal{H}_u}$ is unitary, and $T_c = T|_{\mathcal{H}_c}$ is completely non-unitary.

We consider the Sz. Nagy–Foias model. There is a unitary operator $V_c$ from $\mathcal{H}_c$ onto the function space

$$\mathcal{H}_c = H^2(D) \oplus \overline{AL^2(D) \oplus \{ \Theta(\cdot) W(\cdot) \oplus \Delta(\cdot) W(\cdot), W \in H^2(D) \}},$$

which has the form

$$V_c x = D^*(z - z^*)^{-1} x \oplus \Delta(e^{it}) \sum e^{-i\lambda a_n},$$

for $x = \sum_n T^{n-1} a_n$, $a_n \in D$, such that

$$(V_c T_c V_c^{-1})(u \oplus v) = e^{-it}(u(e^{it}) - u(0)) \oplus e^{-it}v(e^{it})$$

(cf. [19] or [23]). Here $\Theta(\cdot)$ is the characteristic operator function of Livsic and Smulyan (see Sz.-Nagy and Foias [19]),

$$\Theta_T(\lambda) = [-T + \lambda D^*(I - \lambda^*T)^{-1} D] |_{\mathcal{D}}$$

and

$$\Delta(e^{it}) = (I - \Theta_T(e^{it}) \Theta_T(e^{it}))^{1/2}.$$  

The relations

$$TD = D^* T, \quad T^* D^* = DT^*$$

are also useful.

Let $T$ be a contraction on $\mathcal{H}$ and $A$ be a linear operator on $\mathcal{H}$ which is affiliated to $T$ so that

$$TAT^* = A.$$  

In this case, $A$ is said to be a $T^*$-Toeplitz operator. Let $U_+$ be the minimal isometric dilation of $T$ on the space $\mathcal{H}_+$ ($\mathcal{H} \subset \mathcal{H}_+$). Let $U$ be the unitary part in the “Wold decomposition” of $U_+$, acting on the subspace

$$G = \bigcap_{n=0}^{\infty} U^*_+ \mathcal{H}_+$$

of $\mathcal{H}_+$.

Denote by $P_G$ the (orthogonal) projection of $\mathcal{H}_+$ onto its subspace $G$. Let $X = P_G|_{\mathcal{H}}$. Then there is an operator $\hat{A} \in \mathcal{L}(G)$ commuting with $U$ such that

$$A = X^* \hat{A} X$$

(cf. [20]). This $\hat{A}$ is said to be the $T^*$ symbol of $A$. 


Let \( H \) be a hyponormal operator on \( \mathcal{H} \), then there exists a contraction \( T \) on \( \mathcal{H} \) such that

\[
TH = H^*. \tag{2.8}
\]

If \( H \) is invertible, then \( T = H^*H^{-1} \). From (2.8), it follows that

\[
THT^* = H. \tag{2.9}
\]

Thus \( H \) is \( T^* \)-Toeplitz. In \([20, 21]\) it is proved that the symbol \( N = \hat{H} \) is a normal operator satisfying \( \|N\| = \|H\| \) and

\[
NU = UN = N^*. \tag{2.10}
\]

The weak contraction \( T \) satisfies: \( I - T^*T \in \mathcal{L}_1(\mathcal{H}) \), where \( \mathcal{L}_1(\mathcal{H}) \) is the trace ideal of \( \mathcal{L}(\mathcal{H}) \), and there is at least one regular point \( \lambda \) of \( T \) satisfying \( |\lambda| < 1 \). By a linear fractional transformation

\[
T_\lambda = (T - \lambda I)(I - \lambda T)^{-1}
\]
a weak contraction \( T \) is transformed to an invertible weak contraction. Hence, for simplicity, in this paper, when we investigate a weak contraction, we always assume that it is invertible.

3. Determining Functions

1. Let \( T \) be a contraction in \( \mathcal{H} \) and let \( A \in \mathcal{L}(\mathcal{H}) \) be a \( T^* \)-Toeplitz operator, i.e.,

\[
TAT^* = A. \tag{3.1}
\]

The following properties of \( A \) and \( T \) are useful. For \( \lambda \in \rho(T) \cap \{ \lambda : \lambda = 0 \) or \( \lambda^{-1} \in \rho(T^*) \} \)

\[
A(I - \lambda T^*)^{-1} = (T - \lambda I)^{-1}TA. \tag{3.2}
\]

Let \( D \) and \( D_* \) be the defect operators of \( T \). From (3.1), it follows that

\[
[T, A] = TAD^2 \tag{3.3}
\]

and

\[
[T^*, A] = -D^2AT^*. \tag{3.4}
\]

2. Define an \( \mathcal{L}(\mathcal{D}) \)-valued analytic function of two complex variables

\[
D(\lambda, \mu) = (I - \lambda DAT^* (A - \mu I)^{-1} (I - \lambda T^*)^{-1} D)_{\mathcal{D}} \tag{3.5}
\]
for $|\lambda| < 1$ and $\mu \in \rho(A)$. The function $D(\cdot, \cdot)$ has the following properties:

\begin{equation}
DD(\lambda, \mu) = \{ A \mu I, I \lambda T^* \} D,
\end{equation}

where $\{L, M\} = LML^{-1}M^{-1}$ is the multiplicative commutator.

**Proof.** From (3.4), it follows that (1)

\begin{align*}
(A - \mu I)(I - \lambda T^*)(A - \mu I)^{-1}(I - \lambda T^*)^{-1} \\
= I + [A - \mu I, I - \lambda T^*](A - \mu I)^{-1}(I - \lambda T^*)^{-1} \\
= I - \lambda D^2AT^*(A - \mu I)^{-1}(I - \lambda T^*)^{-1}. 
\end{align*}

This implies (3.6).

(2) If $T$ is invertible, then

\begin{equation}
D(\lambda, \mu) D = D \{ T^{-1}, (A - \mu I)^{-1} \} \{ (A - \mu I)^{-1}, (T - \lambda I)^{-1} \}
\end{equation}

for $\mu \in \rho(A)$ and $|\lambda| < 1$.

**Proof.** From (3.3) and (3.2), it follows that

\begin{align*}
T^{-1}(A - \mu I)^{-1} T(T - \lambda I)^{-1} (A - \mu I)(T - \lambda I) \\
= T^{-1}(A - \mu I)^{-1} T(T - \lambda I)^{-1} T AD^2 \\
= I + T^{-1}(A - \mu I)^{-1} T AD^2 - T^{-1}(A - \mu I)^{-1} (T - \lambda I)^{-1} T^2 AD^2 \\
= I - \lambda T^{-1} A(A - \mu I)^{-1} (I - \lambda T^*)^{-1} D^2,
\end{align*}

which proves (3.7).

(3) For $|\lambda| < 1$ and $\mu \in \rho(A)$, the function $D(\lambda, \mu)$ is invertible and

\begin{equation}
D(\lambda, \mu)^{-1} = (I + \lambda DAT^*(I - \lambda T^*)^{-1}(A - \mu I)^{-1} D)|_{\mathcal{D}}.
\end{equation}

3. Define another $L(\mathcal{D})$-valued analytic function of two variables

\begin{equation}
D_*(\lambda, \mu) = (I - D(A^* - \mu I)^{-1} A^*(I - \lambda T^*)^{-1} D)|_{\mathcal{D}}
\end{equation}

for $|\lambda| < 1$ and $\mu \in \rho(A^*)$. This function $D_*(\cdot, \cdot)$ has the following properties.

\begin{equation}
DD_*(\lambda, \mu) = (T^*TA^* - \mu I)(I - \lambda T^*)(A^* - \mu I)^{-1}(I - \lambda T^*)^{-1} D.
\end{equation}

If $T$ is invertible then (3.10) is equivalent to

\begin{equation}
DD_*(\lambda, \mu) = \{ T^*, A^* - \mu I \} \{ A^* - \mu I, I - \lambda T^* \} D.
\end{equation}
(2) If \( \lambda \in \rho(T) \), then
\[
D_*(\lambda, \mu) D = D\{(A^* - \mu I)^{-1}, (T - \lambda I)^{-1}\}. \tag{3.12}
\]

(3) If \( T \) is invertible, then \( D_*(\cdot, \cdot) \) is invertible and
\[
D_*(\lambda, \mu)^{-1} = (I + DA^*T*(I - \lambda T^*)^{-1}(A^* - \mu I)^{-1}T^*D\})|_{\mathcal{D}}. \tag{3.13}
\]

If \( \mu \in \rho(T^*TA^*) \) then \( D_*(\lambda, \mu) \) is invertible and
\[
D_*(\lambda, \mu)^{-1} = (I + DA^*(I - \lambda T^*)^{-1}(T^*TA^* - \mu I)^{-1}D)|_{\mathcal{D}}. \tag{3.14}
\]

(4) The functions \( D(\cdot, \cdot) \) and \( D_*(\cdot, \cdot) \) satisfy
\[
I - D_*(\lambda', \mu') D(\lambda, \mu) = (1 - \lambda' T) D(I - \lambda' T)^{-1}(A - \mu I)^{-1}A(I - \lambda T^*)^{-1}D. \tag{3.15}
\]

If \( T \) is invertible, then
\[
I - D_*(\lambda, \mu) D(\lambda, \mu) = (1 - \lambda T) D(I - \lambda T)^{-1}(A - \mu I)^{-1}A(I - \lambda T^*)^{-1}D. \tag{3.16}
\]

for \( \lambda \in \rho(T) \).

If \( \rho(T) \cap \mathcal{T} \neq \emptyset \), where \( \mathcal{T} = \{ z \in \mathbb{C}, |z| = 1 \} \), then \( D_*(1/\lambda, \mu)^{-1} \) is the analytic continuation of \( D(\lambda, \mu) \) on \( |\lambda| > 1 \).

4. Define an \( \mathcal{L}(\mathcal{D} \to \mathcal{D}) \)-valued analytic function of two complex variables,
\[
\Theta(\lambda, \mu) = (-T + \lambda D_*(A - \mu I)^{-1}A(I - \lambda T^*)^{-1}D)|_{\mathcal{D}}, \tag{3.17}
\]

for \( |\lambda| < 1 \) and \( \mu \in \rho(A) \). This function will be referred to as the determining function of the \( T^*-\)Toeplitz operator \( A \). It is evident that if \( 0 \in \rho(A) \), then
\[
\Theta(\lambda, 0) = \Theta_T(\lambda)
\]
(see (2.3)). The function \( \Theta(\cdot, \cdot) \) has the following properties:

(1) \( \Theta(\lambda, \mu) = -TD(\lambda, \mu). \tag{3.18} \)

(2) If \( \lambda \in \rho(T) \) and \( |\lambda| < 1 \), then
\[
\Theta(\lambda, \mu) D = D_*(A - \mu I)^{-1}(T - \lambda I)^{-1}T(A - \mu I)(\lambda I - T). \tag{3.19}
\]

If \( T \) and \( A \) are invertible and \( |\lambda| < 1 \), then
\[
\Theta(\lambda, \mu) D = D_*(I - \mu A^{-1})^{-1}(I - \lambda T^*)^{-1}(I - \mu A)^{-1}(\lambda I - T). \tag{3.20}
\]
(3) \[ D_\lambda \theta(\lambda, \mu) = T(\mu I - A)(I - \lambda T^*)(A - \mu I)^{-1}(I - \lambda T^*)^{-1} D. \quad (3.21) \]

(4) If \( T \) is invertible, then \( \theta(\lambda, \mu) \) is invertible and
\[
\theta(\lambda, \mu)^{-1} = (-T^{-1} - DAT^*(I - \lambda T^*)^{-1}(A - \mu I)^{-1} T^{-1} D_\lambda)_{|D_\lambda}. \quad (3.22)
\]

**Lemma 1.** If \( T \) is a contraction, \( A \) is a \( T^* \)-Toeplitz operator, and \( T \) and \( A \) are invertible, then
\[
\sigma(T) \subset T. \quad (3.23)
\]

*Proof.* Since \( \theta(\lambda, 0) \) is invertible for \(|\lambda| < 1\), \( \theta_T(\lambda) \) is invertible for \(|\lambda| < 1\). Thus (3.23) holds (cf. [19] or [23]).

5. Define an \( \mathcal{L}(\mathcal{D}_* \to \mathcal{D}) \)-valued analytic function of two complex variables
\[
\theta_*(\lambda, \mu) = (-T^* DA^* T^*(I - \lambda T^*)^{-1}(A^* - \mu I)^{-1} D)_{|\lambda}. \quad (3.24)
\]
for \(|\lambda| < 1\) and \( \mu \in \rho(A^*) \). The function \( \theta_*(\lambda, \mu) \) has the following properties:

(1) \[ D\theta_*(\lambda, \mu) = (I - \lambda T^*) (A^* - \mu I)(I - \lambda T^*)^{-1} T^*(A^* - \mu I)^{-1} D. \quad (3.25) \]

(2) If \( T \) and \( A \) are invertible then
\[
\theta_*(\lambda, \mu) D_\lambda = -DA^* T^*(I - \lambda T^*)^{-1} (A^* - \mu I)^{-1} \times T^*(A^* T^*)^{-1} A^* (I - \lambda T^*)^{-1} D. \quad (3.26)
\]

(3) If \( T \) is invertible, then \( \theta_*(\lambda, \mu) \) is invertible and
\[
\theta_*(\lambda, \mu)^{-1} = -T^* D_\lambda(\lambda, \mu) \\
= (-T^* + D_\lambda T^* - (A^* - \mu I)^{-1} A^*(I - \lambda T^*)^{-1} D)_{|\lambda}. \quad (3.27)
\]

(4) If \(|\lambda| < 1\), \(|\lambda'| < 1\), and \( \mu \in \rho(A) \), then
\[
\theta_*(\lambda', \mu') - \theta(\lambda, \mu) = (1 - \lambda'/\lambda) D_\lambda(\mu I - A)^{-1} (I - \lambda T^*) T A(I - \lambda T^*)^{-1} D. \quad (3.28)
\]

(5) If \( \theta_*(\lambda', \mu) \) is invertible, then
\[
I - \theta_*(\lambda', \mu)^{-1} \theta(\lambda, \mu) \\
= (1 - \lambda'/\lambda) D(I - \lambda' T)^{-1} (A - \mu I)^{-1} A(I - \lambda T^*)^{-1} D. \quad (3.29)
\]
If $T$ and $A$ are invertible, then
\[
I - \Theta(\lambda, \mu) \Theta^*_*(\lambda', \mu')^{-1} - (1 - \bar{\lambda}' \lambda) D_* (\mu I - A)^{-1} (I - \bar{\lambda}' T) T (\mu I - A) T
\times (I - \bar{\lambda}' T)^{-1} A (A - \mu I)^{-1} T^{-1} D_*.
\] (3.30)

6. Let the contraction $T$ be invertible. Define an $\mathcal{L}(\mathcal{D})$-valued analytic function of two variables
\[
E(\lambda, \mu) = (I + \lambda \mu D T^{-1} (I - \lambda T)^{-1} (A - \mu I)^{-1} D)|_\mathcal{D}.
\] (3.31)

This function has the following properties:

1. \[E(\lambda, \mu) D(\lambda, \mu) = (I - \lambda \mu D T^{-1} (I - \lambda T)^{-1} (A - \mu I)^{-1} D)|_\mathcal{D} = -T^{-1} \Theta_T(\lambda).
\] (3.32)

2. \[DE(\lambda, \mu) = (I - \lambda T^{-1} (A - \mu I)(I - \lambda T)^{-1} (A - \mu I)^{-1} D.
\] (3.33)

3. \[E(\lambda, \mu) D = DT^{-1} (I - \lambda T)^{-1} (A - \mu I)^{-1} (T - \lambda I) T^{-1} (A - \mu I) T.
\] (3.34)

4. If $\lambda \in \rho(T)$, then $E(\lambda, \mu)$ is invertible and
\[
E(\lambda, \mu)^{-1} = (I - \lambda \mu DT^{-1} (A - \mu I)^{-1} T (T - \lambda I)^{-1} D)|_\mathcal{D}.
\] (3.35)

If $T^{-1} \in \mathcal{L}(\mathcal{H})$, then define another $\mathcal{L}(\mathcal{D})$-valued analytic function
\[
E_*(\lambda, \mu) = (I + \mu D(I - \lambda T)^{-1} T^*(A^* - \mu I)^{-1} T^*^{-1} D)|_\mathcal{D}.
\] (3.36)

for $|\lambda| < 1$, $\mu \in \rho(A^*)$. The function $E_*(\cdot, \cdot)$ has the following properties:

1. \[E_*(\lambda, \mu) D = D(T^{-1} - \lambda I)^{-1} (A^* - \mu I)^{-1} (T - \lambda I) (A^* - \mu I).
\] (3.37)

2. \[DE_*(\lambda, \mu) = T^*(T - \lambda I) (A^* - \mu I)(T^{-1} - \lambda I)^{-1} (A^* - \mu I)^{-1} T^*^{-1} D.
\] (3.38)

3. \[E_*(\lambda, \mu) D_*(\lambda, \mu) = (I - D(I - \lambda T^*)^{-1} D)|_\mathcal{D} = -T^* \Theta(\lambda).
\] (3.39)

4. The functions $E(\cdot, \cdot)$ and $E_*(\cdot, \cdot)$ are related by
\[
I - E_*(\lambda', \mu')^* E(\lambda, \mu)
= (\lambda \lambda' - 1) \mu DT^{-1} (A - \mu I)^{-1} (I - \lambda' T)^{-1} T (A - \mu I)
\times (I - \lambda T)^{-1} (A - \mu I) D
\] (3.40)

and
\[
I - E(\lambda, \mu) E_*(\lambda', \mu')^*
= (\lambda \lambda' - 1) \mu DT^{-1} (I - \lambda T^*)^{-1} (A - \mu I)^{-1} T (I - \lambda T)^{-1} D.
\] (3.41)
4. Symbols and the Riemann–Hilbert Problem

In this section, we consider the functional model of the contraction $T$. The representation of the $T^*$-Toeplitz operator $A$ and the related operators $X, U, \ldots$ in the functional model are denoted by $A, X, \tilde{U}, \ldots$ respectively. Denote $G_0 = \mathcal{A}L^2(\mathcal{D})$ and $\tilde{X}_0 = P_{G_0} \upharpoonright \mathcal{H}$, i.e., if $u \oplus v \in \mathcal{H}$, then $\tilde{X}_0(u \oplus v) = v$. It is easy to verify that

$$\tilde{X}_0^* h = -\Theta P(Ah) \oplus (I - APA)h, \quad h \in G_0, \quad (4.1)$$

where $P$ is the projection to the Hardy space $H^2(\mathcal{D})$. In fact, if

$$u = -\Theta P(Ah) \quad \text{and} \quad v = (I - APA)h$$

then

$$P(\Theta^*u + Av) = 0,$$

thus $u \oplus v \in \mathcal{H}$. On the other hand, for any $u_1 \oplus v_1 \in \mathcal{H}$,

$$(u \oplus v, u_1 \oplus v_1) = -(Ah, P(\Theta^*u_1 + Av_1)) + (h, v_1) = (h, \tilde{X}_0(u_1 \oplus v_1)),$$

which proves (4.1).

