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ON SOME CLASSES OF HYPONORMAL TUPLES OF COMMUTING OPERATORS

Daoxing Xia*

Dedicated to the memory of Professor Ernst Hellinger

Some theorems on hyponormal or cohyponormal n-tuples of commuting operators are given. Using the analytic model, a trace formula for the subnormal n-tuple of operators is established.

1. Let \( H \) be a Hilbert space, and \( \mathcal{L}(H) \) be the algebra of all linear bounded operators on \( H \). An operator \( H \in \mathcal{L}(H) \) is said to be hyponormal if \( [H^*, H] \geq 0 \), where \( [A, B] \) is the commutator of \( A \) and \( B \). An operator \( C \in \mathcal{L}(H) \) is said to be cohyponormal if \( [C^*, C] \leq 0 \). It is obvious that \( C \) is cohyponormal iff \( C^* \) is hyponormal operator. Recently some authors [1], [14] tried to generalize the concept of hyponormal operators to the n-tuple case. In [1], Athavale introduced a sort of hyponormality for n-tuple of operators. Although, the author still does not know whether this concept is the most natural generalization of the single operator case or not, in the present paper we adopt this hyponormality and study some classes of hyponormal n-tuple of commuting operators.

A tuple \( H = (H_1, \ldots, H_n) \), \( H_j \in \mathcal{L}(H) \) is said to be hyponormal [1], if

\[
\sum_{i,j} ([H_i^*, H_j] x_i, x_j) \geq 0,
\]

for all \( x_j \in H \). The tuple \( H \) is said to be commuting, if \( [H_i, H_j] = 0 \) for \( i, j = 1, 2, \ldots, n \). Similar to the single operator case, a tuple \( A = (A_1, \ldots, A_n) \), \( A_j \in \mathcal{L}(H) \) is said to be cohyponormal if

\[
\sum_{i,j} ([A_i^*, A_j] x_i, x_j) \leq 0
\]

for all \( x_j \in H \). An n-tuple \( A = (A_1, \ldots, A_n) \) is said to be t-hyponormal if \( A^* = (A_1^*, \ldots, A_n^*) \) is cohyponormal. For \( n = 1 \), t-hyponormal is equivalent to hyponormal. The question is whether t-hyponormal is equivalent to hyponormal in the case of \( n > 1 \). It may not be well-known to some mathematician that for the n-tuple of commuting operators t-hyponormal is not equivalent to hyponormal if \( n \geq 2 \). The author gives an elementary example to show that fact in \S 2 of the

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present paper. We also prove that suppose \( \mathcal{H} = (H_1, \ldots, H_n) \) is a commuting irreducible \( n \)-tuple of operators and satisfies the condition that (i) there is no subspace \( \mathcal{H}_0 \) reducing \( H_1 \) such that \( H_1|_{\mathcal{H}_0} \) is a linear combination of a unilateral shift with the identity and (ii)

\[
\text{rank } [H_1^*, H_1] = 1,
\]

then \( \mathcal{H} \) is hyponormal iff \( \mathcal{H} \) is \( t \)-hyponormal (Theorem 4 in §3).

The second problem arises from R. Curto’s work [6], [7]. He proves that if \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \) is not in the Taylor spectrum \( \text{sp}(A) \) of a commuting \( n \)-tuple \( A = (A_1, \cdots, A_n) \), then

\[
\sum_{j=1}^{n} (A_j - \lambda_j I)(A_j^* - \overline{\lambda}_j I) \quad \text{and} \quad \sum_{j=1}^{n} (A_j^* - \overline{\lambda}_j I)(A_j - \lambda_j I)
\]

are invertible. He also shows that this condition is not sufficient for \( \lambda \not\in \text{sp}(A) \). A natural problem is to find some classes of \( n \)-tuples of commuting operators in which that condition is also sufficient. For the hyponormal or \( t \)-hyponormal \( n \)-tuple of operators the invertibility of \( \sum_{j=1}^{n} (A_j - \lambda_j I)(A_j^* - \overline{\lambda}_j I) \) implies the invertibility of \( \sum_{j=1}^{n} (A_j^* - \overline{\lambda}_j I)(A_j - \lambda_j I) \). In §4, we prove that if \( A = (A_1, \cdots, A_n) \) is a \( t \)-hyponormal \( n \)-tuple of commuting operators then the condition that

\[
\sum_{j=1}^{n} (A_j - \lambda_j I)(A_j^* - \overline{\lambda}_j I)
\]

is invertible is a necessary and sufficient condition for \( \lambda = (\lambda_1, \cdots, \lambda_n) \not\in \text{sp}(A) \) (see Theorem 5). It is not known that if this condition is also sufficient for hyponormal commuting \( n \)-tuple of operators for \( n > 1 \).

An important class of hyponormal \( n \)-tuples of commuting operators is the class of all subnormal \( n \)-tuples of operators (cf. [1], [5], [9], [10]). In [17], the author generalized Morrel’s Theorem (cf. [11]) to the subnormal \( n \)-tuple case. Using Morrel’s theorem, Theorem 9 in [17] is equivalent to the following

**THEOREM.** Let \( (S_1, \cdots, S_n) \) be an irreducible subnormal \( n \)-tuple of operators on a Hilbert space \( \mathcal{H} \). If \( \text{rank } [S_1^*, S_1] = 1 \), and \( [S_j^*, S_j] \mathcal{H} \subseteq [S_1^*, S_1] \mathcal{H} \) then there are complex numbers \( \alpha_j, \beta_j \neq 0, j = 2, \cdots, n \) such that

\[
S_j = \alpha_j S_1 + \beta_j I, \quad j = 2, \cdots, n.
\]

In §2 and §3 we prove the following. Let \( \mathcal{H} = (H_1, \cdots, H_n) \) be an irreducible commuting \( n \)-tuple of operators satisfying \( \text{rank } [H_1^*, H_1] = 1 \). If \( \mathcal{H} \) satisfies that either (i) \( \mathcal{H} \) is hyponormal and \( \text{rank } [H_j^*, H_j] \leq 1 \) for \( j = 2, \cdots, n \), and there
is no subspace $\mathcal{H}_0$ reducing $H_1$ such that $H_1|_{\mathcal{H}_0}$ is a linear combination of a unilateral shift with multiplicity one and the identity, (ii) $H$ is hyponormal and $[H_j^*, H_j] \mathcal{H} \subset [H_1^*, H_1] \mathcal{H}$, or (iii) $H$ is cohyponormal or $t$-hyponormal, then there are $\alpha_j, \beta_j \in \mathbb{C}$ such that

$$H_j = \alpha_j H_1 + \beta_j I, \quad j = 2, \cdots, n.$$  

This is a generalization of above theorem. By the way, the analytic model of a hyponormal operator with rank one self-commutator has been given in [12].

For the subnormal $n$-tuple of operators, in [17] the author established an analytic model. Using this analytic model, in §5 of this paper, the author establishes a trace formula for the subnormal $n$-tuple of operators which has some connection with Carey and Pincus theory of local index and principal current (cf. [2], [3], [4]).

2. The following example shows that for the $n$-tuple of commuting operators, the $t$-hyponormality is not equivalent to the hyponormality if $n \geq 2$.

Example 1. Let $\mathcal{H} = H^2(\mathbb{T})$ be the Hardy space. Define operators

$$(H_1 f)(\zeta) = \zeta f(\zeta)$$

and

$$(H_2 f)(\zeta) = M(\zeta) f(\zeta)$$

for $f \in \mathcal{H}$, where $M \in H^\infty(\mathbb{T})$. Then $H_2^*$ is the Toeplitz operator $f \mapsto P(M^* f)$, where $P$ is the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$. It is easy to calculate that

$$([H_2^*, H_2] f, f) = (M f, M f) - (P(M f), M f) = \|P_-(M f)\|^2,$$

for $f \in \mathcal{H}$, where $P_- = I - P$. On the other hand,

$$([H_1, H_1] g, g) = |g(0)|^2 = |P_-(\zeta g)|^2$$

and

$$([H_2^*, H_1] f, g) = (M f, (\zeta) g) - (P(M f), (\zeta) g) = (P_-(M f), P_-(\zeta g)).$$

Therefore

$$|([H_2^*, H_1] f, g)|^2 \leq ([H_1^*, H_1] g, g) ([H_2^*, H_2] f, f).$$

Hence $(H_1, H_2)$ is hyponormal. But if $M(\zeta)$ is not linear function of $\zeta$, $|\zeta| < 1$, then there are functions $f, g \in H^2(\mathbb{T})$ such that

$$|([H_2^*, H_1] g, f)|^2 \leq ([H_1^*, H_1] g, g) ([H_2^*, H_2] f, f).$$
Thus \((H_1, H_2)\) is \(t\)-hyponormal iff there are constants \(c\) and \(b\) such that

\[ M(\zeta) \equiv c\zeta + b. \]

This example is also very useful for Theorem 1.