Hence

$$\tilde{X}_0 \tilde{X}_0^* h = (I - APA)h, \quad h \in G_0. \quad (4.2)$$

If $h \in \ker(\tilde{X}_0^*)$, then

$$(I - APA)h = 0 \quad \text{and} \quad \Theta P(Ah) = 0 \quad (4.3)$$

by (4.1). Thus

$$\|h\|^2 - \|P Ah\|^2 = ((I - APA)h, h) = 0.$$

This implies $Ah \in H^2(\mathcal{D})$ and $\|Ah\| = \|h\|$. Thus

$$h = Ah \in H^2(\mathcal{D}).$$

Hence $\Theta h = 0$ by (4.3). Conversely, if $\Theta h = 0$ and $h \in H^2(\mathcal{D})$, then (4.3) is satisfied. Thus

$$\ker(\tilde{X}_0^*) = \{h \in H^2(\mathcal{D}) : \Theta h = 0\}.$$
In the special situation where $T$ is invertible, then $\Theta(z)$ is invertible for sufficiently small $|z|$. Therefore

$$\ker(\tilde{T}_0^*) = \{0\}$$

if $T$ is invertible.

On the other hand, $\ker(\tilde{T}_0) = \{u \oplus 0: u \in H^2(\mathcal{D}_*) \text{ and } P(\Theta^*u) = 0\}$. If $u \in \ker(\tilde{T}_0)$, then

$$\tilde{A}u = \tilde{T}_0^*\tilde{T}_0 u = 0.$$ 

Hence $\ker(\tilde{T}_0) \subseteq \ker(\tilde{A})$. If $\ker(\tilde{A}) = \{0\}$ then $\ker(\tilde{T}_0) = 0$.

In general, there is a partial isometry $Z_0$ from $\mathcal{H}$ into $G_0$ with initial space $\ker(\tilde{T}_0)^+$ and final space $\ker(\tilde{T}_0^*)^+$ such that

$$\tilde{T}_0 = (I - APA)^{1/2} Z_0.$$ 

Let

$$Z = Z_0 \oplus I_{\mathcal{D}_0}$$

then $Z$ is a partial isometry from $\mathcal{H}$ into $H = G_0 \oplus \mathcal{H}_u$ with initial space $\mathcal{H}_0 = \mathcal{H} \ominus \ker(\tilde{T}_0)$ and final space $G \ominus \ker(\tilde{T}_0^*)$. If $T$ is invertible and $\ker(A) = \{0\}$, then $Z$ is unitary. Define $|\tilde{X}| = (\tilde{X}\tilde{X}^*)^{1/2}$, then

$$\tilde{X} = |\tilde{X}| Z.$$ 

Take an auxiliary Hilbert space $\mathcal{D}_1$ such that

$$\dim(\ker(\tilde{T}_0) \oplus \mathcal{D}_1) = \dim(\ker(\tilde{T}_0^*) \oplus \mathcal{D}_1).$$

If $\dim \ker(\tilde{T}_0) = \dim \ker(\tilde{T}_0^*)$, then $\mathcal{D}_1$ can be $\{0\}$. Define a unitary operator $\hat{Z}$ from $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{D}_1$ onto $\hat{G} = G \oplus \mathcal{D}_1$ such that

$$\hat{Z}x = Zx, \quad \text{for } x \in \mathcal{H} \ominus \ker(\tilde{T}_0).$$

Let $\hat{X} = (|\tilde{X}| \oplus 0_{\mathcal{D}_1}) \hat{Z}$, then $|\hat{X}| = (\hat{X}\hat{X}^*)^{1/2} = |\tilde{X}| \oplus 0_{\mathcal{D}_1}$. Define

$$\hat{A} = |\hat{X}| \hat{A} |\hat{X}|$$

then $\hat{Z}^*(\hat{A} \oplus 0_{\mathcal{D}_1}) \hat{Z} = A \oplus 0_{\mathcal{D}_1}$. It is obvious that $\hat{A} \oplus 0_{\mathcal{D}_1}$ is a $T^*$-Toeplitz operator, where $\hat{T} = \hat{Z}(T \oplus 0_{\mathcal{D}_1}) \hat{Z}^*$. Sometimes we will consider $\hat{A}$ instead of $A$ as the main object of study.

The operator $B \oplus I_{\mathcal{D}_u}$ is denoted by $[B]$, for example,

$$|\tilde{X}| = [(I - APA)^{1/2}].$$
It is obvious that
\[
\lim_{n \to \infty} \bar{U}_0^n P \bar{U}_0^{-n} = 0, \\
\lim_{n \to -\infty} \bar{U}_0^n P \bar{U}_0^{-n} = I |G_0|,
\]
where \((\bar{U}_0 f)(e^{it}) = e^{it} f(e^{it})\), for \(f(e^{it}) \in G_0\). The operator \((I - \Delta PA)^{1/2}\) can be approximated by polynomials of \( \Delta PA \) in the uniform topology. Hence
\[
\lim_{n \to +\infty} \bar{U}_0^n (I - \Delta PA)^{1/2} \bar{U}_0^{-n} = I |G_0| \\
\lim_{n \to -\infty} \bar{U}_0^n (I - \Delta PA)^{1/2} \bar{U}_0^{-n} = (I - A^2)^{1/2} = |\Theta|.
\]
Since \(|X| \tilde{x}_n = |I| \tilde{x}_n\), we conclude that
\[
\lim_{n \to +\infty} \bar{U} |X| \bar{U}^{-n} = I
\]
and
\[
\lim_{n \to -\infty} \bar{U} |X| \bar{U}^{-n} = [|\Theta|].
\]
Hence the following limits exist:
\[
A_+ = \lim_{n \to +\infty} \bar{U}^n \tilde{A} \bar{U}^{-n} = \tilde{A}
\]
(4.4)
and
\[
A_- = \lim_{n \to -\infty} \bar{U}^n \tilde{A} \bar{U}^{-n} = [|\Theta|] \tilde{A}[|\Theta|].
\]
(4.5)
For definiteness, we always choose \(\tilde{A}\) such that \(\ker \tilde{A} \supset \ker(\tilde{X}_0)\) and \(\text{ran}(\tilde{A}) \subset (\ker(\tilde{X}_0^*))^\perp\).

The operators \(A_+\) and \(A_-\) commute with \(\bar{U}\) and they are called the symbols of the operator \(A\). It is obvious that there are operator valued bounded measurable functions of \(e^{it}\) such that
\[
(A_\pm f)(e^{it}) = A_\pm (e^{it}) f(e^{it})
\]
for \(f \in G_0\).

In the following, we only consider the operators \(\tilde{A}\), \(\tilde{T}\), and \(\tilde{T}^*\), etc., and for simplicity of printing we drop the notation "\(\sim\)"

**Lemma 4.1.** The following weak limit exists:
\[
\lim_{n \to \infty} T^n (A \mu I)^{-1} T^* n = X^* (A_+ \mu I)^{-1} X
\]
(4.6)
for \( \mu \in \rho(A) \). Besides,

\[
\lim_{n \to \infty} (T^n(A - \mu I)^{-1} T^*n(I - \lambda T^*)^{-1} Da, Db) = \frac{1}{2\pi i} \int_{|\lambda| = 1} \frac{(A(z)(A(z)_+ - \mu I)^{-1} A(z)_a, b) dz}{z - \lambda}.
\]

(4.7)

for \( a, b \in \mathcal{D}_1, |\lambda| < 1, \) and \( \mu \in \rho(A) \).

**Proof.** Define

\[
\hat{\mathcal{U}} = \hat{T}^*(U \oplus I_{\mathcal{D}_1}) \hat{T}.
\]

From (4.4), we have

\[
s-lim_{n \to \infty} \hat{U}^n(A \oplus 0_{\mathcal{D}_1}) \hat{U}^{-n} = s-lim_{n \to \infty} \hat{T}^*(U^n \hat{A} U^{-n} \oplus 0_{\mathcal{D}_1}) \hat{Z} = \hat{Z}^*(A_+ \oplus 0_{\mathcal{D}_1}) \hat{Z}.
\]

Hence

\[
Z^* A^* Z \oplus 0_{\mathcal{D}_1} = \omega-lim_{n \to \infty} \hat{U}^n(A^* \oplus 0_{\mathcal{D}_1}) \hat{U}^{-n}.
\]

If \( \mu \in \rho(A) \) and \( \mu \neq 0 \), then

\[
\lim_{n \to \infty} (\hat{U}^n(A \oplus 0_{\mathcal{D}_1} - \mu I)^{-1} \hat{U}^{-n} x, y) = \lim_{n \to \infty} (\hat{U}^n(A \oplus 0_{\mathcal{D}_1} - \mu I)^{-1} \hat{U}^{-n} x, y)
\]

\[
= (x, \hat{Z}^*(A_+^* \oplus 0_{\mathcal{D}_1} - \mu I) \hat{Z}^{-1} y) = (\hat{Z}^*(A_+ - \mu I)^{-1} \hat{Z} x, y),
\]

for \( x, y \in \mathcal{H} \oplus \mathcal{D}_1 \). On the other hand, we have

\[
U^n X T^*n = X.
\]

Therefore

\[
(U \oplus I_{\mathcal{D}_1})^n(|X| \oplus 0_{\mathcal{D}_1}) \hat{T} \hat{U}^n(T^* \oplus 0_{\mathcal{D}_1})^n = X \oplus 0_{\mathcal{D}_1}.
\]

Thus

\[
s-lim_{n \to \infty} \hat{U}^n(T^* \oplus 0_{\mathcal{D}_1})^n = \hat{Z}(X \oplus O_{\mathcal{D}_1}).
\]
Hence

\[
\lim_{n \to +\infty} (T^n (A - \mu I)^{-1} T^* x, y) = \lim_{n \to +\infty} (\hat{U}^n (A \oplus 0_{\mathcal{H}} - \mu I)^{-1} (T^* \oplus 0_{\mathcal{H}})^n x, \hat{U}^n (T^* \oplus 0_{\mathcal{H}})^n y)
\]

\[
= (\hat{Z}(A_+ \oplus 0_{\mathcal{H}} - \mu I)^{-1} (X \oplus 0_{\mathcal{H}}) x, \hat{Z}(X \oplus 0_{\mathcal{H}}) y)
\]

\[
= ((A_+ - \mu I)^{-1} X x, X y), \quad \text{for} \quad x, y \in \mathcal{H}. \quad (4.8)
\]

Hence (4.6) for \( \mu = 0 \) may be obtained by letting \( \mu \to 0 \) in (4.6). Substituting

\[x = (I - \lambda T^*)^{-1} Da, \quad Xx = (1 - \lambda e^{-it})^{-1} A(e^{it})a\]

and

\[y = Db, \quad Xy = A(e^{it})b\]

in (4.8), we obtain (4.7).

Denote

\[\xi(e^{it}, \mu) = I - A(e^{it}) (A_+ (e^{it}) - \mu I)^{-1} A(e^{it}), \quad (4.9)\]

for \( \mu \in \rho(A), e^{it} \in T \), and

\[\eta(z, \mu) = D_{\mathcal{H}}(z, \tilde{\mu})^* D(z, \mu), \quad \text{for} \quad \mu \in \rho(A) \text{ and } |z| < 1.\]

**Theorem 4.2.** For every continuous function \( \phi(\cdot) \) on \( T \),

\[
\lim_{\rho \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} (\eta(\rho e^{it}, \mu) - \xi(e^{it}, \mu)) \phi(e^{it}) \, dt = 0. \quad (4.10)
\]

If \( D \) is in the Hilbert–Schmidt class then

\[
\lim_{\rho \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \|\eta(\rho e^{it}, \mu) - \xi(e^{it}, \mu)\|_1 \, dt = 0. \quad (4.11)
\]

**Proof** First, we have to prove that

\[
\mu T(A - \mu I)^{-1} T^* - \mu (A - \mu I)^{-1} = D^2 - F(\mu), \quad (4.12)
\]

where

\[F(\mu) = TA(A - \mu I)^{-1} D^2 A T^*(A - \mu I)^{-1} .\]
Notice that
\[
D^2_\phi (A - \mu I) - TA(A - \mu I)^{-1} D^2 A
= D^2_\phi (A - \mu I) - (T + \mu T(A - \mu I)^{-1}) [A - \mu I, T^*]
= \mu T(A - \mu I)^{-1} T^*(A - \mu I)^* - \mu I,
\]
by (3.1) and (3.4). This implies (4.12). From Lemma 4.1, the following weak limit exists:
\[
L = \lim_{n \to +\infty} \mu DT^n(A - \mu I)^{-1} T^{*n}(I - \lambda T^*)^{-1} D.
\] (4.13)

Then
\[
L = Q_1(\lambda, \mu) + Q_2(\lambda) - Q_3(\lambda, \mu).
\] (4.14)

where
\[
Q_1(\lambda, \mu) = \mu D(A - \mu I)^{-1} (I - \lambda T^*)^{-1} D,
\]
\[
Q_2(\lambda) = \lim_{n \to \infty} \sum_{m=0}^{n} DT^mD^2_\phi T^{*m}(I - \lambda T^*)^{-1} D,
\] (4.15)

and
\[
Q_3(\lambda, \mu) = \lim_{n \to \infty} \sum_{m=0}^{n} DT^mF(\mu) T^{*m}(I - \lambda T^*)^{-1} D,
\] (4.16)

provided the weak limits for \( Q_3 \) and \( Q_3 \) exist, since
\[
T^{n+1}(A - \mu I)^{-1} T^{*n+1} - (A - \mu I)^{-1} = \sum_{m=0}^{n} T^m(D^2_\phi - F(\mu)) T^{*m}
\]
by (4.12). It is easy to see that
\[
Q_1(\lambda, \mu) = I - D(I - \lambda T^*)^{-1} D - [I - DA(A - \mu I)^{-1} (I - \lambda T^*)^{-1} D]
= \Theta_T(0)^* \Theta_T(\lambda) - D_\phi(0, \mu)^* D(\lambda, \mu),
\] (4.17)
since \( D_\phi(0, \mu)^* D(\lambda, \mu) = I - DA(A - \mu I)^{-1} (I - \lambda T^*)^{-1} D \) by (3.16) and
\[
\Theta_T(0)^* \Theta_T(\lambda) = I - D(I - \lambda T^*)^{-1} D.
\]

It is obvious that
\[
\sum_{m=0}^{\infty} \| D_\phi T^{*m} a \|^2 = \| T^* a \|^2 - \lim_{n \to \infty} \| T^{*n} a \|^2.
\]

Thus the weak limit in (4.15) exists and then the weak limit in (4.16) also exists, because of (4.14).
It is easy to see that
\[ \sum_{m=0}^{\infty} DT^m D_* z^{m+1} = (\Theta_T(z) + T)^* \]
and
\[ \sum_{m=0}^{\infty} D_* T^m (I - \lambda T^*)^{-1} Dz^{m+1} = (\Theta_T(z) - \Theta_T(\lambda)) z/(z - \lambda) \]
for \(|z| < 1\). Thus
\[ \frac{1}{2\pi i} \int_0^{2\pi} \frac{\left(\Theta_T(\rho e^{it}) + T\right)^* (\Theta_T(\rho e^{it}) - \Theta_T(\lambda)) \rho e^{it} }{\rho e^{it} - \lambda} \, dt \]
\[ = \sum_{m=0}^{\infty} DT^m D_* T^m (I - \lambda T^*)^{-1} D\rho^{(2m+1)}. \]
Hence,
\[ Q_2(\lambda) = \lim_{\rho \to 1} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\left(\Theta_T(\rho e^{it}) + T\right)^* (\Theta_T(\rho e^{it}) - \Theta_T(\lambda)) \rho e^{it} }{\rho e^{it} - \lambda} \, dt \]
\[ = \lim_{\rho \to 1} \frac{1}{2\pi i} \int_{|z| = \rho} \frac{\Theta_T(z)^* \Theta_T(z) \, dz}{z - \lambda} - \Theta_T(0)^* \Theta_T(\lambda), \quad (4.18) \]
since
\[ \lim_{\rho \to 1} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\Theta_T(\rho e^{it})^* \Theta_T(\lambda) \rho dt}{\rho - \lambda e^{-it}} = \Theta_T(0)^* \Theta_T(\lambda). \]
Next, let us calculate \( Q_3(\lambda, \mu) \). From (3.5) and (3.9),
\[ \sum_{m=0}^{\infty} DT^m (A - \mu I)^{-1} (I - \lambda T^*)^{-1} T^m Dz^m = -\frac{D(\lambda, \mu) - D(z, \mu)}{\lambda - z} \]
and
\[ \sum_{m=0}^{\infty} DT^{m+1} (A - \mu I)^{-1} Dz^{m+1} = D_*(z, \tilde{\mu})^* - D_*(0, \tilde{\mu})^* \]
for \(|z| < 1\). Thus
\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{(D\tilde{\lambda}(\rho e^{it}, \mu)^* - D\tilde{\lambda}(0, \mu)^*) (D(\rho e^{it}, \mu) - D(\lambda, \mu)) \rho e^{it} \, dt}{\rho e^{it} - \lambda} \]
\[ = \sum_{m=0}^{\infty} DT^m F(\mu) T^m (I - \lambda T^*)^{-1} D\rho^{(2m+1)}, \quad (4.19) \]
for $\rho < 1$, $|\lambda| < 1$, and $\mu \in \rho(A)$. Since the weak limit of (4.16) exists, by Abelian's theorem, the weak limit of the right-hand side of (4.19) exists and equals $Q_3(\lambda, \mu)$ as $\rho \to 1^-$. Thus

$$Q_3(\lambda, \mu) = \lim_{\rho \to 1^-} \frac{1}{2\pi i} \int_{|z| = \rho} \frac{\eta(z, \mu)}{z - \lambda} dz - D_*(0, \mu)^* D(\lambda, \mu), \quad (4.20)$$

since taking residues at $\infty$ gives

$$\frac{1}{2\pi} \int_0^{2\pi} D_*(\mu e^{it\omega}, \mu)^* \frac{\rho e^{it\omega}}{\rho e^{it\omega} - \lambda} dt = D_*(0, \mu)^*$$

for $|\lambda| < \rho$. From (4.13), (4.14), (4.18), and (4.20) it follows that

$$L = w-\lim_{\rho \to 1^-} \frac{1}{2\pi i} \int_{|z| = \rho} \frac{\Theta_T(z)^* \Theta_T(z) - \eta(z, \mu)}{z - \lambda} dz. \quad (4.21)$$

Thus

$$w-\lim_{\rho \to 1^-} \frac{1}{2\pi i} \int_{|z| = \rho} \frac{\eta(z, \mu) - \xi(z, \mu)}{z - \lambda} dz = 0 \quad (4.22)$$

for $|\lambda| < 1$.

Now, we consider the equality (4.7) corresponding to $A^*$ and $\tilde{\mu}$, i.e.,

$$\lim_{n \to +\infty} (T^n(A^* - \tilde{\mu}I)^{-1} T^*n(I - \lambda T^*)^{-1} Da, Db) \quad (4.23)$$

for $a, b \in D_1$, $|\lambda| < 1$, and $\mu \in \rho(A)$. Also we have the equality

$$\tilde{\mu} T(A^* - \tilde{\mu}I)^{-1} T^* - \tilde{\mu}(A^* - \tilde{\mu}I)^{-1} = D_* - TA^*(A^* - \tilde{\mu}I)^{-1} D^2A^* T^* (A^* - \mu I)^{-1}; \quad (4.24)$$

which is similar to (4.12). Again we have

$$\eta(z, \mu)^* = I - (1 - |z|^2) D(I - \bar{z} T)^{-1} (A^* - \tilde{\mu}I)^{-1} A^* (I - \bar{\lambda} T^*)^{-1} D. \quad (4.25)$$

By means of (4.23), (4.24), and (4.25) we can prove that

$$w-\lim_{\rho \to 1^-} \frac{1}{2\pi i} \int_{|z| = \rho} \frac{(\eta(z, \mu) - \xi(z, \mu))^*}{z - \lambda} dz = 0 \quad (4.26)$$

for $|\lambda| < 1$. Equation (4.26) implies that (4.22) holds also for $|\lambda| > 1$, if we apply the adjoint operation to (4.22).
Next we have to prove that

\[ \frac{1}{2\pi} \int_0^{2\pi} |(\eta(\rho e^{it}, \mu) a, b)|^2 \, dt \leq K(a, b, \mu), \quad \rho < 1, \quad (4.27) \]

where \( K(a, b, \mu) \) is a constant independent of \( \rho \).

From (3.16), we have

\[ |(\eta(z, \mu) a, b)| \leq 1 + \| A(A - \mu I)^{-1} (1 - |z|^2) (I - zT^*)^{-1} Da \| \| (I - zT^*)^{-1} Db \|. \quad (4.28) \]

On the other hand, it is obvious that

\[ \frac{1}{2\pi} \int_0^{2\pi} \| (I - \rho e^{it} T^*)^{-1} x \|^2 \, dt = \sum_{n=0}^{\infty} \| T^{*n} x \|^2 \rho^{2n} \leq (1 - \rho^2)^{-1} \| x \|^2. \quad (4.29) \]

From (4.28), (4.29), and the Schwarz inequality, it is easy to prove (4.27), with

\[ K(a, b, \mu) = 1 + \| A(A - \mu I)^{-1} Da \| \| Db \|. \]

Since every continuous function on \( T \) may be approximated uniformly by meromorphic functions of the form

\[ \sum_{j=1}^{n} c_j \frac{1}{z - \lambda_j} \]

with \( |\lambda_j| \neq 1, j = 1, 2, ..., n \). From (4.27), it is easy to see that (4.10) holds.

Suppose \( D^2 \) belongs to the trace class. Then, with \( J = DAT(A^* - \mu I)^{-1} \) we have

\[ \frac{1}{2\pi} \int_0^{2\pi} \| I - D(\rho e^{it}, \mu) \|^2 \, dt \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \rho^2 \text{tr}(DAT(A^* - \mu I)^{-1} \sum_{n=1}^{\infty} e^{n-it} \rho^n T^{*n} D^2 \]

\[ \times \sum_{m=0}^{\infty} e^{-m-it} \rho^m (A - \mu I)^{-1} T^{*m} AD) \, dt \]

\[ = \rho^2 \text{tr} \left( J \sum_{n=0}^{\infty} \rho^n T^{*n} D^2 T^n J^n \right) \leq \text{Tr}(JJ^*) < \infty. \]
Thus there is a Hilbert–Schmidt class operator valued function $D(e^{it}, \mu)$ such that
\[
\lim_{\rho \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \|D(\rho e^{it}, \mu) - D(e^{it}, \mu)\|_2^2 \, dt = 0.
\]
Similarly, there is a Hilbert–Schmidt class operator valued function $D_*(e^{it}, \mu)$ such that
\[
\lim_{\rho \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \|D_*(\rho e^{it}, \mu) - D_*(e^{it}, \mu)\|_2^2 \, dt = 0.
\]
Thus the operator $D_*(e^{it}, \mu)^* D(e^{it}, \mu)$ belongs to trace class for almost all $e^{it} \in T$ and there is a sequence $\{\rho_n\}$ satisfying $0 < \rho_n < 1$ and $\rho_n \to 1$ such that
\[
\lim_{n \to \infty} \|D_*(\rho_n e^{it}, \mu) - D(\rho e^{it}, \mu)\|_2 = \lim_{n \to \infty} \|D(\rho_n e^{it}, \mu) - D(\rho e^{it}, \mu)\|_2 = 0.
\]
Thus
\[
\lim_{n \to \infty} \|\eta(\rho_n e^{it}, \mu) - D_*(e^{it}, \mu) D(e^{it}, \mu)\|_2 = 0 \tag{4.30}
\]
for almost all $e^{it} \in T$ and
\[
\lim_{\rho \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \|D_*(\rho e^{it}, \mu)^* D(\rho e^{it}, \mu) - D_*(e^{it}, \mu)^* D(e^{it}, \mu)\|_1 \, dt = 0. \tag{4.31}
\]
From (4.10) and (4.30) we get (4.11).

From (4.9), there exists a sequence $\{\rho_n\}$, $\rho_n \to 1^-$ such that
\[
\lim_{n \to \infty} \|\eta(\rho_n e^{it}, \mu) - \xi(e^{it}, \mu)\|_1 = 0
\]
for almost all $e^{it} \in T$. Thus $\lim_{n \to \infty} D_*(\rho_n e^{it}, \mu)^{-1} \xi(e^{it}, \mu) = \lim_{n \to \infty} D(\rho_n e^{it}, \mu)$.

**Corollary 4.3.** The determining function $\Theta(\lambda, \mu)$ (or $D(\lambda, \mu)$) solves the following Riemann–Hilbert problem with barrier $\xi(e^{it}; \mu)$ (cf. (4.9)):
\[
\lim_{\rho \to 1^-} \Theta(\rho e^{it}, \mu) \xi(e^{it}, \mu) = \lim_{\rho \to 1^-} \Theta(\rho e^{it}, \mu), \tag{4.32}
\]
where $\Theta(\lambda, \mu) = -\overline{T}D_*(1/\lambda, \mu)^*$ for $|\lambda| > 1$. 


Denote
\[ ζ(z, μ) = I + μω(z) Δ(z)(A - μI)^{-1} A(z) ω(z)^*, \quad \text{for } μ ∈ ρ(A), \]
where \( ω(z) \) is the isometry from \( |Θ(z)|^2 \) onto \( Θ(z)^2 \) in the polar decomposition
\[ Θ(z) = ω(z)|Θ(z)|. \]

**Lemma 4.4.** If \( T \) is invertible, then for \( μ ∈ ρ(A) \),
\[ \lim_{n → ∞} (T^∗n(A - μI)^{-1} T^n(I - λI)^{-1} D_∗a, D_∗b) = \frac{1}{2π} \left( \int_0^{2π} \frac{ζ(e^{it}, μ) dt}{1 - λe^{it}} a, b \right). \]

**Proof.** Consider the functional model of \( T \). It is obvious that
\[ V_∗D_∗b = (I - Θ(z)Θ(0)^*) b ⊕ A(z)Θ(0)^*b. \]
By a direct computation, it is easy to see that
\[ V_∗(I - λT)^{-1}D_∗a = \frac{(I - Θ(z)Θ(λ)^*) a ⊕ (-1) A(z)Θ(λ)^*a}{1 - λz}. \]
If fact, if we multiply \( (I - λT) \) from the left on both sides of (4.35), then (4.35) is reduced to the formula (4.34) for the vector \( a \).
Since
\[ \bar{T}^n ⊕ v = (z^n u - ΘP(z^n(Θ^*u + Δv))) ⊕ (z^n v - ΔP(z^n(Θ^*u + Δv))), \]
\[ \bar{T}^n u ⊕ v = P(z^n u) ⊕ z^n v \]
for \( u ⊕ v ∈ H_c \), and \( s\)-limit \( n → ∞ z^n Pz^n = I \), it is easy to see that
\[ \lim_{n → ∞} \bar{T}^n u ⊕ v = P(u - Θ(Θ^*u + Δv))) ⊕ (v - Δ(Θ^*u + Δv)) \]
for \( u ⊕ v ∈ H_c \). If \( u ⊕ v \) is the vector in (4.35), then
\[ \lim_{n → ∞} V_∗T^∗n T^n(I - λT)^{-1}D_∗a = P \left( \frac{(I - Θ(z)Θ(λ)^*) a}{1 - λz} ⊕ \frac{(-1) A(z)Θ(λ)^*a}{1 - λz} \right). \]
The inequality \((1/2π) \int_0^{2π} \|Θ_T(e^{it})^* a\|^2 dt ≥ \|T^∗ a\|^2\) follows immediately by expanding \( Θ_T(ρe^{it}) \) in a power series and taking the limit as \( ρ → 1 \). Hence
ker(\Theta_T(e''')) = \{0\} for almost all \(e'' \in T\). Thus \([\text{range}(\omega(e'') \omega(e'''))] = \ker(\omega(e'') \omega(e''')) = \{0\}, a.e. This means \(\omega(e'') \omega(e''') = I_{\mathcal{S}_-}\) a.e. Hence

\[
\lim_{n \to \infty} V_c T^* T^n (I - \lambda T)^{-1} D_\ast a = P \left( \frac{\omega A \omega a}{1 - \lambda z} \right) \oplus \left( \frac{-1}{1 - \lambda z} A |\Theta| w \ast a \right). \tag{4.38}
\]

Since \(A^k = (\tilde{X}_0 \ast \tilde{A} \tilde{X})^k\), it is easy to see that

\[
P_{\mathcal{S}_-} A^k (u \oplus v) = (-1) \Theta P(\Lambda(z) A_+ ([I - \Delta P A] A_+) k^{-1} v) \oplus ([I - \Delta P A] A_+ k v),
\]

for \(k > 0\). By the same method, computing (4.36) and using (4.37), we have

\[
\lim_{n \to \infty} P_{\mathcal{S}_-} \tilde{T}^* T^n A^k \tilde{T}^* u \oplus v = P((-1) \Theta A([\Theta]^2 A_+) k^{-1} (|\Theta|^2 v - \Delta \Theta \ast u)) \oplus (([\Theta]^2 A_+) k (|\Theta|^2 v - \Delta \Theta \ast u)). \tag{4.39}
\]

If \(u \oplus v = V_c (I - \lambda T)^{-1} D_\ast a\), then (4.38) becomes

\[
\lim_{n \to \infty} V_c P_{\mathcal{S}_-} T^* T^n A^k T^n (I - \lambda T)^{-1} D_\ast a = P \left( \frac{\omega A A^k \omega a}{1 - \lambda z} \right) \oplus \left( \frac{-1}{1 - \lambda z} \Theta A^k \Delta \omega a \right). \tag{4.40}
\]

Notice that \((A - \mu I)^{-1}\) may be approximated by polynomials in \(A\) in the norm topology. From (4.38) and (4.40) it follows that

\[
\lim_{n \to \infty} V_c P_{\mathcal{S}_-} T^* T^n (A - \mu I)^{-1} T^n (I - \lambda T)^{-1} D_\ast a = P \left( \frac{\omega A (A - \mu I)^{-1} \Delta \omega a}{1 - \lambda z} \right) \oplus \left( \frac{-1}{1 - \lambda z} \Theta (A - \mu I)^{-1} \Delta \omega a \right). \tag{4.41}
\]

Note that if \(h(\cdot) \in H^2\), then

\[(Pf, h) = (f, h).\]

From (4.38) and (4.41), we obtain (4.33)

**Theorem 4.5.** For every continuous function \(\phi(\cdot)\) on \(T\),

\[
\lim_{\rho \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} (E(\rho e'', \mu) E_\ast(\rho e'', \bar{\mu}) - \zeta(e'', \mu)) \phi(e'') dt = 0. \tag{4.42}
\]
If $D$ is in the Hilbert–Schmidt class then

$$\lim_{\rho \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \|E(\rho e^{it}, \mu) \ E_* (\rho e^{it}, \tilde{\mu})^\ast - \zeta (e^{it}, \mu)\|_1 dt = 0. \quad (4.43)$$

**Proof.** First, we have to prove that

$$T^* (A - \mu I)^{-1} T - (A - \mu)^{-1} = \mu (A - \mu I)^{-1} D^2 T^{-1} (A - \mu I)^{-1} T. \quad (4.44)$$

Notice that

$$[T^* , (A - \mu I)^{-1}] = (A - \mu I)^{-1} \ [A - \mu I, T^*] (A - \mu I)^{-1}$$

$$= (A - \mu I)^{-1} D^2 T^{-1} A (A - \mu I)^{-1}.$$

Therefore the left-hand side of (4.44) equals

$$(A - \mu I)^{-1} D^2 T^{-1} A (A - \mu I)^{-1} T + (A - \mu I)^{-1} (T^* T - I),$$

which equals the right-hand side of (4.44).

From Lemma 4.4, the following weak limit exists:

$$L = \lim_{n \to \infty} \mu D T^{-1} T_0^n (A - \mu I)^{-1} T^* T^{-1}.$$

It is obvious that

$$L = \lim_{n \to \infty} \mu^2 D T^{-1} T^* T^{-1} (A - \mu I)^{-1} D^2 T^{-1} (A - \mu I)^{-1} T^* T^{-1} (A - \mu I)^{-1} T^{-1}.$$

by (4.44). If $0 < |z| < 1$, then

$$\frac{\mu}{\lambda} \sum_{k=0}^{\infty} D T^{-1} (A - \mu I)^{-1} T^* T^{-1} (A - \mu I)^{-1} D^2 T^{-1} (A - \mu I)^{-1} T^* T^{-1} (A - \mu I)^{-1} T^{-1}.$$

and

$$\lambda \sum_{k=0}^{\infty} D T^{-1} (A - \mu I)^{-1} D^2 T^{-1} (A - \mu I)^{-1} T^* T^{-1} (A - \mu I)^{-1} T^{-1}.$$

Hence (4.45) implies

$$L = (E_*(A, \mu) - I)^* + \lim_{\rho \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \frac{E(\rho e^{it}, \mu) - I (E(\rho e^{it}, \tilde{\mu}) - E_*(A, \tilde{\mu}))}{\rho (\rho - \tilde{\mu} e^{it})} dt$$

$$= \lim_{\rho \to 1^-} \frac{1}{2\pi i} \int_{|z| = \rho} \frac{E(z, \mu) \ E_*(z, \tilde{\mu})^*}{(1 - \tilde{\mu} z) z} dz - I.$$
Thus
\[
\lim_{\rho \to 1} \frac{1}{2\pi} \int \frac{E(\rho e^{it}, \mu) E_*(\rho e^{it}, \overline{\mu})^* - \zeta(\rho e^{it}, \mu)}{1 - \overline{\lambda} \rho e^{it}} \, dt = 0. \quad (4.46)
\]

By the method used in the proof of Theorem 4.2, we can prove (4.42) and (4.43) using (4.46).

**Remark 1.** The functions \(\zeta(\cdot, \mu)\) and \(\xi(\cdot, \mu)\) satisfy the following relation. If \(A_+(z)\) is diagonal in the \(J\)-matrix representation (cf. Section 8 of this paper), and \(\xi(e^{it}, \mu)\) is invertible, then
\[
\zeta(e^{it}, \mu) = \Theta(e^{it}) \xi(e^{it}, \mu) \Theta(e^{it})^*.
\quad (4.47)
\]

In fact, from the definition of \(\zeta(\cdot, \mu)\), it is easy to see that
\[
\xi(e^{it}, \mu) A = A(I - A_+(A_+ - \mu))^{-1}(1 - |\Theta|^2)
\]
\[
= A(A_+ - \mu I)^{-1}(A_+ |\Theta|^2 - \mu I).
\]

Therefore,
\[
\zeta(e^{it}, \mu)^{-1} A = A(A_+ |\Theta|^2 - \mu I)^{-1}(A_+ - \mu I).
\]

Thus
\[
|\Theta| \xi(e^{it}, \mu)^{-1} |\Theta| A = A |\Theta| (A_+ |\Theta|^2 - \mu I)^{-1}(A_+ - \mu I) |\Theta|
\]
\[
= A(A_+ - \mu I)^{-1}(A_+ - \mu |\Theta|^2),
\]

since
\[
[|\Theta|](A_+ |\Theta|^2 - \mu I)^{-1} = ([|\Theta|] A_+ [|\Theta|] - \mu I)^{-1} [|\Theta|].
\]

On the other hand, we have
\[
A(A_+ - \mu I)^{-1}(A_+ - \mu |\Theta|^{-1}) = A(I + \mu(A - \mu I)^{-1} A^2).
\]

Hence
\[
|\Theta| \xi(e^{it}, \mu)^{-1} |\Theta| = I + \mu A(A_+ - \mu I)^{-1} A,
\]

which implies (4.47).

Let \(U_*\) and \(U_{**}\) be the minimal unitary and the minimal isometric dilation of \(T^*\) on the space \(\mathcal{H}\) and \(\mathcal{H}_{**}\), respectively (\(\mathcal{H} \subset \mathcal{H}_{**} \subset \mathcal{H}\)). Let
\[
G_* = \bigcap_{n=0}^{\infty} \mathcal{H}_{*+}^{\perp}.
\]
Theorem 4.6. For \( \mu \in \rho(A) \), the weak limit

\[
A_*(\mu) = \lim_{n \to \infty} T^n(A - \mu I)^{-1} T^n
\]

exists and it is a T-Toeplitz operator. There exists a unitary operator \( \Omega \) (which is independent of \( \mu \)) from \( G_* \) to \( G \) such that the symbol of \( A_*(\mu) \) is

\[
\Omega^{-1}(A - \mu I)^{-1}\Omega.
\]

Proof. Suppose \( T, T^* \), \( \mathcal{H} \), etc., are the corresponding objects in the functional model. Since

\[
w-lim_{n \to \infty} T^n T^n u \oplus v \oplus w = P((I - \Theta \Theta^*) u - \Theta \Delta v) \oplus (|\Theta|^2 v - \Delta \Theta^* u) \oplus w
\]

(cf. (4.36)) and

\[
w-lim_{n \to \infty} T^n A^k T^n u \oplus v \oplus w
\]

\[
= P((-1) \Theta \Delta (A_+ (|\Theta|^2) A_+) A_+ (|\Theta|^2) v - \Delta \Theta^* u))_0
\]

\[
\oplus(|\Theta|^2 v - \Delta \Theta^* u) \oplus w,
\]

where \((\cdot)_0\) denotes the component in \( G_0 \) of \((\cdot)\), (cf. (4.39)), we have

\[
w-lim_{n \to \infty} T^n (A - \mu I)^{-1} T^n u \oplus v \oplus w
\]

\[
= P(\omega A((A - \mu I)^{-1} (\Delta \omega^* u - |\Theta| v))_0)
\]

\[
\oplus(|\Theta|)((A - \mu I)^{-1} (|\Theta| v - \Delta \omega^* u) \oplus w. \quad (4.48)
\]

It is easy to see that

\[
P_{G_*} v \oplus v \oplus w = \omega A(\Delta \omega^* u - |\Theta| v) \oplus |\Theta| ((|\Theta| v - \Delta \omega^* u) \oplus w \quad (4.49)
\]

since

\[
P_{G_*} v = \lim_{n \to \infty} U^n v = \lim_{n \to \infty} T^n h, \quad \text{for} \quad h \in \mathcal{H}
\]

and \( G_* = \bigvee_{n \geq 0} U^n G_* \mathcal{H} \) [20, 21]. Besides,

\[
P_\mathcal{H}((-1) \omega \Delta v) \oplus |\Theta| v \oplus w = P((-1) \omega \Delta v) \oplus |\Theta| v \oplus w. \quad (4.50)
\]

Define the unitary operator \( \Omega \)

\[
\Omega \omega A(\Delta \omega^* u - |\Theta| v) \oplus |\Theta| ((|\Theta| v - \Delta \omega^* u) \oplus w
\]

\[
= (|\Theta| v - \Delta \omega^* u) \oplus w. \quad (4.51)
\]
From (4.48)-(4.51), it follows that

$$A_\ast(\mu) = P_\Phi \Omega (A_\ast - \mu I)^{-1} \Omega P_{\sigma_\ast},$$

which proves the theorem.