A tuple \(H = (H_1, \cdots, H_n)\) of operators on \(\mathcal{H}\) is said to be irreducible, if the only subspaces reducing \(H_1 \cdots, H_n\) are \(\{0\}\) and \(\mathcal{H}\).

**THEOREM 1.** Let \(H = (H_1, \cdots, H_n)\) be a commuting hyponormal irreducible \(n\)-tuple of operators satisfying the condition that \(\text{rank}[H_1^*, H_1] = 1\) and

\[ \text{rank}[H_j^*, H_j] \leq 1, \quad j = 2, \cdots, n. \]

Then either (i) there are numbers \(\alpha_k\) and \(\beta_k\) such that

\[ H_k = \alpha_k H_1 + \beta_k I, \quad k = 2, 3, \cdots, n. \]

or (ii) \(H_1\) is a linear combination of a unilateral shift with multiplicity one and identity and there are numbers \(\alpha_k, \beta_k\) and \(\gamma_k\) such that

\[ H_k = (\alpha_k H_1 + \beta_k I)(\gamma_k H_1 + I)^{-1}, \quad k = 2, 3, \cdots, n. \]

**PROOF.** (1) First let us study the relation between \(H_1\) and \(H_2\).

There exists a vector \(e_1 \in \mathcal{H}, \ e_1 \neq 0\) such that

\[ [H_1^*, H_1]x = (x, e_1)e_1, \quad \text{for } x \in \mathcal{H}. \]  

(2)

From (1), it follows that

\[ |([H_2^*, H_1]x_2, x_1)|^2 \leq ([H_1^*, H_1]x_1, x_1)([H_2^*, H_2]x_2, x_2). \]

Therefore there is a vector \(e_2\) such that

\[ [H_2^*, H_1]x = (x, e_2)e_1 \]  

(3)

and

\[ [H_2^*, H_2]x = c(x, e_2)e_2 \]  

(4)

for \(x \in \mathcal{H}, \ c \geq 1\).

(ii) Let \(\mathcal{H}_1\) be the smallest subspace containing \(e_1\) and reducing \(H_1\). We have to prove that \(\mathcal{H}_1\) is invariant with respect to \(H_2^*\). It is obvious that

\[
\|e_1\|^2 H_2^* e_1 = H_2^* [H_1, H_1] e_1 \\
= H_1^* [H_2^*, H_1] e_1 + [H_1^*, H_1] H_2^* e_1 - [H_2^*, H_1] H_1^* e_1 \\
= (e_1, e_2) H_1^* e_1 + (H_2^* e_1, e_1) e_1 - (H_1^* e_1, e_2) e_1.
\]
Hence $H^*_2 e_1 \in \mathcal{N}_1$. Also we have

$$H^*_2 H_1^* e_1 = H_1^*[H^*_2, H_1]e_1 + H_1^* H_1 H^*_2 e_1$$

$$= \sum_{j=0}^{n-1} H_1^* H_1^{n-j-1} [H^*_2, H_1] H_1^j e_1 + H_1^* H_1 H^*_2 e_1.$$ 

Therefore $H^*_2 H_1^* e_1 \in \mathcal{N}_1$ for $m, n = 0, 1, 2, \ldots$. Thus $\mathcal{N}_1$ is invariant with respect to $H^*_2$.

(iii) According to the decomposition $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_1^\perp$, $H_j$ may be written as matrices

$$H_j = \begin{pmatrix} A_j & 0 \\ B_j & C_j \end{pmatrix}, \quad j = 1, 2,$$ 

where $A_j = H^*_j |_{\mathcal{N}_1}$, $B_1 = 0$, and $C_1$ is normal. The relation $[H_1, H_2] = 0$ is equivalent to

$$[A_1, A_2] = 0, \quad [C_1, C_2] = 0 \quad (5)$$

and

$$B_2 A_1 = C_1 B_2. \quad (6)$$

Let $e_2^1$ be the projection of $e_2$ on $\mathcal{N}_1$ and $e_2^2 = e_2 - e_2^1$. Then (3) is equivalent to

$$[A_2^*, A_1] x = (x, e_2^1) e_1, \quad \text{for } x \in \mathcal{N}_1, \quad (8)$$

$$[B_2^* C_1 - A_1 B_2^*] x = (x, e_2^2) e_1, \quad \text{for } x \in \mathcal{N}_1, \quad (9)$$

and

$$[C_2^*, C_1] = 0. \quad (10)$$

(iv) Let us prove that $e_2^2 = 0$. Suppose on contrary that $e_2^2 \neq 0$.

**LEMMA 1.** If $e_2^2 \neq 0$, then $(zI - A_1)^{-1} e_1$ is an eigenvector of $A_1^*$ for $z \in \rho(H_1)$.

**PROOF.** From (9), we have

$$[B_2^* (zI - C_1)^{-1} - (zI - A_1)^{-1} B_2^*] x = ((zI - C_1)^{-1} x, e_2^2) (zI - A_1)^{-1} e_1, \quad (11)$$

for $z \in \rho(H_1)$. From (7) and (11), we obtain that

$$[B_2^* (zI - C_1)^{-1} C_1^* - (zI - A_1)^{-1} A_1^* B_2^*] x = ((zI - C_1)^{-1} C_1^* x, e_2^2) (zI - A_1)^{-1} e_1 \quad (12)$$
and

\[ [B_2 C_1^*(zI - C_1)^{-1} - A_1^*(zI - A_1)^{-1} B_2^*]x = ((zI - C_1)^{-1}x, e_2^2) A_1^*(zI - A_1)^{-1}e_1, \]

for \( z \in \rho(H_1) \). On the otherhand, it is obvious that

\[ [A_1^*, A_1]x = (x, e_1)e_1. \]

Therefore

\[ [A_1^*, (zI - A_1)^{-1}]x = ((zI - A_1)^{-1}x, e_1)(zI - A_1)^{-1}e_1. \]

From (12), (13), (15) and \([C_1^*, C_1] = 0\) we have

\[ ((zI - A_1)^{-1} B_2^* x, e_1)(zI - A_1)^{-1}e_1 \]
\[ = ((zI - C_1)^{-1} C_1^* x, e_2^2)(zI - A_1)^{-1}e_1 - ((zI - C_1)^{-1} x, e_2^2) A_1^*(zI - A_1)^{-1}e_1. \]

Let

\[ x = (zI - C_1)e_2^2 \]

in (16). Then (16) shows that \((zI - A_1)^{-1}e_1\) is an eigenvector of \( A_1^* \), for \( z \in \rho(H_1) \), which proves Lemma 1.

**COROLLARY 1.** If \( e_2^2 \neq 0 \), then \( \mathcal{H}_0 = \mathcal{H}_1 \), where \( \mathcal{H}_0 \) is the closure of \( \text{span}\{((zI - H_1)^{-1}e_1 : z \in \rho(H_1)\} \).

**PROOF.** It is easy to show that \((zI - H_1)^{-1}e_1 = (zI - A_1)^{-1}e_1\).

By Lemma 1, \( \mathcal{H}_0 \) is invariant with respect to \( H_1^* \). It is obvious that \( e_1 \in \mathcal{H}_0 \), \( \mathcal{H}_0 \subset \mathcal{H}_1 \) and \( \mathcal{H}_0 \) is invariant with respect to \( H_1 \). Therefore \( \mathcal{H}_0 = \mathcal{H}_1 \) which proves Corollary 1.