5. **Principal Functions Associated to Weak Contractions**

$$A_T = s\text{-lim}_{n \to \infty} T^n T^{*n}$$

Let $T$ be a contraction in Hilbert space. It is known that the strong limit

$$A_T = s\text{-lim}_{n \to \infty} T^n T^{*n}$$

exts (cf. [19, 23]); sometimes $A_T$ is simply denoted by $A$. It is obvious that $A_T$ is a $T^*$-Toeplitz operator and its symbol $A_\ast(e'') = I$. Let

$$H_T = A_T^{1/2} T^*,$$  \hspace{1cm} (5.1)

then

$$[H_T^*, H_T] = A_T^{1/2} (I - T^* T) A_T^{1/2} \geq 0,$$

so that $H_T$ is hyponormal. Since $H_T^* H_T = A_T$, there is a partial isometry $W$ from $\mathcal{R}(A_T)$ onto $\mathcal{K}(H_T)$ such that

$$H_T = WA_T^{1/2}.  \hspace{1cm} (5.2)$$

This is the polar decomposition of the hyponormal operator $H_T$. The operator $W$ can be extended to an isometric operator from $\mathcal{K}$ into $\mathcal{K}$ (cf. [23, p. 4]), which is still denoted by $W$. If $T$ is invertible, then $\mathcal{R}(H_T) = \mathcal{R}(A_T)$ and hence we may assume that $W$ is unitary. This hyponormal operator $H_T$ is canonically associated with $T$.

Later, we only consider the case when $T$ is a weak contraction, that is, $T$ is invertible and the defect operator $D \in \mathcal{L}_2(\mathcal{K})$ or equivalently $I - T^* T \in \mathcal{L}_1(\mathcal{K})$. Since the principal function $g(\cdot)$ of $H_T$ is completely determined by $T$, $g_T$ is said to be the associated principal function of the contraction $T$.

**Lemma 5.1.** If $T$ is a weak contraction, $0 \in \rho(T)$ and $\mu \in \rho(A_T)$, then

$$\det(D(\lambda, \mu)) = \exp \int_0^\lambda \int_0^\mu g_T(\rho e^{it}) \rho \, d\rho \, dt \frac{\mu}{(\rho^2 - \mu)(e^{-it} - \lambda)}$$  \hspace{1cm} (5.3)
and
\[
\det(D_*(\lambda, \mu)) = \exp \frac{1}{\pi} \int \int g_T(\rho e^{it}) \frac{d\rho \, dt}{(p^2 - \mu)(1 - \rho e^{it})}, \tag{5.4}
\]
where \(g_T\) is the associated principal function of the contraction \(T\).

**Proof.** From (3.6'), it is easy to see that \(\{ A - \mu I, I - \lambda T^* \}\) has determinant and
\[
P_{\mathcal{H}^+}\{ A - \mu I, I - \lambda T^* \} = P_{\mathcal{H}^+},
\]
where \(P_{\mathcal{H}^+}\) is the projection from \(\mathcal{H}\) onto \(\mathcal{H}^+\). Therefore
\[
\det\{ A - \mu I, I - \lambda T^* \} = \det(D_*(\lambda, \mu)), \tag{5.5}
\]
From (3.6), it follows that
\[
\det D(\lambda, \mu) = \det(\{ A - \mu I, I - \lambda T^* \}|_{\mathcal{H}}).
\]
Hence
\[
\det D(\lambda, \mu) = \det\{ A - \mu I, I - \lambda T^* \}. \tag{5.5}
\]
If \(A_T\) is invertible, then \(T^* = A_T^{-1/2} W A_T^{1/2}\), and by a theorem in [3], [4]. or [23] we have
\[
\det\{ A - \mu I, I - \lambda T^* \} = \exp \frac{1}{2\pi i} \int g_T(\rho e^{it}) \, d\ln(\rho^2 - \mu) \wedge d\ln(1 - \rho e^{it}). \tag{5.6}
\]
This and (5.5) imply (5.3). So we only have to establish (5.6) for general \(A_T\).

Since \(T\) is invertible, we may assume that the operator \(W\) in (5.2) is unitary. It is obvious that
\[
\]
and \([A_T, T^*] \in L_1(\mathcal{H})\) (see (3.4)). On the other hand,
\[
A_T^{1/2}[A_T, T^*] = [A_T, W] A_T^{1/2} \tag{5.7}
\]
and
\[
(I - Q)[A_T, T^*](I - Q) = (I - Q)[A_T, W](I - Q) = 0, \tag{5.8}
\]
where $Q$ is the projection from $\mathcal{H}$ onto the $\mathcal{H}(A_T)$. Let $\{Q_n\}$ be a sequence of projections such that

$$0 \leq Q_1 \leq \cdots \leq Q_n \leq \cdots, \quad Q_n \to Q,$$

$[A_T, Q_n] = 0$, and $A_T Q_n$ is invertible in $\mathcal{L}(Q_n \mathcal{H})$. From (5.7), it follows that

$$A_T^k Q_n [A_T, T^*] Q_n = A_T^k (A_T^{1/2} | Q_n \mathcal{H})^{-1} Q_n [A_T, W] Q_n (A_T^{1/2} | Q_n \mathcal{H}).$$

Therefore

$$\text{Tr}(Q_n A_T^k [A_T, T^*] Q_n) = \text{Tr}(Q_n A_T^k [A_T, W] Q_n).$$

Let $n \to \infty$, then $\text{Tr}(Q A_T^k [A_T, T^*] Q) = \text{Tr}(Q A_T^k [A_T, W] Q)$. This and (5.8) imply

$$\text{Tr}(A_T^k [A_T, T^*]) = \text{Tr}(A_T^k [A_T, W]).$$

A variation of the collapsing functional technique (cf. p. 172 of [23]) enables us to prove easily that for every pair of polynomials $p(\cdot, \cdot)$ and $q(\cdot, \cdot)$

$$\text{Tr}([p(A_T, T^*), q(A_T, T^*)]) = \text{Tr}([p(A_T, W), q(A_T, W)]).$$

Thus

$$\text{Tr}(i[p(A_T, T^*), q(A_T, T^*)]) = \frac{1}{2\pi} \int g_T(\rho e^{i\theta}) \frac{\partial(p(\rho^2, e^{i\theta}), q(\rho^2, e^{i\theta}))}{\partial(\rho, \theta)} d\rho^2 d\theta. \quad (5.9)$$

By means of the standard relation between the determinant formula and trace formula (cf. [3, 4]), (5.6) is proved using (5.9).

**Theorem 5.2.** Let $T$ be a weak contraction and $0 \in \rho(T)$, then

$$\Theta_T^\mu(e^{it}) \Theta_T(e^{it}) - I|_{\mathcal{D}} \in \mathcal{L}_1(\mathcal{D})$$

and

$$\det \left( \frac{\Theta_T(e^{it})^* \Theta_T(e^{it}) - \mu I|_{\mathcal{D}}}{1 - \mu} \right) = \exp \int g_T(r^{1/2} e^{-it}) dr \quad (5.10)$$

for almost all $e^{it}$ on the unit circle and $\mu \in \rho(A_T)$.

**Proof.** From (4.31), there is a sequence $\{\rho_n\}$ satisfying $0 < \rho_n < 1$ and $\rho_n \to 1$ such that

$$\lim_{n \to \infty} \|\eta(\rho_n e^{i\theta}, \mu) \cdot \xi(e^{it}, \mu)\|_1 = 0.$$
Thus
\[
\det(\xi(e^{it}, \mu)) = \lim_{n \to \infty} \det(\eta(\rho_n e^{it}, \mu)).
\]

On the other hand, from (4.9), (5.3), and (5.4)
\[
\lim_{n \to \infty} \det(\eta(\rho_n e^{it}, \mu)) = \lim_{n \to \infty} \exp \left\{ \frac{1}{\pi} \int \left[ \frac{g_T(pe^{it}) \rho \mu}{(\rho^2 - \mu)(\mu - \rho_n e^{it})} \right] \right\
+ \frac{1}{\pi} \int \frac{g_T(pe^{it}) \rho \mu}{(\rho^2 - \mu)(1 - \rho_n e^{it})} \right\}
= \exp \left[ \frac{g_T(pe^{it})}{\rho^2 - \mu} \right].
\]

Since the symbol \( A_+ \) of \( A_T \) is the identity and
\[
\xi(e^{it}, \mu) = I - A(e^{it})^2/(1 - \mu)
= \frac{\Theta(e^{it}) \Theta(e^{it}) - \mu I}{1 - \mu},
\]
we have
\[
\det(\xi(e^{it}, \mu)) = \det \left( \frac{\Theta(e^{it}) \Theta(e^{it}) - \mu I}{1 - \mu} \right),
\]
which proves (5.10).

**Corollary 5.3.** Under the condition of Theorem 5.2, let \( N_T(\rho e^{it}) \) be the number of eigenvalues (counting multiplicity) of \( \Theta_T(e^{it}) \Theta_T(e^{it})^* \) (or of \( \Theta_T(e^{it}) \Theta_T(e^{it})^* \)) which lie in the interval \((0, \rho] \), then
\[
g_T(\rho e^{it}) = -N_T(\rho^2 e^{-it})
\tag{5.11}
\]
for almost all \( e^{it} \) on the unit circle, where \( g_T(\cdot) \) is the associated principal function of the contraction \( T \).

**Proof.** Suppose that the eigenvalues of \( \Theta_T(e^{it}) \Theta_T(e^{it})^* \) are
\[
0 \leq \lambda_1(e^{it}) \leq \lambda_2(e^{it}) \leq \cdots \leq \lambda_n(e^{it}) \leq \cdots \leq 1.
\]
Then
\[
\det \left( \frac{\Theta_T(e^{it}) \Theta_T(e^{it}) - \mu I}{1 - \mu} \right) = \prod_j \frac{\lambda_j(e^{it}) - \mu}{1 - \mu}.
\]
From (5.10), it follows that
\[ \int \frac{-g_T(r^{1/2} e^{-i\theta})}{r - \mu} \, dr = \sum_j \ln \frac{1 - \mu}{\lambda_j(e^{i\mu}) - \mu}. \]
Thus
\[ -g_T(r^{1/2} e^{-i\theta}) = \sum_{\lambda_j(e^{i\mu}) < r} 1 = \mathcal{N}_T(re^{i\theta}), \]
which proves (5.11).

Let \( M_T \) be the family of all functions \( f(\cdot, \cdot, \cdot) \) defined on \( S^1 \times S^1 \times R \) of the form
\[ f(e^{i\theta}, e^{i\phi}, x) = \sum_{m,n} e^{i(m\theta + n\phi)} \int e^{i\xi t} \, d\omega_{m,n}(t), \]
where \( \{\omega_{m,n}(t); m, n = 0, \pm 1, \pm 2, \ldots\} \) are complex measures satisfying
\[ \sum (|m| + |n|) \| T^m T^{*n} \| \int (1 + |t|) \, d|\omega_{m,n}| (t) < +\infty. \]
For these functions \( f \) in \( M_T \), define the operators
\[ f(T, T^*, A_T) = \sum T^m T^{*n} \int e^{i\mu t} \, d\omega_{m,n}(t). \]

**Corollary 5.4.** Under the condition of Theorem 5.2
\[ \text{Tr}([f(T^*, T, A_T), h(T^*, T, A_T)]) = \frac{1}{\pi} \iint g_T(\rho e^{i\theta}) \frac{\partial (f(e^{i\theta}, e^{-i\theta}, \rho^2), h(e^{i\theta}, e^{-i\theta}, \rho^2))}{\partial(\rho, \theta)} \rho \, d\rho \, d\theta, \] (5.12)
for \( f \) and \( h \) in \( M_T \).

The proof of this corollary is similar to that of the corresponding result in [4] cf. [23].

**Theorem 5.5.** Let \( T \) be a weak contraction such that \( T^{-1} \in \mathcal{L}(\mathcal{H}) \). If \( f(\cdot) \) is an analytic function defined on a neighborhood of \( \sigma(T) \cup \sigma(T^*^{-1}) \) then \( f(T) - f(T^*^{-1}) \in \mathcal{L}_1(\mathcal{H}) \). Besides, if also either \( a(T) \neq \{z: |z| = 1\} \) or
\[ \iint \left| \frac{g_T(\rho e^{i\theta})}{\rho} h(\rho^2) \right| \, d\rho \, d\theta < +\infty. \] (5.13)
where $h$ is a continuous function on $[0, 1]$, then

$$\text{Tr}(h(A_T)(f(T) - f(T^{-1}))) = h(0) \sum_{j} n_j (f(\lambda_j) - f(\lambda_j^{-1})) + \frac{1}{\pi} \int_{\partial \Gamma} g_T(\rho e^{-i\theta}) f'(e^{i\theta}) h(\rho^2) \frac{e^{i\theta}}{\rho^2} \, d\rho \, d\theta,$$

(5.14)

where $\{\lambda_j\}$ is the set of eigenvalues of $T$ with multiplicities $\{n_j\}$, respectively.

**Proof.** Suppose that $f(z) = (\lambda - z)^{-1}$ for $z \in \mathbb{C}\setminus\{\lambda\}$ and $\lambda \in \rho(T)$. It is easy to verify that

$$(\lambda I - T)^{-1} A^{1/2}_T = A^{1/2}_T (\lambda I - W^*)^{-1}$$

and

$$A^{1/2}_T (\lambda I - T^{-1})^{-1} = (\lambda I - W^*)^{-1} A^{1/2}_T.$$

Thus

$$A^{1/2}_T ((\lambda I - T)^{-1} - (\lambda I - T^{-1})^{-1}) A^{1/2}_T = [A_T, (\lambda I - W^*)^{-1}].$$

Hence for any polynomial $p(\cdot)$,

$$\text{Tr}(p(\lambda I - T) A_T (A_T + \varepsilon I)^{-1} ((\lambda I - T)^{-1} - (\lambda I - T^{-1})^{-1}))$$

$$= \frac{1}{\pi} \int_{\partial \Gamma} g_T(\rho e^{-i\theta}) \frac{e^{i\theta}}{(\lambda - e^{i\theta})^2} \frac{p(\rho^2) \rho}{\rho^2 + \varepsilon} \, d\rho \, d\theta.$$

(5.15)

If $p(0) = 0$, then putting $\varepsilon \to 0$ in (5.15), we get

$$\text{Tr} \{ p(A_T)(f(T) - f(T^{-1})) \} = \frac{1}{\pi} \int_{\partial \Gamma} g_T(\rho e^{-i\theta}) \frac{e^{i\theta}}{(\lambda - e^{i\theta})^2} \frac{p(\rho^2)}{\rho^2 + \varepsilon} \, d\rho \, d\theta,$$

(5.16)

and (5.14) holds for $h = p$ and $f(z) = (\lambda - z)^{-1}$.

Now suppose $\rho(z) \equiv 1$. If $\sigma(T) \not\subset \{z: |z| = 1\}$, then $\ln z$ may be defined as a regular function on a neighborhood of $\sigma(T) \cup \sigma(T^{-1})$ because $\sigma(T) = \{z: |z| = 1\} \cup \{\lambda_n\}$. From (5.15) it follows that

$$\text{Tr}(A_T (A_T + \varepsilon I)^{-1} (\ln T - \ln T^{-1})) = \frac{1}{\pi} \int_{\partial \Gamma} g_T(\rho e^{-i\theta}) \frac{\rho \, d\rho \, d\theta}{\rho^2 + \varepsilon},$$

where

$$\ln T = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T)^{-1} \ln \lambda \, d\lambda.$$
and \( T \) is a suitable contour. Since
\[
(\lambda - T)^{-1}(\lambda - T^*)^{-1} \in \mathcal{L}(\mathcal{H}) \quad \text{for} \quad \lambda \in \rho(T),
\]
it is easy to see that \( \ln T - \ln T^* \in \mathcal{L}(\mathcal{H}) \). Hence
\[
\frac{1}{\pi} \int |g_T(\rho e^{-i\theta})| \frac{\rho}{\rho^2 + \varepsilon} \, d\rho \, d\theta = -\text{Tr}(A_T(A_T + \varepsilon I)^{-1}(\ln T - \ln T^*-1)).
\]

Let \( \varepsilon \to 0 \), then
\[
\frac{1}{\pi} \int |g_T(\rho e^{-i\theta})| \frac{\rho}{\rho} \, d\rho \, d\theta < +\infty.
\]

Therefore (5.13) holds for every continuous \( h \) in case of \( \sigma(T) \not\subset \{ z : |z| = 1 \} \).

From (5.13) and (5.16), it follows that
\[
\text{Tr}(Qh(A_T)(f(T) - f(T^*-1))) = \frac{1}{\pi} \int g_T(\rho e^{-i\theta}) f'(e^{i\theta}) h(\rho^2) \frac{e^{i\theta}}{\rho} \, d\rho \, d\theta,
\]

for \( f(z) = (\lambda - z)^{-1} \), where \( Q \) is the projection from \( \mathcal{H} \) onto \( \overline{A_T} \).

Let \( Q_n \) be the projection from \( \mathcal{H} \) onto the space spanned by the eigenspaces of \( T \) corresponding to \( \lambda_j \) and the eigenspace of \( T^* \) corresponding to \( \bar{\lambda}_j \), \( j = 1, 2, \ldots, n \). Then \( Q_n \uparrow (I - Q) \) (cf. [16]). Hence
\[
\text{Tr}((I - Q)(f(T) - f(T^*-1))) = \lim_{n \to \infty} \text{Tr}(Q_n(f(T) - f(T^*-1)))
\]
\[
= \lim_{n \to \infty} \sum_{j=1}^{n} n_j(f(\lambda_j) - f(\bar{\lambda}_j^{-1}))
\]

for \( f(\cdot) = (\lambda - (\cdot))^{-1} \). On the other hand, it is obvious that
\[
\text{Tr}((I - Q) h(A_T)(f(T) - f(T^*-1))) = h(0) \text{Tr}((I - Q)(f(T) - f(T^*-1))).
\]

From (5.17), (5.18), and (5.19), it follows (5.14) for \( f(z) = (\lambda - z)^{-1} \).

This proves the result for these special \( f \), but the case of general analytic \( f \) is proved using the Cauchy formula.

Now let us restrict ourselves to the case where \( A_T \) is invertible. In this case the operator \( X \) is invertible. The polar symbols of \( H \) with respect to \( W \) are denoted by
\[
S_{\pm}(W, H) = n-\lim_{n \to \pm \infty} W^* H W^n.
\]
For boundedly invertible $T$, $Z_1 = Z_0 V_c$ is a unitary operator from the completely nonunitary space $\mathcal{H}_c$ onto $\mathcal{A}L^2(\mathcal{D}_*) = L^2(\mathcal{D}_*)$.