Now let us continue to prove theorem 1. By Lemma 1, if \( e_2^2 \neq 0 \), then let \( q(z) \) be the eigenvalue of \( H_1^* \) corresponding to the eigen vector \((zI - H_1)^{-1}e_1\), for \( z \in \rho(H_1) \). As in [12], denote

\[ S(z, w) = ((zI - H_1)^{-1}e_1, (wI - H_1)^{-1}e_1), \quad z, w \in \rho(H). \]

Then

\[ S(z, w) = ((wI - H_1^*)^{-1}(zI - H_1)^{-1}e_1, e_1) \]
\[ = (w - q(z))^{-1}((zI - H_1)^{-1}e_1, e_1). \]

(17)

However, \( S(z, w) = S(w, z) \), therefore

\[ S(z, w) = (z - q(w))^{-1}(e_1, (wI - H_1)^{-1}e_1). \]

(18)
From $H_1^*(zI - H_1)^{-1}e_1 = q(z)(zI - H_1)^{-1}e_1$, it is easy to see that

$$q(\infty) = \lim_{z \to \infty} q(z)$$

exists and is finite. Besides,

$$H_1^*e_1 = q(\infty)e_1.$$ 

Let $H_1$ be replaced by $H_1 - \overline{q(\infty)}$. Therefore we may assume that $q(\infty) = 0$. Multiplying (17) and (18) through by $\overline{w}$ and letting $w \to \infty$, we get

$$((zI - H_1)^{-1}e_1, e_1) = z^{-1}\|e_1\|^2.$$ 

Using the formula

$$[H_1^*, (zI - H_1)^{-1}]e_1 = ((zI - H_1)^{-1}e_1, e_1)(zI - H_1)^{-1}e_1,$$

we obtain that

$$q(z) = \|e_1\|^2z^{-1}.$$ 

Therefore

$$S(z, w) = \|e_1\|^2(\overline{w}z - \|e_1\|^2)^{-1}.$$ 

By multiplying a positive number to $H_1$, we may assume that $\|e_1\| = 1$ and $A_1$ is a unilateral shift with multiplicity one. Thus we assume that $\lambda_1 = H^2(T)$, the Hardy space, and

$$(A_1f)(z) = zf(z), \quad \text{for } f \in H^2(T).$$ 

(v) Now, let us determine the form of the normal operator $C_1$. By (9), we have

$$C_1^*B_2f = B_2((f(\cdot) - f(0))(\cdot)^{-1}) + f(0)e_2^2.$$ 

Hence $C_1^*B_21 = e_2^2$ and $C_1^*B_2(\cdot) = B_21$. Therefore

$$C_1e_2^2 = C_1C_1^*B_21 = C_1^*B_2(\cdot) = B_21$$

and $C_1C_1^*e_2^2 = e_2^2$. Thus for polynomials $p(\cdot)$ and $q(\cdot)$, we have

$$C_1(B_2p(\cdot) + q(C_1^*)e_2^2) = B_2((\cdot)p(\cdot) + q(0)) + q_1(C_1^*)e_2^2,$$ 

by (7), where $q_1(\varsigma) = (q(\varsigma) - q(0))\varsigma^{-1}$. Similarly, we have

$$C_1^*(B_2p(\cdot) + q(C_1^*)e_2^2) = B_2(p(\cdot) - p(0))(\cdot)^{-1} + (p(0) + q(C_1^*)C_1^*)e_2^2.$$ 

Therefore, by (21) and (22) we get
\[ C_1 C_1^*(B_2 p(\cdot) + q(C_1^*) e_2^2) = B_2 p(\cdot) + q(C_1^*) e_2^2. \] (23)

Let \( \mathcal{H}_2 \) be the closure of the set \( \{ B_2 p + q(C_1^*) e_2^2 : p, q \text{ are polynomials} \} \). From (21), (22) and (23), it is easy to see that \( \mathcal{H}_2 \) reduces \( C_1 \) and \( C_1^* \mid \mathcal{H}_2 = U \) is unitary.

Let the spectral resolution of \( U \) be
\[ U = \int e^{i\theta} dE(e^{i\theta}), \]
where \( E(\cdot) \) is a spectral measure on \( T \). Denote
\[ \rho(\cdot) = ||E(\cdot) B_2 1||^2. \]

By (7), it is easy to calculate that
\[
\int |f(e^{i\theta})|^2 d\rho(e^{i\theta}) = \|f(U)B_2 1\|^2 = \|B_2 f(\cdot)\|^2 \\
\leq \|B_2\|^2 \frac{1}{2\pi} \int |f(e^{i\theta})|^2 d\theta.
\]

Therefore \( \rho(\cdot) \) is absolutely continuous. Let \( F(e^{i\theta}) = d\rho(e^{i\theta})/d\theta \). From (20), (21) and (22), it is easy to see that \( B_2 1 \) is the cyclic vector for \( \{U, U^*\} \) in \( \mathcal{H}_2 \). So we may assume that \( \mathcal{H}_2 \) is the Hilbert space of all Baire functions \( f(\cdot) \) on \( T \) satisfying
\[
(f, f) = \frac{1}{2\pi} \int |f(e^{i\theta})|^2 F(e^{i\theta}) d\theta < +\infty,
\]
\[ B_2 f(e^{i\theta}) = f(e^{i\theta}), \quad \text{for } f \in \mathcal{H}_1 \] (24)
and
\[ (U f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta}), \quad \text{for } f \in \mathcal{H}_2. \]

Denote
\[ M(\cdot) = A_2 1. \]

By (8), we may prove that
\[ e_2^1(\xi) = ([A_1^*, A_2] 1)(\xi) = (M(\xi) - M(0))\xi^{-1}. \] (25)

It is easy to see that \( M(\cdot) \in H^\infty(T) \) and
\[ (A_2 f)(\xi) = M(\xi) f(\xi), \quad \text{for } f \in H^2(T), \]
(26)
since \([A_1, A_2] = 0\), and \(((A_1^*, A_2) f)(\zeta) = f(0) (M(\zeta) - M(0)) \zeta^{-1}\), by (8). By the way, \(A_2\) is the same operator \(H_2\) defined in the example 1. Besides, \(A_2^*\) is the Toeplitz operator
\[
A_2^* f = P(M f), \quad f \in H^2(\mathbb{T}).
\] (27)

By (24) we have
\[
(B_2 f, B_2 h) = \frac{1}{2\pi} \int f(e^{i\theta}) \overline{h(e^{i\theta})} F(e^{i\theta}) d\theta, \quad \text{for } f, h \in H^2(\mathbb{T}).
\]

Therefore
\[
B_2^* B_2 f = P(F f). \tag{28}
\]

From (4), it is easy to see that
\[
([A_2^*, A_2] + B_2^* B_2) f = c(f, e_2^1) e_2^1. \tag{29}
\]

From (25)-(29), we conclude that
\[
(P((F + |M|^2) f) - MP(M f))(\zeta) = c(f, M_1(\cdot)) M_1(\zeta), \tag{30}
\]
for \(f \in H^2(\mathbb{T})\), where \(M_1(\zeta) = (M(\zeta) - M(0)) \zeta^{-1}\). Put \(f(\zeta) = (z - \zeta)^{-1}, |z| > 1\) in (30). Notice that
\[
P(M(z - \cdot)^{-1})(\zeta) = \overline{M(z^{-1})}(z - \zeta)^{-1},
\]
\[
((z - \cdot)^{-1}, (M(\cdot) - M(0))(\cdot)^{-1}) = \overline{M(z^{-1})} - M(0)
\]
and
\[
P(h(z - \cdot)^{-1})(\zeta) = (P(h)(\zeta) + P_-(h)(\zeta))(z - \zeta)^{-1}
\]
where \(h = F + |M|^2\), and \(P_- = I - P\). Therefore (30) implies that
\[
P(F + |M|^2)(\zeta) + P_-(F + |M|^2)(z)
= M(\zeta) \overline{M(z^{-1})} + c(z - \zeta) \zeta^{-1} (M(\zeta) - M(0)) \overline{M(z^{-1})} - M(0)). \tag{31}
\]
for \(|\zeta| < 1\) and \(|z| > 1\). For \(\xi \in \mathbb{T}\), letting \(\zeta \rightarrow \xi\) and \(z \rightarrow \xi\) in (31), we obtain that
\[
F(\xi) = 0 \quad \text{for a.e. } \xi \in \mathbb{T}.
\]
Thus \(B_2 1 = 0\), and hence \(e_2^2 = 0\). This is a contradiction.
(vi) Now, we only have to consider the case \(e_2^2 = 0\). Under this condition, we have
\[
B_2^* C_1 - A_1 B_2^* = 0, \tag{32}
\]
by (9). From (7) and (32), it is easy to see that

\[ B_2[A_1^*, A_1] = [C_1^*, C_1]B_2 = 0. \]

Thus \( B_2 e_1 = 0 \). Besides

\[ B_2 A_1^{*m} A_1^n e_1 = C_1^{*m} C_1^n B_2 e_1 = 0. \]

Therefore \( B_2 = 0 \) and \( \mathcal{H}_1 \) reduces \( H_2 \). Similarly, \( \mathcal{H}_1 \) also reduces \( H_3, \ldots, H_n \). Therefore \( \mathcal{H}_1 = \mathcal{H} \), otherwise \( \mathcal{H} \) is reducible. Hence \( H_1 \) is pure on \( \mathcal{H} \).

(vii) By the identity \([A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0\), we have that

\[ [H_2, [H_1^*, H_1]] = [H_1, [H_1^*, H_2]]. \] (33)

therefore

\[ H_2[H_1^*, H_1]e_1 - [H_1^*, H_1]H_2 e_1 = H_1[H_1^*, H_2]e_1 - [H_1^*, H_2]H_1 e_1. \] (34)

Using (34), (2) and (3), we obtain that

\[ (H_2 - k_2 I)e_1 = (H_1 - k_1 I)e_2, \] (35)

where

\[ k_j = (H_j e_1, e_1)\|e_1\|^{-1}, \quad j = 1, 2. \]

By (2), (3) and (35), it is easy to show that

\[ (H_1 - k_1 I)[H_1^*, H_2 - k_2 I] = (H_2 - k_2 I)[H_1^*, H_1 - k_1 I]. \] (36)

By the identity \([A, BC] = B[A, C] + [A, B]C\), it is easy to see that (36) is equivalent to

\[ [H_1^*, (H_1 - k_1 I)](H_2 - k_2 I) = [H_1^*, (H_2 - k_2 I)](H_1 - k_1 I). \]

Therefore

\[ ((H_2 - k_2 I)x, e_1)e_1 = (H_1 - k_1 I)x, e_1) e_2. \] (37)

If \((H_1^* - \overline{k}_1 I)e_1 = 0\), then from (37), we have

\[ H_2^* e_1 = \overline{k}_2 e_1, \]

For simplicity, in this case we may assume that \( k_1 = k_2 \) and \( \|e_1\| = 1 \). Then

\[ T(z, w) = ((\overline{w}I - H_1^*)^{-1} e_1, (\overline{z}I - H_1^*)^{-1} e_1) = z^{-1}\overline{w}^{-1}. \]
By the commutator property (e.g. cf. [12]), we have

\[ S(z, w) = T(z, w)(1 - T(z, w))^{-1} = (z\overline{w} - 1)^{-1}. \]

Thus we may assume that \( \mathcal{H}_1 = \mathcal{H}^2(\mathbb{T}) \) and that \( H_1 \) and \( H_2 \) are the operators in the example 1. Now, we have to prove theorem 1 for the case of the example 1. It is easy to see that

\[ e_2(\zeta) = (M(\zeta) - M(0))\zeta^{-1}. \]

Thus \( H_2^*e_1 = 0 \) implies that \( P(M) = 0 \), i.e. \( M(0) = 0 \). Let \( M_1(\zeta) = M(\zeta)\zeta^{-1} \). Then (4) becomes

\[ P(|M|^2f) - MP(Mf) = c(f, M_1)M_1 \tag{38} \]

Let \( f = 1 \), in (38), then

\[ P(|M_1|^2) = cM_1(0)M_1. \]

However \( P_- (|M_1|^2)(z) = P(|M_1|^2)(\frac{1}{z}) - P(|M_1|^2)(0) \). Therefore

\[ P_- (|M_1|^2)(z) = cM_1(0)\overline{(M_1(\zeta^{-1}) - M_1(0))}. \]

Let \( f(\zeta) = (z - \zeta)^{-1}, |z| > 1 \) in (38). Then we get (31) with \( F \equiv 0 \), i.e.

\[ \frac{cM_1(0)(\overline{M_1(\zeta^{-1}) - M_1(0)}) + cM_1(0)M_1(\zeta)}{M_1(\zeta)M_1(\zeta^{-1})(1 - c)\zeta z^{-1} + c} \]

If \((c - 1)M_1(\cdot) \neq 0\), then

\[ M_1(\zeta) = bM_1(0)(\zeta + b)^{-1} \]

where \( b \) is a constant satisfying

\[ b = c(M_1(0) - \overline{M_1(\zeta^{-1})})z(c - 1)\overline{M_1(\zeta^{-1})}^{-1} \]

for \(|z| > 1\). It is easy to see that \(|b|^2 = c(c - 1)^{-1} > 1\). Thus

\[ H_2 = bM_1(0)(H_1 + bI)^{-1}. \]

(viii) If \( (H_1^* - \overline{k}_1I)e_1 \neq 0 \) then (37) implies that there is a \( \alpha \in \mathbb{C} \) such that

\[ e_2 = \alpha e_1. \tag{39} \]

From (35) there is a number \( \beta \in \mathbb{C} \) such that

\[ H_2e_1 = (\alpha H_1 + \beta I)e_1. \tag{40} \]
Now we have to prove that

$$H_2 = \alpha H_1 + \beta I. \quad (41)$$

From (2), (3) and (39), we get 

$$[H^*_1, H_2] = [H^*_1, \alpha H_1 + \beta I].$$

Therefore, by (40),

$$H_2 (\lambda I - H^*_1)^{-1}(\mu I - H_1)^{-1} e_1$$

$$= (\lambda I - H^*_1)^{-1}(\mu I - H_1)^{-1} H_2 e_1 + (\lambda I - H^*_1)^{-1}[H_2, H_1] (\lambda I - H^*_1)^{-1}(\mu I - H_1)^{-1} e_1$$

$$= (\lambda I - H^*_1)^{-1}(\mu I - H_1)^{-1}(\alpha H_1 + \beta I) e_1$$

$$+ (\lambda I - H^*_1)^{-1}[\alpha H_1 + \beta I, H_1^*] (\lambda I - H^*_1)^{-1}(\mu I - H_1)^{-1} e_1$$

$$= (\alpha H_1 + \beta I) (\lambda I - H^*_1)^{-1}(\mu I - H_1)^{-1} e_1,$$

for $\lambda, \mu \in \rho(H_1)$, which proves (41). Similarly we may study the relation between $H_1$ and $H_i$, $i > 2$. Theorem 1 is proved.

**Remark.** Example 1 also shows that there is a commuting hyponormal irreducible pair $(H_1, H_2)$ of operators satisfying the condition that rank$[H^*_1, H_1] = 1$ and rank$[H^*_2, H_2] > 1$. Besides, there is a commuting hyponormal irreducible pair $(H_1, H_2)$ of operators such that the case (ii) of theorem 1 does occur with $\gamma_2 \neq 0$.

From the proof of Theorem 1, it is easy to see the following

**Corollary 2.** Let $(H_1, \cdots, H_n)$ be an irreducible hyponormal n-tuple of commuting operators. If $[H^*_1, H_1] = 1$ and $[H^*_j, H_j] \mathcal{H} \subset [H^*_1, H_1] \mathcal{H}, j = 2, \cdots, n$, then there are complex numbers $\alpha_j, \beta_j$ such that $H_1 = \alpha_1 H_1 + \beta_1 I, j = 2, 3, \cdots, n$.

**Theorem 2.** Let $n > 1$ and $H = (H_1, \cdots, H_n)$ be a commuting hyponormal irreducible n-tuple of operators satisfying the condition that rank$[H^*_1, H_1] = 1$. Then either there is a subspace $\mathcal{H}_0$ which reduces $H_1$ such that $H_1 |_{\mathcal{H}_0}$ is a linear combination of an unilateral shift with multiplicity one and the identity, or there are numbers $\alpha_k$ and $\beta_k$ such that

$$H_k = \alpha_k H_1 + \beta_k I, \quad k = 2, 3, \cdots, n.$$

3. Let $H = (H_1, \cdots, H_n)$ be a tuple of operators on $\mathcal{H}$. It is obvious that $H$ is t-hyponormal if and only if

$$\sum_{i, j}([H^*_j, H_i] x_i, x_j) \geq 0, \quad (42)$$
for all \( x_j \in \mathcal{H} \).