**Theorem 5.6.** If $A_T$ is invertible, then

\[
S_+(W, H_T) = W, \tag{5.20}
\]

\[
S_-(W, H_T)|_{\mathcal{H}_u} = W|_{\mathcal{H}_u} = T^*|_{\mathcal{H}_u}, \tag{5.21}
\]

and

\[
(Z_1 S_-(W, H_T)|_{\mathcal{H}_u} Z_1^{-1} f)(e^{it}) = e^{-it} (\Theta_T(e^{it})*\Theta(e^{it}))^{1/2} f(e^{it}). \tag{5.22}
\]

**Proof.** Since $T^* = A_T^{-1/2} W A_T^{1/2}$, it is obvious that

\[
A_T = s\text{-}\lim_{n \to \infty} T^n T^{*n} = s\text{-}\lim_{n \to \infty} A_T^{1/2} W^n A_T^{-1/2} W^n.
\]

Therefore $s\text{-}\lim_{n \to \infty} W^n A_T^{-1} W^n = I$, which implies

\[
s\text{-}\lim_{n \to \infty} W^n A_T^{1/2} W^n = I, \tag{5.23}
\]

since $A_T \geq 0$ and $A_T^{-1} \in \mathcal{L}(\mathcal{H})$. From (5.23), we get (5.20).

We turn now to the calculation of $S_-(W, H_T)$. It is obvious that in the unitary subspace $\mathcal{H}_u$

\[
A_T|_{\mathcal{H}_u} = I|_{\mathcal{H}_u} \quad \text{and} \quad H_T|_{\mathcal{H}_u} = W|_{\mathcal{H}_u} = T^*|_{\mathcal{H}_u},
\]

which imply (5.21).

It is obvious that

\[
Z_1 A_T^{1/2} Z_1^{-1} = (I - A P A)^{1/2},
\]

\[
Z_1 T^* Z_1^{-1} = (I - A P A)^{-1/2} e^{-it} (I - A P A)^{1/2}.
\]

Therefore

\[
s\text{-}\lim_{n \to -\infty} (Z_1 W^n A_T W^n Z_1^{-1} f)(e^{it}) = s\text{-}\lim_{n \to -\infty} e^{nit} (I - A P A) e^{-nit} f(e^{it}) = \Theta_T(e^{it})*\Theta_T(e^{it}) f(e^{it}), \tag{5.24}
\]

which implies

\[
s\text{-}\lim_{n \to -\infty} (Z_1 W^n A_T^{1/2} W^n Z_1^{-1} f)(e^{it}) = (\Theta_T(e^{it})*\Theta_T(e^{it}))^{1/2} f(e^{it}). \tag{5.25}
\]

Thus (5.22) is proved.
Corollary 5.7. Let $H$ be a completely hyponormal operator in $\mathcal{H}$. Suppose $H$ is invertible. Denote the polar decomposition of $H$ by

$$H = W |H|,$$

where $W$ is unitary and $|H| \geq 0$. If

$$S_-(W, |H|) = \lim_{n \to \infty} W^* W^n = c I,$$  \hspace{1cm} (5.26)

where $c$ is a constant, then the principal function $g(\cdot)$ of $H$ is a complete unitary invariant for the unitary operator $W$.

Proof. Without loss of generality, we may assume that the constant $c$ in (5.26) is 1. It is obvious that $|H|$ is invertible, since $H$ is invertible. Define an operator

$$T = |H|^* |H|^{-1},$$

then

$$I - T^* T = |H|^{-1} |H^*, H| |H|^{-1} \geq 0.$$  

Thus $T$ is a contraction. From (5.26), it follows that

$$S_-(W, |H|^{-1}) = I,$$

$$|H|^2 = \lim_{n \to \infty} T^n T^* n.$$  

Therefore $|H| = A_T$ and $H = H_T$. In this case $\mathcal{H}_w = \{0\}$, since $H$ is completely hyponormal. From Corollary 5.3, (5.24), and (5.25) it is easy to see that $g(\cdot)$ is a complete unitary invariant, since it determines the spectral multiplicity of $W$.

6. Phase Shift Formulae for Dissipative Operators

Let $K$ be a dissipative operator, i.e.,

$$\text{Im}(Kf, f) \geq 0 \quad \text{for every } f \in \mathcal{D}(K).$$

If $K$ has no proper dissipative extension, then $K$ is called maximal dissipative (cf., for example, [19]). In this section, we always assume that $K$ is maximal dissipative and there is a constant $k$ such that

$$\text{Im}(Kf, f) \leq k \|f\|^2, \quad \text{for every } f \in \mathcal{D}(K).$$  \hspace{1cm} (6.1)
It is then easy to see that \( \{ e^{ikt}, 0 \leq t < +\infty \} \) is a semi-group of contractions, the limit

\[
B_K = \lim_{t \to \infty} e^{ikt} e^{-iK^*t},
\]

exists, and \( 0 \leq B_K \leq I \). We associate a contraction \( T \) to the maximal dissipative operator through the Cayley transform

\[
T = (K - iI)(K + iI)^{-1}.
\]

**Lemma 6.1.** Let \( K \) be a maximal dissipative operator satisfying (6.1) and \( T \) be its Cayley transform, then

\[
B_K = A_T. \tag{6.2}
\]

**Proof.** The proof is similar to that of Theorem 2.4 in [23]. It follows from the definition of \( B_K \) that

\[
B_K K^* = KB_K. \tag{6.3}
\]

Thus

\[
TB_K T^* = B_K. \tag{6.4}
\]

From (6.1), Im \( K \) is bounded. It is obvious that

\[
\frac{d}{dt} e^{ikt} e^{-iK^*t} = -2e^{ikt} \text{Im} Ke^{-iK^*t}
\]

for \( f \) in a dense subset of \( \mathcal{H} \). Hence

\[
I = B_K + 2 \int_0^\infty e^{ikt} \text{Im} Ke^{-iK^*t} \, dt.
\]

Equation (6.4) implies that

\[
T^n T^{*n} = B_K + 2 \int_0^\infty T^n e^{-ikt} \text{Im} Ke^{-iK^*t} T^{*n} \, dt,
\]

and hence

\[
T^n T^{*n} \geq B_K.
\]

If we take the limits in this inequality, then we get

\[
A_T \geq B_K. \tag{6.5}
\]
On the other hand, $KA_T = A_T K^*$, since $A_T$ is a $T^*$-Toeplitz operator. Thus
\[ e^{iKt}A_T e^{-iK^*t} = A_T. \] (6.6)

It is obvious that
\[ I = A_T + \sum_{n=0}^{\infty} T^n D_n^2 T^* n. \] (6.7)

From (6.6) and (6.7), it follows that
\[ e^{iKt} e^{-iK^*t} = A_T + \sum_{n=0}^{\infty} e^{iKt} T^n D_n^2 T^* n e^{-iK^*t}. \]

Hence
\[ e^{iKt} e^{-iK^*t} \geq A_T, \]

which implies $B_K \succeq A_T$. This inequality and (6.5) imply (6.2).

Define a principal function $g_K(\cdot)$ of the maximal dissipative operator $K$ satisfying (6.1) as
\[ g_K(x + iy) = g_T((x + i)(x - i)^{-1}y^{1/2}), \quad y > 0, \] (6.8)
and $g_K(x + iy) = 0$ for $y < 0$, where $g_T(\cdot)$ is the principal function of the Cayley transform $T$ in case of $[H^*, H_T] \in \mathcal{L}_1(\mathcal{H})$.

**Corollary 6.2.** If the dissipative operator $K$ is bounded, $\text{Im } K \in \mathcal{L}_1(\mathcal{H})$, and $B_K$ is invertible, then $g_K(\cdot)$ is the principal function of the hyponormal operator $H_K = X + i B_K$, where $X$ is a self-adjoint operator $B_K^{-1/2} K B_K^{1/2}$.

**Proof.** From (6.3), it is easy to prove that $X$ is self-adjoint.

Let $T$ be the Cayley transform of $K$ and
\[ W = (X + iI)(X - iI)^{-1} \] (6.9)
then
\[ H_T = A_T^{1/2} T^* = W B_K^{1/2}. \]

From (6.3), it follows that
\[ i[X, B_K] = 2B_K^{1/2} \text{Im } K B_K^{1/2}. \]

Thus $H_K = X + iB_K$ is hyponormal and $[H^*, H_K]$ is in the trace class. On the other hand,
\[ [H^*, H_T] = (X - i)^{-1} [H^*, H_K] (X + i)^{-1} \in \mathcal{L}_1(\mathcal{H}). \]
Thus $g_K(\cdot)$ equals the principal function of the hyponormal operator $H_K$ through the transformation law of the principal function, (6.2), and (6.9).

The characteristic operator function for the maximal dissipative operator $K$ satisfying (6.1) is defined as

$$
\Theta_K(z) = (I - E(K + iI)^{-1}(K^* - iI)^{-1}E)^{1/2}
\times \left[ I - \frac{z-i}{2i} E(K - iI)^{-1}(K^* - zI)^{-1} E \right]_\delta
$$

(6.10)

for $\text{Im} \, z \geq 0$, where $E = 2(\text{Im} \, K)^{1/2}$ and $\delta = \overline{E \mathcal{H}}$. The relationship between $\Theta_K(\cdot)$ and the $\Theta_T(\cdot)$, where $T$ is the Cayley transform of $K$, is given by

$$
\Theta_K(z) = V \Theta_T((z-i)(z+i)^{-1}) U,
$$

(6.11)

where $U: \delta \to \mathcal{Q}$ and $V: \mathcal{Q} \to \delta$ are unitary operators (cf. [19]).

We can therefore state a version of Theorem 5.2, Corollary 5.3, Theorem 5.5, and Corollary 5.6 for the dissipative case.

**Theorem 6.3.** Let $K$ be a maximal dissipative operator satisfying the conditions that $\text{Im} \, K \in \mathcal{L}_1(\mathcal{H})$ and the Cayley transform $T$ of $K$ is invertible, then

$$
\Theta_K(x)^* \Theta_K(x) - I|_\delta \in \mathcal{L}_1(\delta)
$$

and

$$
\det \left( \frac{\Theta_K(x)^* \Theta_K(x) - \mu I|_\delta}{1 - \mu} \right) = \exp \int \frac{g_K(x + iy) \, dy}{y - \mu}
$$

for almost all real $x$ and $\mu \in \rho(B_K)$.

The proof is an immediate consequence of Theorem 5.2, Lemma 6.1, (6.8), and (6.11).

**Corollary 6.4.** Under the conditions of Theorem 6.3, let $N_K(x + iy)$ be the number of the eigenvalues of $\Theta_K(x)^* \Theta_K(x)$ which are in the interval $[0, y]$. Then

$$
g_K(x + iy) = -N_K(x + iy).
$$

**Theorem 6.5.** Let $K$ be a dissipative operator satisfying the condition of Theorem 6.3. If $\Phi(\cdot)$ is an analytic function defined on a neighborhood of $\sigma(K) \cup \sigma(K^*)$ in the Riemann sphere then $\Phi(K) = \Phi(K^*) \in \mathcal{L}_1(\mathcal{H})$. Besides if also either $K$ is bounded or
\[
\int \int \frac{|g_K(x + iy) h(y)|}{y} \, dx \, dy < +\infty,
\]
where \( h \) is a continuous function on \([0, 1]\), then

\[
\text{Tr}(h(B_K)(\Phi(K) - \Phi(K^*)))) = h(0) \sum_j n_j (\Phi(\lambda_j) - \Phi(\bar{\lambda}_j))
- \frac{1}{2\pi i} \int \int g_K(x + iy) \Phi'(x) h(y) y^{-1} \, dx \, dy,
\]
(6.12)

where \( \{\lambda_j\} \) are eigenvalues of \( K \) with multiplicities \( n_j \), respectively.

**Remark 6.6.** Let \( H = u + iv \) be a completely hyponormal operator with \([H^*, H] \in C_1 \) satisfying \( v \geq cl \), where \( c \) is a constant. If

\[
s_-(u, v) = s-lim_{t \to -\infty} e^{int} e^{-iut} = cI
\]

then the principal function \( g(\cdot) \) of \( H \) is a complete unitary invariant for \( u \).

Remark 6.6 may be proved independently from the machinery of this paper in the following way. Here we sketch this proof.

**Proof.** For simplicity, let \( c = 1 \).
Let \( s_+(u, v) = s-lim_{t \to +\infty} e^{int} e^{-iut} \). Then (cf. [9])

\[
\omega_\lambda(l) = \exp \int g(v + i\lambda) \frac{dv}{v - l}
= \det(l + (s_-(u, v)(\lambda) - s_+(u, v)(\lambda))(s_+(u, v) - ll)(\lambda)^{-1})
= \det \left(1 + \int \frac{dR(l,v)}{v - l} \right),
\]
where \( dR(l, \cdot) \) is absolutely continuous with respect to its trace \( d \text{Tr}(R_l(\cdot)) \) and the corresponding Radon–Nikodym derivative \( R'_l(\cdot) \) has its eigenvalues in the interval \([0, 1]\). Let the \( j \)-th eigenvalues of \( R'_l(v) \) be denoted by \( \mu_j(v, \lambda) \), each eigenvalue appearing in this enumeration according to its multiplicity, in such a way that

\[
0 < \cdots \leq \mu_2(v, \lambda) \leq \mu_1(v, \lambda) \leq 1.
\]

Define for each Borel set \( \Delta \) of \( R^1 \) the scalar measures

\[
\sigma^j(\Delta) = \int_\Delta \mu_j(v, \lambda) (d \text{Tr}(R_l(v))
\]
and let $H^j(\lambda) = L^2(d\sigma^{(j)}_\lambda)$. In [9], it was proved that

$$m(\lambda) = \sum_j \dim(H^j(\lambda))$$

is the spectral multiplicity a.e. of $\lambda \in \sigma(u)$.

But the hypothesis $s_-(u, v) = 1$ enables us to compute these dimensions. By the Aronszjan-Weinstein formula (cf. p. 247 of [8]), we define for $B \in \mathcal{L}(\mathcal{H})$ satisfying $B = B^*$ the function $\tilde{v}(\zeta, B)$ as follows. If $\zeta \notin \sigma(B)$ then $\tilde{v}(\zeta, B) = 0$. If $\zeta \in \sigma_p(B)$ then $\tilde{v}(\zeta, B)$ is the dimension of the eigenspace corresponding to $\zeta$. And $\tilde{v}(\zeta, B) = \infty$ for $\zeta \in \sigma(B) \setminus \sigma_p(B)$. Then

$$\tilde{v}(\zeta, s_+(u, v)(\lambda)) = \tilde{v}(\zeta, s_-(u, v)(\lambda)) + v(\zeta, \omega),$$

where $v(\zeta, \omega) = k$ if $\zeta$ is a zero of $\omega$ of order $k$, $v(\zeta, \omega) = -k$ if $\zeta$ is a pole of $\omega$ of order $k$, and $v(\zeta, \omega) = 0$ for other points.

Since $s_-(u, v)(\lambda)$ has only the point 1 in its spectrum, we see that the phase shift determines the multiplicities, i.e., $\sum \dim H^j(\lambda)$ equals the sum of the multiplicities of all the eigenvalues of $s_+(u, v)(\lambda)$.

Each zero $v^{(n)}$ of $\omega$ of order $k$ is an eigenvalue of $s_+(u, v)(\lambda)$ with multiplicity exceeding by $k$ the multiplicity of $v^{(n)}$ as an eigenvalue of $s_-(u, v)(\lambda)$. Thus

$$\sum_{j=1}^m g(\lambda, v^{(j)}) - g(\lambda, v^{(j-1)}) = m(\lambda),$$

where $m$ is the highest number. Thus Corollary 6.6 is proved.

**Remark 6.7.** If $B_K^{-1} \in \mathcal{L}(\mathcal{H})$, then $S_+(X, B_K) = I$, where $X = B_K^{-1/2}KB_K^{1/2}$. In fact,

$$e^{iXt} = B_K^{-1/2} e^{iKt} B_K^{1/2}.$$

Hence

$$e^{-Xt}B_K e^{iXt} = B_K^{1/2} e^{-iKt} e^{iKt} B_K^{1/2}.$$

But

$$\text{s-lim}_{t \to -\infty} e^{-iKt} e^{iKt} = B_K^{-1}.$$

Therefore $S_+(X, B_K) = I$.

**Remark 6.8.** We turn now to a discussion of a simple illustration of Theorem 6.5.
Suppose that $K = R + Q$, with $R = R^*$ and $Q = Q_R + iQ_I$, where $Q_R$ and $Q_I$ are self-adjoint, trace class and $Q_I \geq 0$. Then $K$ is maximal dissipative and $\mathcal{E} \equiv \overline{Q_I \mathcal{H}}$, the operator valued function

$$\hat{\Theta}_K(z) = I_\mathcal{E} + 2i \sqrt{Q_I} (K^* - zI)^{-1} \sqrt{Q_I |\mathcal{E}}$$

defined for $\text{Im} \ z \geq 0$, is easily seen to be the Lifshitz characteristic operator function of the restriction of $K$ to the closed subspace $\mathcal{H}(R + Q_K)\mathcal{E}$, generated by functions $f$.

The contractive analytic function $\det \hat{\Theta}_K(\cdot)$ has the representation

$$\det \hat{\Theta}_K(z) = \beta B(z) \exp \left( -i \int \frac{1 + tz}{t - z} \frac{d\sigma(t)}{1 + t^2} \right) e^{-iaz},$$

where $a > 0$, $|\beta| = 1$,

$$\int \frac{d\sigma(t)}{1 + t^2} < +\infty,$$

and

$$B(z) = \prod_j \left\{ b_j(z) \overline{b_j(-i)} |b_j(-i)|^{-1} \right\}^{m_j}$$

for $b_j(z) = (z - \lambda_j)(z - \bar{\lambda}_j)^{-1}$ and $\{\lambda_j\}$ is an enumeration of the non-real zeros of $\det \hat{\Theta}_K(\cdot)$ with multiplicities $\{m_j\}$.

In fact, this representation can be refined:

$$\det \hat{\Theta}_K(z) = x B(z) \exp \left( -i \int \frac{d\sigma(t)}{t - z} \right),$$

where $|x| = 1$ and $\int d\sigma(t) < \infty$.

This is a consequence of the observations that

(a) $\lim_{y \to \infty} \det \hat{\Theta}_K(x + iy) = 1$,

(b) $\hat{\Theta}_K(z)^{-1} = I_\mathcal{E} - 2i \sqrt{Q_I} (K - zI)^{-1} \sqrt{Q_I}$, and

(c) $|\det \hat{\Theta}_K(z)|^{1/2} \leq \exp(2 \|\sqrt{Q_I} (K - zI)^{-1} \sqrt{Q_I} \|_{\mathcal{L}_1})$ for $\text{Im} \ z > \|Q_I\|_{\mathcal{L}_1}$.

It now follows from Theorem 6.3 or 6.5 above that the Radon–Nikodym derivative $\sigma'(\cdot)$ of $\sigma$ with respect to Lebesgue measure satisfies

$$\sigma'(x) = -\frac{1}{2\pi} \int_{\sigma(b_j)} g_K(x + iy) \frac{dy}{y},$$

for $b_j = (x - \lambda_j)(x - \bar{\lambda}_j)^{-1}$ and $\{\lambda_j\}$ is an enumeration of the non-real zeros of $\det \hat{\Theta}_K(\cdot)$ with multiplicities $\{m_j\}$.
for a.e. \( x \) so that we set a special case of (6.12), cf. [1, 16].

\[
\text{Tr}(\Phi(K) - \Phi(K^*)) = -i \int \Phi'(t) \sigma'(t) \, dt.
\]

We may also conclude that

\[
\text{Tr}(B_k^{n+1} Q_t) = -\frac{1}{\pi} \iint g_k(x + iy) y^n \, dx \, dy
\]

for non-negative integer \( n \).

If \( E_\lambda \) denotes the spectral resolution of \( B_k \), this last equation becomes

\[
\int y^n \, d\text{Tr}(E, Q_t) = \int y^n \left( -\frac{1}{\pi y} \int g_k(x + iy) \, dx \right) \, dy.
\]

Thus

\[
\frac{d}{dy} \text{Tr}(E, Q_t) = -\frac{1}{\pi y} \int g_k(x + iy) \, dx
\]

for a.e. \( y \in [0, 1] \).

In particular, when \( Q_t \) has rank one,

\[
Q_t = (e, e).
\]

We find

\[
\frac{d}{dy} (E, e, e) = -\frac{1}{\pi y} \int_{\sigma(t)} g_k(t + iy) \, dt.
\]

Finally, we note that because

\[
|\Theta_k(x)| = e^{-\pi \sigma(x)} \quad \text{for real } x,
\]

Corollary 6.4 tells us that

\[
g_k(x + iy) = -1
\]

for \( e^{-2\pi \sigma(x)} < y < 1 \) and

\[
g_k(x + iy) = 0
\]

elsewhere.
7. Determinant Formulae for the Determining Functions

Let $H$ be a hyponormal operator in the Hilbert space $\mathcal{H}$. Then there is a contraction in $\mathcal{H}$ such that

$$TH = H^*.$$  \hspace{1cm} (7.1)

Hence $H$ is a $T^*$-Toeplitz operator

$$THT^* = H.$$  \hspace{1cm} (7.2)

From now on, the operator $A$ in the formulas of Sections 3-4 we refer to will be changed to the operator $H$. If $T^{-1} \in \mathcal{L}(\mathcal{H})$ then $\ker(X^*) = \{0\}$. Since $H$ is completely hyponormal, $Hx = 0$ implies $x = 0$, hence $\ker(X) = \{0\}$.

**Lemma 7.1.** If $H$ is hyponormal, invertible and $[H^*, H] \in \mathcal{L}_1(\mathcal{H})$ then

$$\det D(\lambda, \mu) = \exp \left\{ \frac{1}{\pi} \int \frac{\lambda \xi g(\xi) \, dm(\xi)}{\xi(\xi - \mu)(\xi - \lambda \xi)} \right\}.$$  \hspace{1cm} (7.3)

and

$$\det D_*(\lambda, \mu) = \exp \left\{ \frac{1}{\pi} \int \frac{\xi g(\xi) \, dm(\xi)}{\xi(\xi - \mu)(\xi - \lambda \xi)} \right\}.$$  \hspace{1cm} (7.4)

for $|\lambda| < 1$ and $\mu \in \rho(H)$, where $g(\cdot)$ is the principal function of the operator $H$ and $m(\cdot)$ is the planar Lebesgue measure.

**Proof.** By the same method used in proving (5.6), we can prove that

$$\det D(\lambda, \mu) = \det \{H - \mu I, I - \lambda H^*^{-1} H\}$$

$$= \exp \left\{ \frac{1}{2\pi i} \int g(\rho e^{it}) \, d\ln(\rho e^{it} - \mu) \wedge d\ln(1 - \lambda e^{2it}) \right\},$$

which proves (7.3). Similarly, we can prove (7.4).

Let $N_+$ and $N_-$ be the normal symbols of $H$ defined in Section 4, then $N_\pm$ are multiplication operators

$$(N_\pm f)(e^{it}) = N_\pm(e^{it})f(e^{it}).$$

**Theorem 7.2.** If $H$ is hyponormal, $H$ is invertible, and $[H^*, H] \in \mathcal{L}_1$, then

$$\det \xi(e^{it}, \mu) = \exp \int_0^\infty \left( \frac{g(re^{-it/2})}{r - \mu e^{-it}z} + \frac{g(-re^{-it/2})}{r + \mu e^{it/2}} \right) \, dr.$$  \hspace{1cm} (7.5)
and
\[ \det \xi(e^\mu, \mu) = \det((N_+ (e^\mu) - \mu I)^{-1} (N_- (e^\mu) - \mu I)), \quad \mu \in \rho(H), \quad (7.6) \]
where \( \xi(e^\mu, \mu) \) is defined in (4.9) (for \( H \)) and \( g(\cdot) \) is the principal function of \( H \).

**Proof.** (a) From (4.31), it follows that
\[
\det \xi(e^\mu, \mu) = \lim_{n \to +\infty} \det \eta(\rho_n e^\mu, \mu),
\]
where \( 0 < \rho_n < 1 \) and \( \rho_n \to 1 \). By means of Lemma 7.1, it is easy to see that
\[
\det \eta(\rho_n e^\mu, \mu) = \det \frac{D^*(\rho_n e^\mu, \mu)}{\det D(\rho_n e^\mu, \mu)} - \exp \left\{ \frac{1}{\pi} \int \int \frac{g(\zeta) |\zeta|^2 (1 - \rho_n^2) dm(\zeta)}{\zeta(\zeta - \mu) |\zeta - \rho_n e^\mu \zeta|^2} \right\}.
\]
Putting \( n \to \infty \), we obtain (7.5).

(b) To prove (7.6), let \( b \) be a negative semi-definite operator such that \( [\Theta(e^\mu)] = \exp b \). It is evident that \( b \) is in the trace class since \( |\Theta(e^\mu)|^2 = I - A^2 \) is in the trace class for almost all \( e^\mu \).

For convenience, the matrix \( e^{b_x} N_+ (e^\mu) e^{b_x} \) is denoted by \( N_x \), where \( x \in [0, 1] \). It is obvious that
\[
u(x, x') = \det (N_x - \mu I)^{-1} (N_x - \mu I)
\]
exists and is a continuous function of \( x \) and \( x' \). Moreover,
\[
\det((N_+ (e^\mu) - \mu I)^{-1} (N_- (e^\mu) - \mu I)) = u(0, 1). \quad (7.7)
\]
A simple calculation shows
\[
u(x, x + Ax) = \det(I + (N_x - \mu I)^{-1} (e^{bAx} N_x e^{bAx} - N_x))
\]
\[
= \det(I + (N_x - \mu I)^{-1} (bN_x + N_x b) Ax + R(x, Ax)),
\]
where \( |Ax| \) is a small number and \( \|R(x, Ax)\|_1/(Ax)^2 \) is bounded for \( Ax \to 0 \). Thus
\[
u(x + Ax) = \exp \{ \text{Tr}(2(N_x - \mu)^{-1} N_x b) Ax + O(Ax^2) \}.
\]
Hence
\[
u(0, 1) = \lim_{n \to +\infty} \prod_{j=0}^{n-1} u \left( \frac{j}{n}, \frac{j+1}{n} \right)
\]
\[
= \exp \left\{ 2 \int_0^1 \text{Tr}((N_x - \mu I)^{-1} N_x b) dx \right\}. \quad (7.8)
\]
Along the same lines, we change $N_x$ to $N(e^{it}) e^{2bx}$. We can prove that
\[
\det((N_+(e^{it}) - \mu I)^{-1}(N_+(e^{it}) e^{2bx} - \mu I)) = \exp \left( 2 \int_0^1 \text{Tr}((N_+(e^{it}) e^{2bx} - \mu I)^{-1} N_+(e^{it}) e^{2bx} b) \, dx \right). \tag{7.9}
\]

It is obvious that
\[
\text{tr}((N(e^{it}) e^{2bx} - \mu I)^{-1} N(e^{it}) e^{2bx} b) = \text{tr}((N_x - \mu I)^{-1} N_x b).
\]

Thus (7.8) and (7.9) are equal and
\[
u(0, 1) = \det((N_+(e^{it}) - \mu I)^{-1}(N_+(e^{it})[|\Theta(e^{it})|^2] - \mu I)) = \det(I + (N_+(e^{it}) - \mu I)^{-1} N_+(e^{it})([\Theta(e^{it})*\Theta(e^{it})] - I)) = \det(I - N(e^{it})(N(e^{it}) - \mu I)^{-1} \Delta(e^{it})^2),
\]
which equals $\det \xi(e^{it}, \mu)$. We conclude (7.4).

8. $J$-Matrix Representation of Hyponormal Operators

Let $H$ be a hyponormal operator in the Hilbert space $\mathcal{H}$, and $T$ be a contraction in $\mathcal{H}$ satisfying
\[
TH = H^* \tag{8.1}
\]
Let $N$, $X$, $U$ be the operators introduced by Sz.-Nagy and Foias in $[21]$ or cf. Section 2.

Let $\mathcal{H}^{(n)} (n \geq 1)$ be the minimal subspace which reduces $T$ and contains
\[
\bigcup_{j=0}^{n-1} H^j\left[H^*, H\right] \mathcal{H}
\]
as a subset. Then the subspace $\bigvee_{n=1}^\infty \mathcal{H}^{(n)}$ is invariant with respect to $H$ and $T$ and hence also $H^*$ by (8.1). Thus
\[
\mathcal{H} = \bigvee_{n=1}^\infty \mathcal{H}^{(n)},
\]
since $H$ is completely hyponormal. Denote $\mathcal{H}_0 = \mathcal{H}^{(0)} = \mathcal{H}$, and
\[
\mathcal{H}_n = \mathcal{H}^{(n)} \ominus (\mathcal{H}^{(n-1)} \vee \mathcal{H}_0), \quad n > 1.
\]
Then $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n. \quad (8.2)$$

It may be helpful to explain this decomposition a little more fully. $\mathcal{H}_e$ is the closed span of

$$\{ f(\mathcal{T}, \mathcal{T}^*) D^2 \mathcal{H} \vee h(\mathcal{T}, \mathcal{T}^*) D^2 \mathcal{H} \}$$

obtained from polynomials $f(\mathcal{T}, \mathcal{T}^*)$ and $h(\mathcal{T}, \mathcal{T}^*)$.

Because $[\mathcal{T}, \mathcal{H}] \mathcal{H} = [\mathcal{H}, \mathcal{H}^*]$, we can easily deduce, if $\mathcal{H}$ is invertible, that

$$\mathcal{H} \mathcal{D}^2 = [\mathcal{H}, \mathcal{H}^*](\mathcal{H}^* \mathcal{H})^{-1} \mathcal{H}$$

$$\mathcal{H} \mathcal{D}^2 = \mathcal{T}^*[\mathcal{H}, \mathcal{H}^*] \mathcal{H}^{-1}.$$ 

Using these facts it is easy to see that

$$H \mathcal{H}^{(j)} \subset \bigcup_{i=1}^{j+1} \mathcal{H}^{(i)} \quad \text{and} \quad H \mathcal{H}_i \subset \mathcal{H}^{(1)}.$$

When $\mathcal{H}$ is not boundedly invertible, we can use the Moore–Penrose inverse $\mathcal{H}^+$ to get similar identities, and the same decomposition, because $[\mathcal{H}, \mathcal{H}^+] \mathcal{H}^+$ is bounded.

According to this decomposition, the operator $\mathcal{H}$ may be written as a matrix $\mathcal{H} = (H_{ij})$, where $H_{ij}$ is an operator from $\mathcal{H}_j$ into $\mathcal{H}_i$. From the structure of $\mathcal{H}_j$

$$H \mathcal{H}_j \subset \bigoplus_{i=0}^{j+1} \mathcal{H}_i,$$

it follows that $H_{ij} = 0$ for $i > j + 1$. Since $\mathcal{H}^{(n)}$ reduces $\mathcal{T}$, $\mathcal{H}_n$ reduces $\mathcal{T}$. From (8.1), it follow that

$$H^* \mathcal{H}_j = \mathcal{T} H \mathcal{H}_j \subset \bigoplus_{i=0}^{j+1} \mathcal{H}_i.$$ 

It also follows that $H_{ij} = 0$ for $j > i + 1$. Thus $(H_{ij})$ is a tridiagonal matrix, i.e.,

$$H_{ij} = 0 \quad \text{for} \quad |j - i| > 1.$$ 

The matrices above are called $J$-matrices.
Let $P_u$ be the projection from $H$ onto $H_u = \sum_{n \geq 1} \oplus \mathbb{H}_n$, $T_u = T|_{\mathbb{H}_u}$ and $H_u = P_u H|_{\mathbb{H}_u}$. Since $H_u$ reduces $T$, from (7.1) it follows that

$$T_u H_u = H_u^*.$$ 

Since $T_u$ is unitary, it is obvious that $H_u H_u^* = H_u T_u H_u = H_u^* H_u$, which means

$$H_u = (H_{ij})_{i \geq 1, j \geq 1}$$

is normal. Since each $\mathbb{H}_j$ reduces $T$, the operator $T$ may be written as a diagonal matrix $(T_j \delta_j)$, where $T_0 - T_c$ is the completely non-unitary part of $T$ and all other $T_i = T_u|_{\mathbb{H}_i}$ are unitary. Thus we can choose the space $G$ and the operators $X$, $U$, and $N$ in the Sz.-Nagy–Foias theorem (cf. [21]) such that

$$G = \sum_{j \geq 0} \oplus G_j,$$

and the matrices

$$N = (N_{ij}), \quad X = (X_i \delta_j), \quad U = (U_i \delta_j)$$

possess the following properties: $X_i$, $i \geq 1$, and $U_i$, $i \geq 0$, are unitary operators, and $N$ is also a $J$-matrix. For convenience, we take

$$G_0 = \overline{\Delta L^2(\mathbb{D})},$$

$$X_0 = P_{G_0} V_0,$$

where $P_{G_0}$ is the projection from $H^2(\mathbb{D}^*) \oplus \overline{\Delta L^2(\mathbb{D})}$ onto $G_0$, and take $U_0$ as the multiplication operator,

$$(U_0 f)(e^{it}) = e^{it} f(e^{it}), \quad f \in G_0,$$

i.e., $U_0 = V|_{G_0}$, where $V$ is the minimal isometric dilation of the operator $V_c T_c V_c^{-1}$.

Sometimes, for simplicity, without loss of generality we may also suppose that $0 \notin \sigma(H)$ and that there is also some half plane

$$L_{a, \alpha} = \{ z : \Re (ze^{-i\alpha}) \geq a \},$$

where $\alpha$ is real and $a$ is positive such that $\sigma(H) \subset L_{a, \alpha}$. In fact, if $\sigma(H)$ does not satisfy the above condition then $H$ may be shifted by a constant $c \in C$ such that the hyponormal operator $H - cI$ does satisfy the condition. Let $T = \{ z : |z| = 1 \}$.

**Lemma 8.1.** Suppose $\sigma(H) \subset L_{a, \alpha}$. Then
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\[ \sigma(T) \cap \mathbb{T} \subset \gamma, \quad (8.5) \]
\[ \operatorname{supp} A(\cdot) \subset \gamma, \quad (8.6) \]

where \( \gamma = \{ e^{i\theta} : |\theta + 2\alpha| < \pi \} \), and
\[ \sigma(N) \subset L_{a, x}. \quad (8.7) \]

Proof. Without loss of generality, we may suppose \( x = 0 \). By the cut down property (cf. [15, 23]) of the spectrum of a hyponormal operator \( H = u + iv \) the spectrum of the real part \( u \) is contained in the interval \([a, \infty)\), where \( a > 0 \). Thus \( u \) is invertible. Then
\[ T - I = (H^* + H) H^{-1} \]
is invertible and \(-1 \in \rho(T)\), i.e., (8.5) holds. Hence the characteristic operator function \( \Theta_T(\lambda) \) is regular and invertible in a neighborhood \( O \) of 1. Hence,
\[ \Theta_T(\lambda)^{-1} = \Theta(\lambda)^* \]
for \( \lambda \in O \cap \mathbb{T} \). Thus \( \Delta(e^{it}) = 0 \) for \( e^{it} \in O \cap \mathbb{T} \), which proves (8.6).

From (4.4), it follows that \( \sigma((N + N^*)/2) \subset [a, \infty) \), since \( \sigma(u) \subset [a, \infty) \). This proves (8.7).

Since \( N_{00} U_0 = U_0 N_{00} = N_{00}^* \), \( N_{00} \) is also a multiplication operator
\[ (N_{00} f)(e^{i\theta}) = N_{00}(e^{it}) f(e^{it}), \]
where \( N_{00}(e^{i\theta}) = M_{00}(e^{i\theta}) e^{-i\theta/2}, e^{i\theta/2} \) is a well-defined measurable function, and \( M_{00}(e^{i\theta}) \) is a bounded measurable \( \mathcal{L}(\mathcal{D}) \)-valued function satisfying
\[ M_{00}(e^{i\theta}) = M_{00}(e^{it})^*. \]

If \( \sigma(H) \subset L_{a, x} \), then from (8.5) the value of the function \( e^{i\theta/2} \) may be chosen so that \( e^{i\theta/2} \in L_{a, x} \) for \( e^{i\theta} \in \gamma \). In this case
\[ M_{00}(e^{i\theta}) \geq 0, \]
since \( \sigma(N_{00}(e^{i\theta})) \subset \sigma(N) \subset L_{a, x} \).

From (7.1), it is easy to see that
\[ X_1^* N_{10} U_0^* X_0 f = H_{10} T_0^* f = P_{\mathcal{H}_1} H T^* f = P_{\mathcal{H}_1} T^{-1} H f \]
\[ = T_1^{-1} P_{\mathcal{H}_1} H f = T_1^{-1} X_1^* N_{10} X_0 f \]
for \( f \in \mathcal{H} \), where \( P_{\mathcal{H}_1} \) is the projection from \( \mathcal{H} \) onto \( \mathcal{H}_1 \). Thus
\[ N_{10} U_0^* = X_1 T_1^{-1} X_1^* N_{10}. \quad (8.8) \]
Let $E(\cdot)$ and $F(\cdot)$ be the spectral measures of $U_0$ and $X_1 T_1 X_1^*$, respectively. From (8.8), we obtain

$$\| F(M) x \| \leq \| N_{10} \| \| E(M) x \|$$

(8.9)

for every $x \in \mathcal{H}_0$ and Borel set $M \subset T$. Note that

$$[H^*, H] = HD^2 \ast T^* - I H,$$

therefore $[H^*, H] \mathcal{H} = H \mathcal{D}_*,$. Hence $H_{10} \mathcal{H}_0$ is dense in $\mathcal{H}_1$, since $\mathcal{H}_0 \oplus \mathcal{H}_1$ is the minimal subspace which reduces $T$ and contains $H \mathcal{D}_*$ as a subspace. Therefore $N_{10} G_0$ is dense in $X_1 \mathcal{H}_1$. Thus from (8.9) it follows that $F(\cdot)$ is absolutely continuous with respect to $E(\cdot)$.