**Theorem 3.** Let \( \mathcal{H} = (H_1, \ldots, H_n) \) be a commuting irreducible \( n \)-tuple of operators. If \( \mathcal{H} \) is either cohyponormal or \( t \)-hyponormal, and \( \mathcal{H} \) satisfies the condition that

\[
\operatorname{rank}[H_1^*, H_1] = 1,
\]

then there are numbers \( \alpha_k \) and \( \beta_k \) such that

\[
H_k = \alpha_k H_1 + \beta_k I, \quad k = 2, \ldots, n.
\]

**Proof.** We only consider the case that \( \mathcal{H} \) is \( t \)-hyponormal. The proof of this theorem is similar to, different from but much simpler than that of theorem 1. It is obvious that there are \( e_j \in \mathcal{H}, e_j \neq 0 \) such that (2) still holds good but (3) becomes

\[
[H_2^*, H_1]x = (x, e_1)e_2. \tag{43}
\]

Let \( \mathcal{H}_1 \) be the smallest subspace containing \( e_1 \) and reducing \( H_1 \). By the same method in the proof of Theorem 1 we may prove that \( H_2 e_1 \in \mathcal{H}_1 \) and then \( \mathcal{H}_1 \) is invariant with respect to \( H_2 \). According to the decomposition \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp \), we have

\[
H_j = \begin{pmatrix} A_j & B_j \\ 0 & C_j \end{pmatrix}, \quad j = 1, 2. \tag{44}
\]

where \( B_1 = 0 \) and \( C_1 \) is normal. The relation \( [H_1, H_2] = 0 \) is equivalent to (5), (6) and

\[
A_1 B_2 = B_2 C_1. \tag{45}
\]

Let \( e_2^1 \) be the projection of \( e_2 \) on \( \mathcal{H}_1 \) and \( e_2^2 = e_2 - e_2^1 \). Then (43) is equivalent to (10),

\[
[A_1^*, A_2]x = (x, e_2^1)e_1 \tag{46}
\]

and

\[
(A_1^* B_2 - B_2 C_1^*)x = (x, e_2^2)e_1. \tag{47}
\]

It is obvious that (47) implies

\[
(\bar{w}I - A_1^*)^{-1}B_2 x - B_2(\bar{w}I - C_1^*)^{-1}x = ((\bar{w}I - C_1^*)^{-1}x, e_2^2)(\bar{w}I - A_1^*)^{-1}e_1, \tag{48}
\]

for \( w \in \rho(H_1) \). From (45) and (48), we get

\[
[(\bar{w}I - A_1^*)^{-1}(zI - A_1)^{-1}B_2 - B_2(\bar{w}I - C_1^*)^{-1}(zI - C_1)^{-1}]x \\
= ((\bar{w}I - C_1^*)^{-1}(zI - C_1)^{-1}x, e_2^2)(\bar{w}I - A_1^*)^{-1}e_1 \tag{49}
\]
and
\[
[(zI - A_1)^{-1}(wI - A_1^*)^{-1}B_2 - B_2(zI - C_1)^{-1}(wI - C_1^*)^{-1}]x
= ((wI - C_1^*)^{-1}x, e_2^2)(zI - A_1)^{-1}(wI - A_1^*)^{-1}e_1.
\] (50)

It is similar to (15) that
\[
[(wI - A_1^*)^{-1}, (zI - A_1)^{-1}]B_2x
= ((wI - A_1^*)^{-1}(zI - A_1)^{-1}B_2x, e_1)(zI - A_1)^{-1}(wI - A_1^*)^{-1}e_1.
\] (51)

Subtracting (50) from (49) and using (51) and $[C_1^*, C_1] = 0$, we get
\[
((wI - A_1^*)^{-1}(zI - A_1)^{-1}B_2x, e_1)(zI - A_1)^{-1}(wI - A_1^*)^{-1}e_1
= ((wI - C_1^*)^{-1}(zI - C_1)^{-1}x, e_2^2)(wI - A_1^*)^{-1}e_1
- ((wI - C_1^*)^{-1}x, e_2^2)(zI - A_1)^{-1}(wI - A_1^*)^{-1}e_1.
\] (52)

If $e_2^2 \neq 0$, then letting $x = (zI - C_1)(wI - C_1^*)e_2^2$ for $z, w \in \rho(H_1)$ in (52) we see that $(wI - A_1^*)^{-1}e_1$ is an eigenvector for $A_1$ corresponding to $w \in \rho(H_1)$. Let
\[
\xi = (wI - H_1^*)^{-1}e_1 = (wI - A_1^*)^{-1}e_1
\]
where $w \in \rho(H_1)$. Then $\xi \neq 0$, $\xi \in \mathcal{H}_1$ and
\[
H_1\xi = A_1\xi = \alpha\xi
\]
where $\alpha$ is the eigenvalue corresponding to $\xi$. Thus
\[
H_1^*\xi = \overline{\alpha}\xi,
\]
since $H_1$ is hyponormal. Therefore
\[
0 = [H_1^*, H_1]\xi = (\xi, e_1)e_1,
\]
i.e. $\xi \perp e_1$. Similarly, we have
\[
(\xi, H_1^*mH_1^n e_1) = \alpha^m\overline{\alpha}^n (\xi, e_1) = 0.
\]
i.e. $\xi \perp \mathcal{H}_1$. This is a contradiction. Therefore $e_2^2 = 0$. Then $\mathcal{H}_1$ reduces $H_2$ and similarly $H_3, \cdots, H_n$. Hence $\mathcal{H}_1 = \mathcal{H}$.

Now, we have to show that $e_2$ is a multiple of $e_1$. By (43), it is easy to calculate that
\[
[H_1^*H_1, H_2]x = [H_1^*, H_2]H_1x = (H_1x, e_2)e_1
\] (53)
and
\[ [H_1 H_1^*, H_2^*] x = H_1 [H_1^*, H_2^*] x = (x, e_2) H_1 e_1. \] 

(54)

On the otherhand, by (2), we have

\[
[[H_1^*, H_1], H_2] x = [H_1^*, H_1] H_2 x - H_2 [H_1^*, H_1] x
= (H_2 x, e_1) e_1 - (x, e_1) H_2 e_1.
\]

(55)

From (53), (54), and (55), we may conclude that

\[ H_2 e_1 = \alpha(x) H_1 e_1 + \beta(x) e_1 \]

where

\[ \alpha(x) = (x, e_2) (x, e_1)^{-1} \]

and

\[ \beta(x) = ((H_2 x, e_1) - (H_1 x, e_2)) (x, e_1)^{-1}. \]

It is obvious that \( H_1 e_1 \) and \( e_1 \) are linearly independent. Therefore \( \alpha(x) \) and \( \beta(x) \) must be independent of \( x \). Thus there are numbers \( \alpha \) and \( \beta \) such that \( e_2 = \alpha e_1 \) and (40) is satisfied. Theorem is proved.

**Theorem 4.** Let \( \mathcal{H} = (H_1, \cdots, H_n) \) be a commuting irreducible n-tuple of operators satisfying the condition that (i) there is no subspace \( \mathcal{K}_0 \) reducing \( H_1 \) such that \( H_1|_{\mathcal{K}_0} \) is a linear combination of a unilateral shift with multiplicity one and the identity and (ii)

\[ \text{rank}[H_1^*, H_1] = 1. \]

Then \( \mathcal{H} \) is hyponormal iff \( \mathcal{H} \) is t-hyponormal.

4. In this section, we are studying the Taylor spectrum [13] of cohyponormal or t-hyponormal operators. First, let us review some basic technique (cf. [13], [6], [7]) in the theory of Taylor spectrum.

Let \( \Lambda \) be the exterior algebra of \( n \) generators \( e_1, \cdots, e_n \) with identity \( e_0 \) over the complex field \( \mathbb{C} \) equipped with an inner product such that

\[ \{e_{i_1} \cdots e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\} \cup \{e_0\} \]

is an othonormal basis. Let \( E_i, \quad i = 1, 2, \cdots, n \) be the creation operators

\[ E_i \xi = e_i \xi, \quad \xi \in \Lambda. \]

Then \( E_i E_j + E_j E_i = 0 \), and

\[ E_i^* E_j + E_j^* E_i = \delta_{ij} I. \] 

(56)
For \( 1 \leq k \leq n \), let \( N_k \) be the set of \( k \)-tuple of integers \( \alpha = (i_1, \cdots, i_k) \) satisfying

\[ 1 \leq i_1 < \cdots < i_k \leq n, \]

and \( N_0 = \{0\} \). Let \( e_\alpha = e_{i_1} \cdots e_{i_k} \) for \( \alpha = (i_1, \cdots, i_k) \in N_k \). Besides, let \( \Lambda_k \) be the subspace of \( \Lambda \) spanned by \( \{e_\alpha : \alpha \in N_k\} \).

Notice that for \( \alpha \in N_k, \ell, m = 1, 2, \cdots, n \), we have

\[ (e_\ell e_\alpha, e_m e_\beta) = 0, 1 \text{ or } -1, \]

if \( \binom{\ell}{m}^\alpha \beta \) is not a permutation, is an even permutation or an odd permutation respectively. If \( \ell \neq m \) then

\[ (E^*_\ell e_\alpha, E^*_m e_\beta) = -(e_m e_\alpha, e_\ell e_\beta) \]

by (56). If \( \ell \in \alpha \), then \( (E^*_\ell e_\alpha, E^*_m e_\beta) = 0 \) or 1 for \( \alpha \neq \beta \) or \( \alpha = \beta \) respectively.