We choose $X_1$ such that the operator $U_1 = X_1 T_1^{-1} X_1^*$ is a multiplication operator

$$(U_1 f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$$

(8.10)

and $G_1 = \Delta_1 L^2(\mathcal{D}_1)$, where $\Delta_1(e^{i\theta})$ is a bounded function with support in $\gamma$ and $\Delta_1(e^{i\theta}) \geq 0$. From $U N = N U$, we have

$$N_{10} U_0 = U_1 N_{10}.$$  

(8.11)

From (8.4), (8.10), and (8.11), the operator $N_{10}$ acts multiplicatively,

$$(N_{10} f)(e^{i\theta}) = N_{10}(e^{i\theta}) f(e^{i\theta}), \quad f \in G_0,$$

where $N_{10}(e^{i\theta})$ is a bounded measurable operator valued function.

From $N U = N^*$, we obtain

$$N_{01} = U_0^* N_{10}.$$

Hence

$$(N_{01} f)(e^{i\theta}) = N_{01}(e^{i\theta}) f(e^{i\theta})$$

and

$$N_{01}(e^{i\theta}) = e^{-i\theta} N_{10}(e^{i\theta})^*.$$  

Thus there are bounded measurable operator valued functions $M_{10}(e^{i\theta})$ and $M_{01}(e^{i\theta})$ such that

$$N_{10}(e^{i\theta}) = e^{i\theta/2} M_{10}(e^{i\theta}), \quad N_{01}(e^{i\theta}) = e^{-i\theta/2} M_{01}(e^{i\theta})$$

and

$$M_{10}(e^{i\theta}) = M_{01}(e^{i\theta})^*.$$  

Continuing this process, the following theorem can be proved immediately.
Theorem 8.2. If $H$ is a hyponormal operator then there is a decomposition of $G$ in (8.3) such that $N = (N_{ij})$ is a $J$-matrix and $N_{ij}$ is a multiplication operator
\[(N_{ij} f)(e^{i\theta}) = N_{ij}(e^{i\theta}) f(e^{i\theta}), \quad f \in G_j,\]
where $N_{ij}(e^{i\theta/2}) = M_{ij}(e^{i\theta}) e^{-i\theta/2}$ is a bounded measurable $\mathcal{L}(\mathcal{D})$-valued function satisfying
\[M_{ij}(e^{i\theta}) = M_{ij}(e^{i\theta})^*.\]
If $\sigma(H) \subset L_{a,\gamma}$, then $e^{-i\theta/2} \in L_{a,\alpha}$ for $e^{i\theta} \in \gamma$, $M_{ij}(e^{i\theta}) \geq 0$, and the $J$-matrix $M = (M_{ij}(e^{i\theta}))$ is positive semi-definite for every $e^{i\theta}$ in $\text{supp} \Delta(\cdot)$.

9. Comparison of the Two Mosaics of a Hyponormal Operator

Let $H$ be the hyponormal operator satisfying the condition that $\sigma(H) \subset L_{a,\gamma}$. Let $N(e^{i\theta})$ be its symbol, then there is an $\mathcal{L}(\mathcal{D})$-valued bounded measurable function $M(e^{i\theta})$ satisfying
\[M(e^{i\theta}) \geq 0\]
such that $N(e^{i\theta}) = M(e^{i\theta}) e^{-i\theta/2}$ for $e^{i\theta} \in \gamma$.

We consider the function
\[M(e^{i\theta}, z) = \xi(e^{i\theta}, ze^{-i\theta/2}) = I - \Delta(e^{i\theta})(M(e^{i\theta})(M(e^{i\theta}) - z)^{-1})_{00} \Delta(e^{i\theta})\]
of $z$, which satisfies the following conditions: (1) it is analytic on the complex $z$-plane except for a bounded closed set in the real axis, (2) $\lim_{z \to \infty} M(e^{i\theta}, z) = I$, and (3) if $\text{Im} \, z \geq 0$ then
\[\text{Im} \, M(z^{ii}, z) \leq 0.\]

By methods similar to [3], or [8], there exists a measurable $\mathcal{L}(\mathcal{D})$-valued function $B(\rho e^{-i\theta/2})$ on $(\rho, e^{i\theta}) \in \delta \times \gamma$, where $\delta$ is a bounded closed set in $(0, \infty)$, satisfying
\[0 \leq B(\rho e^{-i\theta/2}) \leq I\]
such that
\[\xi(e^{i\theta}, ze^{i\theta/2}) - M(e^{i\theta}, z) = \exp \left[ - \frac{B(\rho e^{-i\theta/2})}{\rho - z} \right]. \quad (9.1)\]
Thus the function $B(\rho e^{-it/2})$ is a new mosaic of the hyponormal operator. From Theorem 7.2, if $[H^*, H] \in L_1$, then since $\sigma(H) \subseteq L_{a,\infty}$,

$$\text{Tr}(B(\zeta)) = -g(\zeta) \quad (9.2)$$

for $\zeta \in \sigma(H)$. (Thus $B(\cdot)$ is supported on the spectrum of $H$.)

Of course, mosaics could also be introduced in the same way as in [4, 9].

This mosaic possesses global unitary invariance in the following sense: If $H$ and $H'$ are two hyponormal operators in $\mathcal{H}$ and $\mathcal{H}'$, respectively, and there is a unitary operator $R$ from $\mathcal{H}$ onto $\mathcal{H}'$ such that

$$H' = RHR^{-1} \quad (9.3)$$

there is a unitary operator $S$ from $\mathcal{D}$ onto $\mathcal{D}'$ such that

$$B_{H'}(z) = SB_{H}(z) S^{-1} \quad \text{for a.e. } z \in \sigma(H). \quad (9.4)$$

In fact, let $T'H' = H'^*$, $D' = (I - T'^* T')^{1/2}$, $\mathcal{D}' = D' \mathcal{H}$, and

$$S = R|_{\mathcal{D}'}$$

then

$$A_{\mathcal{T}_r}(e^{it}) = SA_{\mathcal{T}_r}(e^{it}) S^{-1} \quad \text{for all } e^{it} \in \gamma. \quad (9.5)$$

Denote the matrix $N(e^{it})$ corresponding to $\mathcal{H}'$ by $N'(e^{it})$. Since $N_{00}(e^{it})$ and $N'_{00}(e^{it})$ are determined by the functional model we have

$$N'_{00}(e^{it}) = SN_{00}(e^{it}) S^{-1}. $$

Denote

$$\hat{N}_{01}(e^{it}) = (N_{01}(e^{it}) 0 \cdots 0),$$

$$\hat{N}_{10}(e^{it}) = \begin{pmatrix} N_{10}(e^{it}) \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

$$\hat{N}_{11}(e^{it}) = \begin{pmatrix} N_{11}(e^{it}) & N_{12}(e^{it}) & \cdots \\ N_{21}(e^{it}) & N_{22}(e^{it}) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$
and those corresponding to $H'$ by $\hat{N}_{01}(e''), \hat{N}_{10}(e''),$ and $\hat{N}_{11}(e''),$ respectively. Then there is a unitary operator $\zeta(e'')$ which may depend on $e''$ such that

$$\hat{N}_{01}(e'') = SN_{01}(e'') \zeta(e''), \quad \hat{N}_{10}(e'') = \zeta(e'') N_{10}(e'') S^{-1}$$

and

$$\hat{N}_{11}(e'') = \zeta(e'') \hat{N}_{11}(e'') \zeta(e'')^{-1}.$$

Thus

$$\hat{N}_{01}(e'')(\hat{N}_{11}(e'') - \mu)^{-1} \hat{N}_{10}(e'') = S\hat{N}_{01}(e'')(\hat{N}_{11}(e'') - \mu)^{-1} \hat{N}_{10}(e'') S^{-1}.$$ 

On the other hand,

$$(N(e'')(N(e'')(\mu)^{-1})_{oo} = I + \mu(N_{00}(e'') - \mu I - \hat{N}_{01}(e'')(\hat{N}_{11}(e'') - \mu I)^{-1} \hat{N}_{10}(e''))^{-1}.$$ 

Therefore

$$I - \Delta'(e'')(N'(e'')(N'(e'') - \mu I)^{-1})_{oo} \Delta'(e'') = S(I - \Delta(e'')(N(e'')(N(e'') - \mu I)^{-1})_{oo} \Delta(e'')) S^{-1},$$

which proves the global unitary invariance of $B(z)$.

Furthermore, $B(z)$ is also a complete unitary invariant. This means if there are two mosaics $B(z)$ and $B'(z)$ of the hyponormal operators $H$ and $H'$, respectively, and there is a unitary operator $S$ from $\mathcal{D}$ onto $\mathcal{D}'$ satisfying (9.4), then there is a unitary operator $\mathcal{R}$ from $\mathcal{H}$ onto $\mathcal{H}'$ satisfying (9.3).

In fact, from (9.1) and (9.4), it is easy to show (9.5). On the other hand, since $M(e'')$ is a $J$-matrix, $((M(e'') - \mu I)^{-1})_{oo}$ has a continued fraction expansion,

$$((M(e'') - \mu)^{-1})_{oo} = (M_{00}(e'') - \mu I + M_{01}(e'')(M_{11}(e''))

- \mu I + M_{12}(e'')(M_{22}(e'') \mu I + \cdots)^{-1}

\times M_{21}(e''))^{-1} M_{10}(e''))^{-1}.$$  

By (9.5) and (9.6), from the unitary equivalence of

$$SA(e'')(M(e'') - \mu I)^{-1})_{oo} \Delta(e'')^{-1} S = \Delta'(e'')((M'(e'') - \mu I)^{-1}) \Delta'(e'')$$

it is easy to show that there is a diagonal unitary matrix $W = (W_{ij})$ with $W_0 = S$ such that

$$WM(e'' W^{-1} = M'(e'').$$  

(9.7)
Since $A$ and hence $X$ are determined as the residue at infinity of the barrier, $H = XNX, H' = X'N'X'$, (9.5) and (9.7), it is easy to prove (9.3) for a certain $R$. In conclusion, we have proved the following.

**THEOREM 9.1.** The mosaic $B(z), z \in \sigma(H)$, of the hyponormal operator $H$ defined by the normal symbols as in (9.1) is a complete global unitary invariant for the operator $H$.

Furthermore, the new mosaic has the following important rotation covariance. Let $B_H(\cdot)$ denote the mosaic of $H$.

**THEOREM 9.2.** Take $e^{it} \in T$. If $H$ is hyponormal then
\[ B_{e^{it}H}(z) = B_H(e^{-it}z), \quad z \in \sigma(e^{it}H). \] (9.8)

**Proof.** Let $T_z = (e^{it}H)^*(e^{it}H)^{-1} = e^{-2it}T$. Then from (2.4) and (2.5),
\[ \Theta_{T_z}(\lambda) = \Theta_T(\lambda e^{2it}) e^{-2it} \]
and
\[ \Delta_{T_z}(e^{i\theta}) = \Delta_T(e^{i(\theta + 2\pi)}). \] (9.9)

Let $N_z, M_z(e^{i\theta}, z)$ be the normal operator and the function (9.1) corresponding to $e^{it}H$. Then from (9.9) and Theorem 8.2
\[ N_z(e^{i\theta}) = e^{i\theta}N(e^{i(\theta + 2\pi)}). \]

Thus the function
\[ M_z(e^{i\theta}, z) = M_z(e^{i(\theta + 2\pi)}, z), \]

which proves (9.8).

Next, we will show that the original mosaic $\tilde{B}(z)$ of a hyponormal operator introduced by Carey and Pincus (cf. [4] or [23]) is not unitarily equivalent to this new one, since the original mosaic $\tilde{B}(z)$ does not have the rotation covariance.

**LEMMA 9.3.** Let $H = u + iv$ be a hyponormal operator and $m$ a non-negative integer. Then
\[ \text{tr}(v[i[u, v] u^m]) = \frac{1}{2\pi} \int g(x + iy) x^m y \, dx \, dy \]
\[ + \frac{1}{8\pi^2 i} \int \int \int \int g(x_1 + iy_1, x_2 + iy_2) \]
\[ \times (x_1^{m-1} + x_1^{m-2}x_2 + \cdots + x_2^{m-1}) \, dx_1 \, dy_1 \, dx_2 \, dy_2, \] (9.10)
where \( g(z) \) is the principal function of \( H \) and

\[
g(z, z') = \text{tr}(\tilde{B}(z) \tilde{B}(z')),
\]

where \( \tilde{B}(z) \) is the original mosaic introduced by Carey and Pincus.

**Proof.** Without loss of generality, we may assume that the operator \( H \) is an operator in the singular integral model (cf. [23]), i.e.,

\[
(uf)(x) = xf(x)
\]

and

\[
(vf)(x) = \beta(x)f(x) + \frac{\alpha(x)}{2\pi i} \int \frac{\alpha(s)f(s) ds}{x - (s + i0)}, \tag{9.11}
\]

where \( \alpha(\cdot) \) and \( \beta(\cdot) \) are certain bounded \( \mathcal{L}(\mathcal{D}) \)-valued functions and \( \mathcal{D} \) is an auxiliary Hilbert space. By means of Plemelj's formula, it is easy to calculate that

\[
\text{tr}(iv[u, v]u^m) = \frac{1}{2\pi} \int \int \text{tr} \left( \alpha(x) \beta(x) \alpha(x) + \frac{1}{2} \alpha(x)^4 \right) x^m dx
d + \frac{1}{4\pi^2 i} \int \int \text{tr} \left( \alpha(x)^2 \beta(x)^2 \right) \frac{x^m}{x - s} dx ds, \tag{9.12}
\]

and since

\[
\exp \left( \int \frac{\tilde{B}(\lambda + it) dt}{i - l} \right) = I + \alpha(\lambda)(\beta(\lambda) - II)^{-1} \alpha(\dot{\lambda})
\]

(cf. p. 107 of [23], or [10]), we have

\[
\int \tilde{B}(x + iy) dy = \alpha(x) \tag{9.13}
\]

and

\[
\int \tilde{B}(x + iy) y dy = \alpha(x) \beta(x) \alpha(x) + \frac{1}{2} \alpha(x)^4. \tag{9.14}
\]

From (9.11), (9.12), (9.13), and (9.14), it is easy to check (9.10).
LEMMA 9.4. Let \( H = u + iv \) be a hyponormal operator then

\[
\text{tr}(v^2[u, v]u) = \frac{1}{2\pi} \int g(x + iy) xy^2 \, dx \, dy
\]

\[
+ \frac{1}{8\pi^2} \int \int \int \int g(x_1 + iy_1, x_2 + iy_2)(y_1 + y_2) \, dx_1 \, dy_1 \, dx_2 \, dy_2
\]

\[
- \frac{1}{(2\pi)^3} \lim_{\varepsilon \to 0^+} \int \int \int \frac{\text{tr}(\alpha(x)^2 \alpha(s)^2 \alpha(t)^2)}{|x - s|} \, dx \, ds \, dt,
\]

(9.15)

where \( \alpha(\cdot) \) is the function in the singular integral model of \( H \) (see (9.11)).

Proof. It is obvious that

\[
\text{tr}[u^2, v^2] = 6 \text{tr}(v^2[u, v]u) - 6 \text{tr}(v[u, v]^2).
\]

Thus

\[
\text{tr}(iv^2[u, v]u) = \frac{1}{2\pi} \int g(x + iy) xy^2 \, dx \, dy + i \text{tr}(v[u, v]^2).
\]

(9.16)

By means of (9.11), it is easy to calculate that

\[
-\text{tr}(v[u, v]^2) = \frac{1}{(2\pi)^3} \int \int \text{tr}((\alpha(x) \beta(x) \alpha(x) + \frac{1}{2}(\alpha(x))^4 \alpha(x)^2) \, dx \, dt
\]

\[
+ \frac{1}{8\pi^3} \lim_{\varepsilon \to 0^+} \int \int \int \frac{\text{tr}(\alpha(x)^2 \alpha(s)^2 \alpha(t)^2)}{|x - s|} \, dx \, ds \, dt,
\]

which implies (9.15) by (9.16).

THEOREM 9.5. There exists a hyponormal operator \( H \) such that the original mosaic \( \mathcal{B}_{ihH}(z) \) corresponding to \( iH \) is not unitarily equivalent to \( \mathcal{B}_{ihH}(1/i)z \) by a constant unitary map.

Proof. Let \( u_1 + iv_1 \) be the Cartesian decomposition of \( iH = -v + iu \), then \( u_1 = - \) and \( v_1 = u \). Denote

\[
g_1(z) = \text{tr}(\mathcal{B}_{ihH}(z))
\]

\[
g_1(z, z') = \text{tr}(\mathcal{B}_{ihH}(z) \mathcal{B}_{ihH}(z')).
\]

If \( \mathcal{B}_{ihH}(x + iy) \) is unitarily equivalent via a constant unitary map to

\[
\mathcal{B}_H \left( \frac{1}{i} (x + iy) \right) = \mathcal{B}_H (-y + ix),
\]
then
\[ g_1(x + iy) = g(-y + ix) \]
and
\[ g_1(x_1 + iy, x_2 + iy) = g(-y_1 + ix_1, -y_2 + ix_2). \]

Applying Lemma 9.3 to the hyponormal operator \( u_1 + iv_1 \), we have
\[
\text{tr}(ui[u, v] v^2) = \text{tr}(v_1 i[u, v] u_1^2)
\]
\[
= \frac{1}{2\pi} \int \int g_1(x + iy) x^2 y \, dx \, dy + \frac{1}{8\pi^2 i} \int \int \int \int g_1(x_1 + iy_1, x_2 + iy_2)
\]
\[
\times (x_1 + x_2) \, dx_1 \, dx_2 \, dy_1 \, dy_2
\]
\[
= \frac{1}{2\pi} \int \int g(x + iy) x^2 y \, dx \, dy - \frac{1}{8\pi^2 i} \int \int \int \int g(x_1 + iy_1, x_2 + iy_2)
\]
\[
\times (y_1 + y_2) \, dx_1 \, dx_2 \, dy_1 \, dy_2. \tag{9.17}
\]

On the other hand, we have
\[
\text{tr}(ui[u, v] v^2) = \overline{\text{tr}(v_2 i[u, v] u)}. \tag{9.18}
\]

From (9.15), (9.17), and (9.18), it follows that
\[
\lim_{\epsilon \to 0^-} \int \int \tr(\alpha(x)^2 \alpha(s)^2 \alpha(t)^2) \left[1 - \frac{1}{x - s}\right] \, dx \, ds \, dt = 0.
\]

This is equivalent to
\[
\lim_{\epsilon \to 0^+} \int \int \tr \left(\left[ 1 - \frac{1}{x - s} \right] \alpha(t)^2 \right) \, dx \, ds \, dt = 0. \tag{9.19}
\]

But it is evident that there exists an \( \mathcal{L}(\mathcal{D}) \)-valued bounded measurable function satisfying \( \alpha(t) \geq 0 \) such that (9.19) does not hold. Thus for this \( \alpha(\cdot), \mathcal{B}_{iH}(z) \) is not unitarily equivalent to \( \mathcal{B}_{iH}((1/i)z) \).

The \( n \)-point function
\[
\tr(B(z_1) B(z_2) \cdots B(z_n))
\]
will be investigated further elsewhere. We record one more result:
**Theorem 9.6.** If \( H \) is hyponormal, \([H^*, H]\) is finite rank, \( 0 \in \rho(H) \), and \( T = H^*H^{-1} \). Then \( T^*h \to 0 \) for all \( h \in \mathcal{H} \) (i.e., \( T \in \mathcal{C}_1 \)). Furthermore,

\[
\exp \left\{ - \int B(\rho e^{-it/2}) \rho^{-1} \, d\rho \right\} = \Theta_T(e^{it})^* \Theta_T(e^{it})
\]

and \( \Theta_T(\lambda) \) is the essentially unique factorization of the positive operator in the left-hand side of the above equation.