Let \( \mathcal{H} = (H_1, \cdots, H_n) \) be a commuting \( n \)-tuple of operators on \( \mathcal{H} \). Denote \( \tilde{\mathcal{H}}_k = \mathcal{H} \otimes \Lambda_k \) and \( \tilde{\mathcal{H}} = \mathcal{H} \otimes \Lambda \). For \( k = 0, 1, 2, \cdots, n - 1 \), let \( D_k \) be the operator in \( \mathcal{L}(\tilde{\mathcal{H}}_k \rightarrow \tilde{\mathcal{H}}_{k+1}) \) defined by

\[ D_k x \otimes \xi = \xi = \sum_{j=1}^{n} H_j x \otimes e_j \xi \quad \text{for } x \otimes \xi \in \tilde{\mathcal{H}}_k. \]

Denote

\[
\hat{\mathbf{H}} = \begin{pmatrix}
D_0 & D_1^* & 0 & \cdots & 0 \\
0 & D_2 & D_3^* & \cdots & 0 \\
0 & 0 & D_4 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & D_{n-1}^*
\end{pmatrix},
\]

for even \( n \) and

\[
\hat{\mathbf{H}} = \begin{pmatrix}
D_0 & D_1^* & 0 & \cdots & 0 \\
0 & D_2 & D_3^* & \cdots & 0 \\
0 & 0 & D_4 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & D_{n-1}
\end{pmatrix},
\]

for odd \( n \).

By Curto's Theorem [7], \( 0 = (0, \cdots, 0) \) is not in the Taylor's spectrum \( \text{sp}(\hat{\mathbf{H}}) \) iff \( \hat{\mathbf{H}} \) is invertible.
THEOREM 5. Let \( \mathcal{H} = (H_1, H_2, \ldots, H_n) \) be a commuting \( t \)-hyponormal \( n \)-tuple of operators on \( \mathcal{H} \). Then \( (\lambda_1, \ldots, \lambda_n) \) is in the Taylor spectrum \( sp(\mathcal{H}) \), iff
\[
\sum_{j=1}^{n} (H_j - \lambda_j I)(H_j^* - \bar{\lambda}_j I)
\]
is not invertible in \( \mathcal{L}(\mathcal{H}) \).

PROOF. (i) Without loss of generality, we may assume that \( \lambda_j = 0, j = 1, 2, \ldots, n \). By a theorem of Curto [7], if \( (0, \ldots, 0) \) is not in the \( sp(\mathcal{H}) \), then
\[
\sum_{j=1}^{n} H_j H_j^*
\]
is invertible in \( \mathcal{L}(\mathcal{H}) \) which is equivalent to the condition that there is a positive number \( K \) such that
\[
\sum_{j=1}^{n} \|H_j^* x\|^2 \geq K^2 \|x\|^2, \quad \text{for } x \in \mathcal{H}.
\]
Hence we only have to prove that
\[
\|\hat{\mathcal{H}} x\| \geq K \|x\|, \quad x \in \hat{\mathcal{H}}
\]
and
\[
\|\hat{\mathcal{H}}^* x\| \geq K \|x\|, \quad x \in \hat{\mathcal{H}}
\]
under the condition (59).

(ii) In order to give the estimate of \( \|\hat{\mathcal{H}} x\| \) or \( \|\hat{\mathcal{H}}^* x\| \) from below, we have to calculate
\[
S_j = \|D_j \sum_{\alpha \in N_j} x_\alpha \otimes e_\alpha + D_{j+1}^* \sum_{\alpha \in N_{j+2}} x_\alpha \otimes e_\alpha\|^2
= S_{j1} + S_{j2} + S_{j3},
\]
for \( j = -1, 1, \ldots, n - 1 \), where
\[
S_{j1} = \|\sum_{m=1}^{n} \sum_{\alpha \in N_j} H_m x_\alpha \otimes e_m e_\alpha\|^2,
\]
\[
S_{j2} = \|\sum_{m=1}^{n} \sum_{\alpha \in N_{j+2}} H_m^* x_\alpha \otimes E_m^* e_\alpha\|^2,
\]
and
\[ S_{j3} = 2 \mathcal{R} \sum_{\ell, m = 1}^{n} \sum_{\alpha \in N_j, \beta \in N_{j+2}} (H_{\ell} x_\alpha \otimes e_\ell e_\alpha, H_m^* x_\beta \otimes E_m^* e_\beta). \]

Besides, \( S_{(-1)1}, S_{(-1)3}, S_{(n-1)2} \) and \( S_{(n-1)3} \) are zeros. It is easy to see that

\[ \| \hat{H} x \|^2 = \sum_{k=0}^{[(n-1)/2]} S_{2k}. \]

where \( x = \sum_{k=0}^{[n/2]} \sum_{\alpha \in N_{2k}} x_\alpha \otimes e_\alpha \). It is obvious that

\[ S_{j3} = 2 \mathcal{R} \sum (H_m H_{\ell} x_\alpha, x_\beta) (e_m e_\ell e_\alpha, e_\beta) = 0, \]

since \( H_m H_{\ell} = H_{\ell} H_m \) and \( e_m e_\ell = -e_\ell e_m \). On the other hand,

\[ S_{j1} = \sum_{\ell \notin \alpha} \sum_{m \notin \beta} (H_{\ell} x_\alpha, H_m x_\beta) (e_\ell e_\alpha, e_m e_\beta) \]

\[ = S_j^{(1)} + S_j^{(2)}, \]

where

\[ S_j^{(1)} = \sum_{\alpha \in N_j} \sum_{\ell \notin \alpha} \| H_{\ell} x_\alpha \|^2, \]

\[ S_0^{(2)} = 0, \] and

\[ S_j^{(2)} = \sum_{\alpha, \beta \in N_j} \sum_{\ell \notin \alpha, m \notin \beta, \ell \neq m} (H_m^* H_{\ell} x_\alpha, x_\beta) (e_\ell e_\alpha, e_m e_\beta), \]

for \( j \geq 1 \). Similarly, we have

\[ S_{j2} = \sum_{\ell \in \alpha \in N_{j+2}} \sum_{m \in \beta \in N_{j+2}} (H_{\ell}^* x_\alpha, H_m^* x_\beta) (E_{\ell}^* e_\alpha, E_m^* e_\beta) \]

\[ = S_j^{(3)} + S_j^{(4)}, \]

where

\[ S_j^{(3)} = \sum_{\ell \in \alpha \in N_{j+2}} \| H_{\ell}^* x_\alpha \|^2 \]

and

\[ S_j^{(4)} = -\sum_{\alpha, \beta \in N_{j+2}} \sum_{\ell \neq m, \ell \notin \alpha} (H_m H_{\ell}^* x_\alpha, x_\beta) (e_m e_\alpha, e_\ell e_\beta). \]
Thus
\[ S_{j+2}^{(2)} + S_j^{(4)} = \sum_{\alpha, \beta \in \mathbb{N}_{j+2}} \sum_{\ell \neq m, \ell \not\in \alpha, m \not\in \beta} \langle [H_{m}^*, H_{\ell}] x_{\alpha}, x_{\beta} \rangle (e_\ell e_\alpha, e_m e_\beta) \]
and
\[ S_{j+2}^{(1)} + S_j^{(3)} = \sum_{\ell=1}^{n} \sum_{\alpha \in \mathbb{N}_{j+2}} \| H_{\ell}^* x_{\alpha} \|^2 + \sum_{\ell \not\in \alpha \in \mathbb{N}_{j+2}} \langle [H_{\ell}^*, H_{\ell}] x_{\alpha}, x_{\alpha} \rangle. \]
Hence
\[ S_{j+2}^{(1)} + S_j^{(2)} + S_j^{(3)} + S_j^{(4)} = \sum_{\ell=1}^{n} \sum_{\alpha \in \mathbb{N}_{j+2}} \| H_{\ell}^* x_{\alpha} \|^2 + S_{j+2}^{(0)}, \]
for \( j = 0, 2, \cdots, 2([(n - 1)/2] - 1) \), where
\[ S_j^{(0)} = \sum_{\alpha, \beta \in \mathbb{N}_{j}} \sum_{\ell \not\in \alpha, m \not\in \beta} \langle [H_{m}^*, H_{\ell}] x_{\alpha}, x_{\beta} \rangle (e_\ell e_\alpha, e_m e_\beta). \]

Therefore
\[ \| \hat{H} \|^2 \geq K \| x \|^2 + \sum_{k=1}^{[(n-1)/2]} S_{2k}^{(0)}, \]
by (59), since
\[ S_{01} = \sum_{m=1}^{n} \| H_m x_0 \|^2 \geq \sum_{m=1}^{n} \| H_m^* x_0 \|^2 \]
and
\[ S_{(n-2)2} = \sum_{m=1}^{n} \| H_m^* x_{12\cdots n} \|^2, \]
for even \( n \).