**Proof.** From (3.19) and (7.3), it is obvious that

\[
\det(\Theta(\lambda, \mu) \Theta(0, \mu)^{-1}) = \exp \left\{ \frac{1}{2\pi} \int \frac{\xi + \lambda \xi}{\xi - \lambda \xi} g(\xi) \, d\kappa(\xi) \right\}
\]

\[
= \exp \left\{ \frac{1}{2\pi} \int e^{-2i\theta} + \lambda \left( e^{2i\theta} + \frac{g(\rho e^{i\theta})}{\rho - \mu e^{i\theta}} \right) \, d\theta \right\}.
\]

This is an outer function of \( \lambda \). By Theorem 6.2 of Chapter V of Sz.-Nagy and Foias [19], the function \( \Theta_T(e^{it}) \) is outer. The unicity of the factorization up to an isometric factor then follows easily.

This theorem was obtained by the authors in an unpublished paper in 1983 (cf. [12]). The first part of this theorem was also obtained by Sz.-Nagy and Foias in the same year (cf. [20]).

**10. Hypormal Operators \( \mathcal{H} \) with Rank One Self-Commutator and Contractions \( T \) so That \( H^* = TH \)**

In this section we mainly consider the case where \([H^*, H]\) is rank one.

**Theorem 10.1.** Let \( H \) be a completely hyponormal operator satisfying the conditions that \( H^{-1} \in \mathcal{L}(\mathcal{H}) \) and the rank of the self-commutator of a hyponormal operator \( H \) is one. Then the contraction \( T = H^*H^{-1} \) is completely non-unitary, iff the negative of the principal function \( g(z) \) is the characteristic function of \( \sigma(H) \) and for almost all \( e^{i\theta} \) the intersection of the ray \( \{pe^{i\theta}, p \geq 0\} \) with \( \sigma(H) \) is either a segment or is empty.

**Proof.** Since the dimensions of \( \mathcal{D} \) and \( \mathcal{D}_* \) are 1, we may identify both \( \mathcal{D} \) and \( \mathcal{D}_* \) with \( \mathbb{C} \). The functions \( \Theta(e^{it}), \zeta(e^{it}, \mu), \) and \( N_{ii}(e^{it}), \) etc., are complex valued functions.

If \( T \) is completely non-unitary then

\[
\zeta(e^{it}, \mu) = (N_{00}(e^{it}) - \mu)^{-1}(N_{00}(e^{it}) |\Theta(e^{it})|^2 - \mu)
\]
has exactly one zero and one pole for $|\Theta(e^{it})| \neq 1$. From Theorem 7.2, it is easy to see that $-g(\rho e^{-it/2})$ as a function of $\rho$ is the characteristic function of the interval

\[ [N_{00}(e^{it}) |\Theta(e^{it})|^2, N_{00}(e^{it})] \]

Thus $g(\cdot)$ satisfies those conditions, then $-g(\rho e^{-it/2})$ as a function of $\rho$ is the characteristic function of an interval $[a(e^{it}), b(e^{it})]$. From Theorem 7.2,

\[ \zeta(e^{it}, \mu) = (b(e^{it}) e^{-it/2} - \mu)^{-1} (a(e^{it}) e^{it/2} - \mu). \]

Thus $A^2(e^{it}) = (b(e^{it}) - a(e^{it})) (a(e^{it}))^{-1}$ and

\[ ((M(e^{it}) - zI)^{-1})_{00} = (b(e^{it}) - z)^{-1}. \]

Since $M(e^{it})$ is a $J$-matrix, $((M(e^{it}) - z)^{-1})_{00}$ is a continued fraction of the form (9.6). This continued fraction can equal $(b(e^{it}) - z)^{-1}$ only if $M_{00}(e^{it}) = b(e^{it})$ and $M_{01} = M_{10} = 0$ so $\bigoplus_{i=1}^{\infty} \mathcal{H}$ is a reducing subspace of $H$ on which $T$ is unitary. But since $H$ is completely non-normal we conclude that $\bigoplus_{i=1}^{\infty} \mathcal{H}_{i} = \{0\}$, i.e., $T$ is completely non-unitary. This implies that $\mathcal{H}_{i} = \{0\}$, $i \geq 1$, i.e., $T$ is completely non-unitary.

**Theorem 10.2.** (i) Let $H$ be completely hyponormal, and if $T$ is a contraction satisfying $TH = H^*$, then (i) the minimal unitary dilation $\mathcal{U}$ of $T$ has a purely absolutely continuous spectrum.

(ii) If the rank of $[H^*, H]$ is one, then there exists a contraction $T$ satisfying $TH = H^*$ such that the spectral multiplicity $m_\mathcal{U}(e^{it})$ of $\mathcal{U}$ is given in the following way: if $-g(\rho e^{-it/2})$ is $L^2$ equivalent to the characteristic function of $n$ disjoint intervals, then $m_\mathcal{U}(e^{it}) = n$; otherwise $m_\mathcal{U}(e^{it}) = \infty$.

This theorem was obtained by the authors in 1983. But in the same year the first part of this theorem also appeared in a published paper of Sz.-Nagy and Foiaš (cf. [20]).

**Proof.** The proof involves a discussion of several cases.

First, we consider the case where $0 \notin \sigma(H)$. In this case, since $M_{ij} = c_{ij}$ is not identically zero, $G_j$ is an infinite dimensional Hilbert space of certain square integrable functions, and $\mathcal{U}$ is unitarily equivalent to multiplication by $e_{ij}$, thus $\mathcal{U}$ has purely absolutely continuous spectrum.

The spectral multiplicity $m_\mathcal{U}(e^{it})$ is a sum of two parts, the spectral multiplicity of the minimal unitary dilation $\mathcal{U}_e$ of $T_e$, and the spectral multiplicity of $T_\nu$. From Theorem 6.1 of Chapter VI of Sz.-Nagy and Foiaš [19], we have $m_\mathcal{U}(e^{it}) = 1 + n_e(e^{it})$, where $n_e(e^{it}) = 1$ if $A(e^{it}) > 0$ or $n_e(e^{it}) = 0$ if $A(e^{it}) = 0$. 
Now if \( A(e^{it}) > 0 \), then \( m_T(e^{it}) + n_\mu(e^{it}) \) is equal to the number of poles of the function \( \xi(e^{it}, \mu e^{-i\theta/2}) \) when this barrier is meromorphic in \( \mu \); otherwise it is infinite, since this number is the rank of the \( J \)-matrix \( M(e^{it}) \).

But \( \xi(e^{it}, \mu e^{-i\theta/2}) \) is meromorphic with \( n \) poles if and only if \(-g(\rho e^{it/2})\) is the characteristic function of \( n \)-disjoint \( (\rho) \) intervals.

Now we treat the cases where \( 0 \in \sigma (H) \).

If \( 0 \in \sigma (H^*) \), then \( T = H^* H^{-1} \) is densely defined on \( \mathcal{H} \). Since it is bounded, it extends to a bounded operator on \( \mathcal{H} \). We denote this extension by \( T \) still. In this case the defect space \( D_1 \) is 1 dimensional, since \( H^* D^2 H = [H^*, H] \) is rank one. If \( e_j, j = 1, 2 \), are eigenvectors of \( D_1 \), then \( T^* e_j, j = 1, 2 \), are eigenvectors of \( D \). Hence \( T^* e_1 \) and \( T^* e_2 \) are parallel, and there are numbers \( c_j \in C, j = 1, 2 \), \( |c_1|^2 + |c_2|^2 > 0 \) such that \( T^*(c_1 e_1 + c_2 e_2) = 0 \) Thus

\[
H(c_1 e_1 + c_2 e_2) = H^* T^*(c_1 e_1 + c_2 e_2) = 0.
\]

This implies \( c_1 e_1 + c_2 e_2 = 0 \), since \( 0 \notin \sigma (H) \). We conclude that both \( D \) and \( D_1 \) are 1 dimensional.

If \( 0 \in \sigma (H^*) \) and range \( [H^*, H] \subset \text{range } H \), then there is a vector \( \eta \) such that \( H \in \text{range } [H^*, H] \) and \( \| N \eta \| = 1 \). Since \( [H^*, H] \) is 1 dimensional, \( \ker (H^*) \) is also 1 dimensional. Choose a vector \( a \in \ker (H^*) \) such that

\[
\| a \|^2 = \| \eta \|^2 + 1 / \| [H H^*, H] \|.
\]

Define a linear bounded operator \( T \) on \( \mathcal{H} \) such that

\[
T(H x + \lambda a) = H^* x + \lambda \eta, \quad x \in \mathcal{H}, \quad \lambda \in C.
\]

Then

\[
\| T(H x + \lambda a) \|^2 = \| H^* x \|^2 + 2 \Re \lambda (x, H \eta) + |\lambda|^2 \| \eta \|^2
\]

\[
= \| H x \|^2 - |(x, H \eta)|^2 \| [H^*, H] \| + 2 \Re \lambda (x, H \eta) + |\lambda|^2 \| \eta \|^2
\]

\[
\leq \| H x \|^2 + |\lambda|^2 \| a \|^2 = \| H x + \lambda a \|^2.
\]

Thus \( T \) is a contraction satisfying \( TH = H^* \). Since \( H^* D^2 H = [H^*, H] \), there is a vector \( \eta_1 \perp \ker (H^*) \) such that \( H^* \eta_1 = H \eta \). A simple calculation shows that

\[
(D^2 H x, a) = (H x, a) - (T^* H^* x, a) = -(x, H \eta)
\]

and \( [H^*, H] x = \| [H^*, H] \| (x, H \eta) H \eta \). Hence

\[
D^2 H x = \left( \| [H^*, H] \| \eta_1 - \frac{a}{\| a \|^2} \right) (x, H \eta).
\]
On the other hand, $H^*T^*\eta = H\eta$. Hence $T^*\eta = \eta_1 + (T^*\eta, a)(a/\|a\|^2)$. Thus

$$D^2a = a - T^*\eta = \left(1 - \frac{\|\eta\|^2}{\|a\|^2}\right) a - \eta_1$$

$$= (\|a\|^2 - \|\eta\|^2)(a \frac{a}{\|a\|^2} - \|H^*, \eta_1\| \eta_1).$$

This means $\mathcal{D}_1 = \{\lambda(a/\|a\|^2) \|[H^*, \eta_1]\| \eta_1): \lambda \in C\}$. Therefore, $\mathcal{D}$ is 1-dimensional and then $\mathcal{D}_*$ is also 1-dimensional.

Now, we consider the case of $0 \in \sigma_p(H^*)$ and

$$\text{range}[H^*, H] \notin \text{range} H. \quad (10.1)$$

In the analytic model [14], range $[H^*, H]$ is the 1-dimensional subspace of constant functions, and range $[H^*H] \subset \text{range} H$ iff

$$\int_{\sigma(H)} \frac{g(\zeta)}{|\zeta|^2} dm(\zeta) < +\infty.$$ 

Thus (10.1) means

$$\int_{\sigma(H)} \frac{g(\zeta)}{|\zeta|^2} dm(\zeta) = +\infty. \quad (10.2)$$

The condition (10.2) and $0 \in \sigma_p(H^*)$ imply that the operator $T$ satisfying $TH = H^*$ must also satisfy $T(\ker H^*) = \{0\}$. In fact $T^*$ is an isometry and $\dim \mathcal{D} = 2$.

The fact that $T^*$ is an isometry under these conditions was first observed several years ago by R. Carey. The result follows as a consequence of the characterization of the eigenvalues of $H^*$ given in [5], and was not published. But see [7], where this was also observed.

Since $\dim \mathcal{D}_* = 0$ in this case, we have $\Theta(\cdot) = 0$ and $\Delta(\cdot) = I$. Thus, the functional model is

$$\tilde{H}_c = L^2(\mathcal{D}) \ominus H^2(\mathcal{D}).$$

Let $P_-$ be the projection from $L^2(\mathcal{D})$ onto $L^2(\mathcal{D}) \ominus H^2(\mathcal{D})$. Then $\tilde{H}_c = V_c H V_c^{-1}$ is

$$\tilde{H}_c = P_- M_0 e^{-iw^2} P_-$

where $M_0(\cdot)$ is a $2 \times 2$ matrix valued self-adjoint function. Since

$$\tilde{H}_c^* \tilde{H}_c = \tilde{H}_c \tilde{H}_c^* = P_- M_0 e^{-iw^2} P_0 M_0 e^{iw^2} P.$$
where \( P_0 = e^{it} P_- e^{-it} - P_- \) is the projection from \( L^2(\mathcal{D}) \) to the constant function, \( P_0 M_0 e^{it/2} P_- \) is rank one. Thus there are constants \( a_n, b_n, v_1, v_2 \) such that

\[
P_0 M_0 e^{it/2} P_- e^{-int} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (a_n x_1 + b_n x_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.
\]

It is easy to show that the matrix function \( M_0(\cdot) = (M_{0ij}(\cdot)) \) satisfies

\[
M_{011}(e^{it}) = 2R \left( \sum_{n=1}^{\infty} a_n v_1 e^{nit} e^{-it/2} \right)
\]

\[
M_{021}(e^{it}) = \sum_{n=1}^{\infty} a_n v_2 e^{nit} e^{-it/2} + \sum_{n=1}^{\infty} b_n \bar{v}_1 e^{-nit} e^{it/2},
\]

\[
M_{022}(e^{it}) = 2R \left( \sum_{n=1}^{\infty} b_n v_n e^{nit} e^{-it/2} \right)
\]

and \( M_{021}(e^{it}) = \overline{M_{012}(e^{it})} \).

Since the eigenvector of \( H^* \) is in \( \mathcal{D} \). We can choose the coordinates so that

\[
H^* e^{-it} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.
\]

This implies

\[
\sum_{n=1}^{\infty} b_n \bar{v}_2 e^{-int} = 0
\]

and \( \sum_{n=1}^{\infty} a_n v_2 e^{-int} = 0 \). Hence \( H_{071}(e^{it}) = 0 \). Since

\[
\xi(e^{it}, ze^{-it/2}) = z^2 (z^2 - M_{011}(e^{it}) z - |M_{012}(e^{it})|^2)^{-1},
\]

we see that \( \det \xi(e^{it}, \mu) \) determines \( M_{011}(e^{it}) \) and \( |M_{012}(e^{it})|^2 \). The function \( M_{012}(e^{it}) = \eta(e^{it}) e^{-it/2} \), where \( \eta(\cdot) \in H^2 \). The outer factor of \( \eta \) is determined by \( |M_{012}(e^{it})|^2 \). If \( \phi(\cdot) \) is an inner factor of \( \eta(\cdot) \), then \( P_-(\Phi^{-1}) = \Phi^{-1} \) and

\[
H^* \begin{pmatrix} 0 \\ \Phi^{-1} \end{pmatrix} = 0,
\]

thus \( \Phi^{-1} = \text{const} e^{-it} \). Therefore, the function \( M_{012}(\cdot) \) is completely determined by \( \det \xi(e^{it}, e^{-it/2}) \).

Let \( \eta_0 \) be the eigenvector of \( [H^*, H] \) and \( \eta_1 \) be the projection of \( \eta_0 \) to \( \mathcal{H}_1 \). If \( \eta_1 \neq 0 \) then \( \mathcal{H}_1 \) is generated by \( \{ T^u \eta_1 \} \). Therefore, we can choose \( M_{10} \) as a \( 1 \times 2 \) matrix function, and all the other functions \( M_{11}, M_{12}, M_{22}, \ldots \), etc., are complex functions.
The conclusions about multiplicity follow again by observing that the number of poles of \( \det(\zeta(e^i\theta, z e^{-i\theta/2})) \) is given as before in terms of the principal function.

**Remark 10.3.** If \( 0 \in \sigma_p(H^*) \), range \([H^*, H] \neq \text{range} \ H \), and \([H^*, H] \) is rank one, then for every \( a \in \ker(H^*) \) satisfying

\[
\|a\|^2 \geq \|\eta\|^2 + (\|[H^*, H]\|)^{-1},
\]

we can define a contraction \( T_a \) such that \( T_a H = H^* \) and

\[
T_a a = \eta.
\]

And every contraction \( T \) satisfying \( TH = H^* \) must be one of such \( T_a \). For every \( a \in \ker(H^*) \) satisfying

\[
\|a\|^2 > \|\eta\|^2 + (\|[H^*, H]\|)^{-1}
\]

the defect space \( D_{T_a} \) is 2 dimensional. In fact \( D_{T_a} \) is spanned by two linearly independent vectors \((1 - \|\eta\|^2) a - \eta_1 \) and \( \|[H^*, H]\| \eta_1 - a \). In this case \( D_\ast \) is not \( \{0\} \). The barrier is a \( 2 \times 2 \) matrix function which depends on \( a \) and cannot be completely determined by the principal function \( g(\cdot) \) of \( H \).

**Remark 10.4.** For the case where \( \dim R([H^*, H]) \) is one, if \( 0 \notin \sigma(H) \), then we have seen that \( \Theta(\cdot) \) is scalar valued and outer. Hence, results of S. O. Sickler [17] give us a complete description of the invariant subspaces of \( H^* H^{-1} \) in this case.

**Remark 10.5.** It follows by Theorem 6.3 of Chapter VII in Sz.-Nagy and Foias [19] that (when \( 0 \notin \sigma(H) \)) the contraction \( T = H^* H^{-1} \) is in class \( C_{11} \) and is quasisimilar to \( \mathcal{U} \).

The following theorem shows us that \( T \) is often similar to \( \mathcal{U} \).

**Theorem 10.6.** If \([H^*, H]\) is in the trace class, \( T \) is invertible and there is a constant \( c \) such that

\[
\frac{1}{2\pi} \int \rho^{-1} |g(\rho e^{i\theta})| \, d\rho \leq c \tag{10.3}
\]

for almost all \( \theta \), where \( g(\cdot) \) is the principal function of the hyponormal operator. Then \( T = H^* H^{-1} \) is similar to a unitary operator.

**Proof.** By a theorem of Sz.-Nagy and Foias (cf. [19] or [23]) we only need to prove that \( \|\Theta_T(\lambda)^{-1}\| \) is uniformly bounded in \(|\lambda| < 1\).
From (3.19) and (5.3), it follows that

$$\left| \det(- T^{-1} \Theta_T(\lambda)) \right| = c_1 \exp \left\{ \frac{1}{2\pi i} \int \frac{g(\zeta)}{\zeta} \frac{\lambda \zeta}{\lambda \zeta - \zeta} \, d\mu(\zeta) \right\},$$

where

$$c_1 = \exp \left\{ -\frac{1}{2\pi} \int g(\zeta) |\zeta|^{-2} \, d\mu(\zeta) \right\}.$$

It is obvious that

$$\left| \det(- T^{-1} \Theta_T(\lambda)) \right| \geq c_1 e^{-c}, \quad |\lambda| < 1.$$

Thus $\Theta_T(\lambda)$ is invertible and $\|\Theta_T(\lambda)^{-1}\|$ is uniformly bounded.

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