(iii) In order to prove (60), we only have to prove that
\[ S_{2k}^{(0)} \geq 0, \ k = 1, 2, \cdots, [(n - 1)/2]. \] (62)

Assume \( 1 \leq j \leq n - 1 \). Let \( \gamma = (\ell_1, \cdots, \ell_{j+1}) \in \mathbb{N}_{j+1}, \) and \( \gamma_p = (\ell_1, \cdots, \ell_{p-1}, \ell_{p+1}, \cdots, \ell_{j+1}) \in \mathbb{N}_j \). Then
\[ \sum_{i, k \in \gamma} \langle [H_{\ell_k}^*, H_{\ell_i}] x_{\gamma_i}, x_{\gamma_k} \rangle (e_\ell e_{\gamma_i}, e_{\ell_k} e_{\gamma_k}) \]
\[ = \sum_{i, k \in \gamma} \langle [H_{\ell_k}^*, H_{\ell_i}] \hat{x}_{\gamma_i}, \hat{x}_{\gamma_k} \rangle \geq 0. \]
by (42), where \( \hat{e}_{\gamma_i} = (e_{\gamma_i}, e_{\xi}, e_{\gamma_i}) \). Thus (62) is proved.

(iv) It is easy to see that

\[
\|\hat{H}^* x\|^2 = \sum_{k=0}^{[n/2]} S_{2k-1},
\]

where \( x = \sum_{k=0}^{(n-1)/2} \sum_{\alpha \in N_{2k+1}} x_{\alpha} \otimes e_{\alpha} \). Using the same method in the step (ii) and (iii) we may prove

\[
\|\hat{H}^* x\|^2 = \sum_{k=0}^{(n-1)/2} \left( \sum_{\ell=1}^{n} \sum_{\alpha \in N_{2k+1}} \|H_{\ell}^* x_{\alpha}\|^2 + S_{2k+1}^{(0)} \right)
\]

where \( S_{n}^{(0)} = 0 \) and hence (61). Theorem is proved.

THEOREM 6. Let \( H = (H_1, \cdots, H_n) \) be a commuting cohyponormal \( n \)-tuple of operators on \( \mathcal{H} \). Then \( (\lambda_1, \cdots, \lambda_n) \) is in the Taylor spectrum \( \text{sp}(H) \), iff

\[
\sum_{j=1}^{n} (H_j^* - \bar{\lambda}_j I)(H_j - \lambda_j I)
\]

is not invertible in \( \mathcal{L}(\mathcal{H}) \).

5. In this section, a trace formula for the subnormal \( n \)-tuple of operators is given. A commuting \( n \)-tuple of bounded operators \( S = (S_1, \cdots, S_n) \) on a separable Hilbert space \( \mathcal{H} \) is subnormal if there is a commuting \( n \)-tuple of normal operators \( N = (N_1, \cdots, N_n) \) on a Hilbert space \( \mathcal{K} \) containing \( \mathcal{K} \) as a subspace satisfying \( N_j \mathcal{H} \subset \mathcal{H} \) and

\[
S_j = N_j |_\mathcal{H}, \quad j = 1, 2, \cdots, n.
\]

This \( N \) is said to be a normal extension of \( S \). Assume \( S \) is a subnormal \( n \)-tuple of operators with normal extension \( N \). Let \( M = \bigvee_{i,j} [S_i^*, S_j] \mathcal{H} \), then \( M \) is an invariant subspace of \( S_j^* \), \( j = 1, 2, \cdots, n \). Let \( C_{kj} \) be the operators in \( \mathcal{L}(M) \) defined by

\[
C_{kj} x = [S_k^*, S_j] x, \quad x \in M.
\]
There exist operators $\Lambda_j \in \mathcal{L}(M)$ such that

$$\Lambda_j^* x = S_j^* x, \quad x \in M.$$

Let $E(\cdot)$ be the spectral measure of the normal extension $N$ on the Taylor spectrum $\text{sp} (N)$. Define a positive semi-definite $\mathcal{L}(M)$-valued measure

$$e(\cdot) = P_M E(\cdot) |_M$$
on $\text{sp}(N)$, where $P_M$ is the projection from $K$ to $M$. In this section, we use the analytic model [17] for $S$ and notions in [17] without further explanation. The trace ideal of the operator algebra $\mathcal{L}(\mathcal{H})$ is denoted by $\mathcal{L}^1(\mathcal{H})$.

**Lemma 2.** Let $S = (S_1, \cdots, S_n)$ be a subnormal $n$-tuple of operators. If $[S_i^*, S_i] \in \mathcal{L}^1(\mathcal{H}), i = 1, 2, \cdots, n$ then $[\prod_{i=1}^n (\overline{w}_i I - S_i^*)^{-1}, \prod_{j=1}^n (z_j I - S_j)^{-1}] \in \mathcal{L}^1(\mathcal{H})$ and

$$\text{tr} \left( \prod_{i=1}^n (\overline{w}_i I - S_i^*)^{-1}, \prod_{j=1}^n (z_j I - S_j)^{-1} \right) \prod_{k=1}^n (\overline{u}_k I - S_k^*)^{-1}$$

$$= \sum_{i=1}^n \text{tr} \left( \int \frac{e(du)(\overline{w}_i I - \Lambda_i^*)}{(\overline{w}_i - \overline{u}_i) \prod_{j=1}^n ((\overline{w}_j - \overline{u}_j)(\overline{v}_j - \overline{u}_j)(z_j - u_j))} \right).$$

(63)

for $z, v, w \in \rho(S_i), i = 1, 2, \cdots, n$.

**Proof.** For simplicity, let $A_i = \overline{w}_i I - S_i^*, B_i = z_i I - S_i$ and $C_i = \overline{v}_i I - S_i^*$. Then

$$[A_i, B_j] = C_{ij} P_M.$$

It is easy to calculate that

$$\left[ A_i^{-1}, \prod_{j=1}^n B_j^{-1} \right] = \sum_{j=1}^n A_i^{-1} \left( \prod_{p=j}^n B_p^{-1} \right) C_{ij} P_M \left( \prod_{q=1}^j B_q^{-1} \right) A_i^{-1}.$$

Therefore

$$\left[ \prod_{j=1}^n A_j^{-1}, \prod_{k=1}^n B_k^{-1} \right]$$

$$= \sum_{i, j=1}^n \prod_{u=1}^i A_u^{-1} \prod_{p=j}^n B_p^{-1} C_{ij} P_M \prod_{q=1}^j B_q^{-1} \prod_{t=i}^n A_t^{-1},$$

where $0 < \rho(S_i) < 1$ for $i = 1, 2, \cdots, n$. The last result is obtained from (63) and the spectral theorem.
and it belongs to $L^1(H)$. Thus

$$
\text{tr} \left( \left[ \prod_{i=1}^{n} A_i^{-1}, \prod_{j=1}^{n} B_j^{-1} \right] \prod_{k=1}^{n} C_k^{-1} \right) = \sum_{i,j=1}^{n} \text{tr} \left( \prod_{k=1}^{n} C_k^{-1} \prod_{\ell=1}^{n} A_{\ell}^{-1} \prod_{p=j}^{n} B_p^{-1} C_{ij} P_M \prod_{q=1}^{j} B_q^{-1} \right). \quad (64)
$$

It is obvious that $P_M B_j^{-1} = (z_j I - \Lambda_j)^{-1} P_M$ and

$$
P_M \prod_{k=1}^{n} C_k^{-1} \cdot \prod_{\ell=1}^{n} A_{\ell}^{-1} \cdot \prod_{p=j}^{n} B_p^{-1} a = \int \frac{e(du) a}{(\tilde{w}_i - \tilde{u}_i) \prod_{k=1}^{n}([\tilde{w}_k - \tilde{u}_k](\tilde{v}_k - \tilde{u}_k)) \prod_{p=j}^{n}(z_p - u_p)}.\nonumber
$$

Therefore (64) equals

$$
\sum_{j,i,j=1}^{n} \text{tr} \left( \int \frac{e(du) C_{ij} \prod_{q=1}^{j} (z_q I - \Lambda_q)^{-1}}{(\tilde{w}_i - \tilde{u}_i) \prod_{k=1}^{n}([\tilde{w}_k - \tilde{u}_k](\tilde{v}_k - \tilde{u}_k)) \prod_{p=j}^{n}(z_p - u_p)} \right). \quad (65)
$$

By (13) of [17], $e(du) C_{ij} = e(du)(\tilde{u}_i I - \Lambda_i^*)(u_j I - \Lambda_j)$. Therefore (65) equals

$$
\sum_{i=1}^{n} \text{tr} \left( \int \frac{e(du)(\tilde{u}_i I - \Lambda_i^*)}{(\tilde{w}_i - \tilde{u}_i) \prod_{j=1}^{n}([\tilde{w}_j - \tilde{u}_j](\tilde{v}_j - \tilde{u}_j))(z_j - u_j)} \right).
$$

$$
\sum_{j=1}^{n} \left( \prod_{q=1}^{j} (z_q I - \Lambda_q)^{-1} \prod_{p=j+1}^{n}(z_p - u_p) \right) \right) = \sum_{i=1}^{n} \text{tr} \left( \int \frac{e(du)(\tilde{u}_i I - \Lambda_i^*)}{(\tilde{w}_i - \tilde{u}_i) \prod_{j=1}^{n}([\tilde{w}_j - \tilde{u}_j](\tilde{v}_j - \tilde{u}_j))(z_j - u_j)} \right)

- \int \frac{e(du)(\tilde{u}_i I - \Lambda_i^*) \prod_{q=1}^{n} (z_q I - \Lambda_q)^{-1}}{(\tilde{w}_i - \tilde{u}_i) \prod_{j=1}^{n}([\tilde{w}_j - \tilde{u}_j](\tilde{v}_j - \tilde{u}_j))(z_j - u_j)} \right)
$$

which equals the right-hand side of (63), by [17]. Lemma 2 is proved.
LEMMA 3. If \( [S_i^{*}, S_i]^{1/2} \in \mathcal{L}^1(\mathcal{H}) \), then for every bounded Baire function \( f(\cdot) \) on \( sp(\mathbb{N}) \),

\[
\int f(u)e(du)(\bar{u}_i I - \Lambda_i^{*}) \in \mathcal{L}^1(M),
\]

and there is a complex measure \( \nu_i(\cdot) \) with finite total variation such that

\[
\text{tr}(\int f(u)e(du)(\bar{u}_i I - \Lambda_i^{*})) = \int f(u)\nu_i(du).
\]

This measure \( \nu_i(du) \) is formally denoted by \( \text{tr}(e(du)(\bar{u}_i I - \Lambda_i^{*})) \). Similarly, we may define the complex measure \( \text{tr}((u_i I - \Lambda_i)e(du)) \).

PROOF. Notice that

\[
\int |(e(du)(\bar{u}_i I - \Lambda_i^{*})a, b)| \leq \left( \int \|\sqrt{e(du)}(\bar{u}_i I - \Lambda_i^{*})a\|^2 \int \|\sqrt{e(du)}b\|^2 \right)^{1/2} = \|b\| \|C_{ii}^{1/2}a\|
\]

for \( a, b \in M \). From (68), it follows (66). Define a complex set function

\[
\nu_i(F) = \text{tr} \left( \int_F e(du)(\bar{u}_i I - \Lambda_i^{*}) \right), \quad F \in \mathcal{B},
\]

on \( sp(\mathbb{N}) \), where \( \mathcal{B} \) is the Borel field of all Borel sets in \( sp(\mathbb{N}) \). Then it is easy to see that

\[
\sum_{j=1}^{\infty} |\nu_i(F_j)| = \text{tr} \left( \int f(u)e(du)(\bar{u}_i I - \Lambda_i^{*}) \right),
\]

where \( \{F_j\} \) is any sequence of multurally disjoint Borel sets and

\[
f(u) = \sum_j 1_{F_j}(u)\text{sign} (\nu_i(F_j)).
\]

From (68) and (69), it is easy to see that

\[
\sum_j |\nu_i(F_j)| \leq \text{tr}(C_{ii}^{1/2}),
\]

which proves the lemma.
It is unknown whether the condition \([S_i^*, S_i]^{1/2} \in L^1(\mathcal{H})\) in Lemma 3 may be replaced by a weaker condition \([S_i^*, S_i] \in L^1(\mathcal{H})\) or not.

Let \(R(S)\) be the algebra of the functions of variables \((u_1, \ldots, u_n)\) generated by \((\overline{u}_i - \overline{z})^{-1}\) and \((u_i - z)^{-1}\), \(z \in \rho(S_i), i = 1, 2, \ldots, n\). For \(f = f(u_1, \ldots, u_n; u_1, \ldots, u_n) \in R(S)\), let

\[
f(S^*, S) = f(S_1^*, \ldots, S_n^*; S_1, \ldots, S_n).
\]

It is obvious that, in general, this operator \(f(S)\) depends on the ordering of product.

**THEOREM 7.** Let \(S\) be a subnormal \(n\)-tuple of operators satisfying \([S_i^*, S_i]^{1/2} \in L^1(\mathcal{H}), i = 1, \ldots, n\) and \(\mathcal{N}\) be its normal extension. Then there are complex measures with finite total variations

\[
\nu_j(du) = \text{tr}((u_j I - \Lambda_j)e(du)), \quad j = 1, \ldots, n,
\]

on \(\text{sp}(\mathcal{N})\) such that for \(f, h \in R(S)\), and any ordering of the product of \(f(S^*, S)\) and \(h(S^*, S)\),

\[
\text{tr}(i[f(S^*, S), h(S^*, S)]) = \int_{\text{sp}(\mathcal{N})} f(\overline{u}, u)d_\nu h(\overline{u}, u),
\]

where

\[
d_\nu h(\overline{u}, u) = \sum_{j=1}^n \left( \frac{\partial h(\overline{u}, u)}{\partial u_j} \nu_j(du) + \frac{\partial h(\overline{u}, u)}{\partial \overline{u}_j} \nu_j(du) \right).
\]

**PROOF.** From Lemma 3 and 4, it is easy to see that for \(f(\overline{u}), h(u), q(\overline{u}) \in R(S)\), we have

\[
\text{tr}(i[f(S^*), h(S)]q(S^*)) = -\sum_{j=1}^n \int_{\text{sp}(\mathcal{N})} q(\overline{u})h(u) \frac{\partial f(\overline{u})}{\partial \overline{u}_j} \nu_j(du)
\]

(71)

Take the conjugate of both sides of (71), it is easy to verify that for \(r(u) \in R(S)\) we have

\[
\text{tr}(i[f(S^*), h(S)]r(S)) = \sum_{j=1}^n \int_{\text{sp}(\mathcal{N})} r(u)f(\overline{u}) \frac{\partial h(u)}{\partial u_j} \nu_j(du).
\]

(72)
Therefore, for \( f_1(\bar{u}), h_1(\bar{u}), f_2(u), h_2(u) \in R(S) \), we have

\[
\text{tr}([f_1(S^*)f_2(S), h_1(S^*)h_2(S)]) \\
= \text{tr}(h_1(S^*)[f_1(S^*), h_2(S)]f_2(S)) + \text{tr}(f_1(S^*)[f_2(S), h_1(S^*)]h_2(S)) \\
= \text{tr}([h_1(S^*)f_1(S^*), h_2(S)]f_2(S)) - [h_1(S^*), h_2(S)]f_1(S^*)f_2(S) \\
+ \text{tr}(f_1(S^*)[f_2(S), h_1(S^*)] - f_1(S^*)f_2(S)[h_2(S), h_1(S^*)]) \\
= \text{tr}([h_1(S^*)h_1(S^*), h_2(S)]f_2(S)) + \text{tr}([f_2(S)h_2(S), h_1(S^*)]f_1(S^*)) \tag{73}
\]

Since \( \text{tr}([h_1(S^*), h_2(S)]f_1(S^*)f_2(S)) + \text{tr}(f_1(S^*)f_2(S)[h_2(S), h_1(S^*)]) = 0 \). From (71), (72) and (73), it is easy to see that (70) holds good for \( f(\bar{u}, u) = f_1(\bar{u})f_2(u) \) and \( h(\bar{u}, u) = h_1(\bar{u})h_2(u) \). Then the general case may be proved by linear operation.

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References


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