ON THE REPRESENTATIONS OF THE LOCAL CURRENT ALGEBRA AND THE GROUP OF DIFFEOMORPHISMS (I)

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ABSTRACT

In this paper, we develop a fundamental analytic expression of Radon-Nikodym derivative of measure on the space of generalized functions, which are quasi-invariant with respect to a group of diffeomorphisms. By means of this expression, a method of determining a class of representations for local current algebra is given.

Recently, several physicists and mathematicians\textsuperscript{[1-5]} have investigated the representations of the local current algebra and the group of diffeomorphisms, that are motivated by the theory of quantum physics and statistical physics. These representations are closely related to quasi-invariant measures\textsuperscript{[1-5],[6].} But the investigation on the measure, which is quasi-invariant with respect to the group of diffeomorphisms, began only a few years ago. This theory of quasi-invariant measures will be very different from the theory of invariant measures on the locally compact group and is more difficult. There are a few papers\textsuperscript{[1-2],[3],[4]} which deal with this problem. In this paper, firstly we give a fundamental theorem of measure on the space of generalized functions, which is quasi-invariant with respect to the group of diffeomorphisms, i.e. the analytic expression of the Radon-Nikodym derivative. Secondly, by means of this expression we give a method of finding out a class of representations of local current algebra in the theory of quantum physics. In this paper, we only consider the group of diffeomorphisms of $R^n$. The method used here, can be extended to the case of the group of diffeomorphisms in $n$-dimensional differentiable manifold, which will be discussed in another paper.

I

Let $\varphi$ be a diffeomorphism of $n$-dimensional Euclidean space $R^n$ onto itself, which is identical mapping outside some compact set depending upon $\varphi$. The set of all such mappings is denoted by $\text{Diff} (R^n)$. The set $\text{Diff} (R^n)$ becomes a group by introducing the composition of mappings as group operation. Following [2], the topology in $\text{Diff} (R^n)$ is defined such that the necessary and sufficient condition for $\varphi_n \rightarrow \varphi$ is that (i) $\{\varphi_n\}$ are identical mappings outside a certain compact set $K$ and (ii) the functions $\{\varphi_n\}$ and their partial derivatives of each order are uniformly convergent to $\varphi$ and its corresponding partial derivatives respectively in $K$. Let $K^n_\varphi$ be the space of...
all test functions that are $C^\infty$ mappings from $R^*$ to $R^\infty$ with compact supports. $K^*_n$ becomes linear topological space, if it is endowed with Schwartz's topology[9]. In quantum physics, it is reasonable to consider the subgroup $D_0(R^*)$ of all \( \varphi \in \text{Diff}(R^*) \) such that there is a continuous function, \( \varphi_t \in \text{Diff}(R^*), \ t \in [0,1] \), with piecewise continuous derivative \( \varphi_t' \), satisfying \( \varphi_0 = I \) (the identical mapping) and \( \varphi_1 = \varphi \). This connected subgroup of topological group \( \text{Diff}(R^*) \) is an infinite dimensional Lie group. In the linear space $K^*_n$, we introduce a non-associative operation as follows, \( [g, h] = \hbar \partial_t g - g^t \partial_t h \). Then $K^*_n$ becomes the Lie algebra of the Lie group $D_0(R^*)$. The exponential mapping can be expressed as follows. For $g \in K^*_n$, the solution \( \varphi_t \) of the differential equation

\[
\frac{d}{dt} \varphi_t = g \circ \varphi_t
\]

with initial condition $\varphi_0 = I$ is denoted by $\varphi^g_t$. The exponential mapping is $\exp: g \mapsto \varphi^g_t$, i.e. $\exp [tg] = \varphi^g_t$. In quantum physics, when the local current algebra$^{[3-10],[3-4]}$ is investigated, the following infinite dimensional Lie group $\mathfrak{g}$ must also be considered. Let $K^*_n$ be the additive group with usual addition as group operation. We consider $D_0(R^*)$ as a transformation group of $K^*_n$. When $\varphi \in D_0(R^*)$ and $f \in K^*_n$, we define $f \circ \varphi(x) = f(\varphi(x))$. The topological group of all pairs of elements $(f, \varphi)$, $f \in K^*_n$, $\varphi \in D_0(R^*)$ with multiplication defined as $(f_1, \varphi_1) \circ (f_2, \varphi_2) = (f_1 + f_2 \circ \varphi_1, \varphi_2 \circ \varphi_1)$ and with the product topology of $K^*_n \times D_0(R^*)$ is called the cross-product group and is denoted by $\mathfrak{g} = K^*_n \cdot D_0(R^*)$.

In the theory of local current algebra, the continuous unitary representation $W:(f, \varphi) \mapsto W(f, \varphi)$ of the group $\mathfrak{g}$ is considered. Let $U(f) = W(f, I)$, $V(\varphi) = W(0, \varphi)$. It is obvious that the mapping $(f, \varphi) \mapsto W(f, \varphi)$ becomes group representation, if and only if $U$ and $V$ satisfy the following commutation relations

\[
U(f_1)U(f_2) = U(f_1 + f_2), \quad V(\varphi)fU(f) = U(f \circ \varphi)V(\varphi),
\]

\[
V(\varphi_1)V(\varphi_2) = V(\varphi_1 \circ \varphi_2).
\]

When $W$ is a continuous unitary representation, we denote $U(tf) = e^{itf(1)}, V(\exp [tg]) = e^{itg(\varphi)}$. Define the operator-valued generalized functions $\rho(x)$ and $J_j(x)$ by the equations

\[
\rho(f) = \int \rho(x)f(x)dx, \quad J_j(g) = \int g^j(x)J_j(x)dx
\]

respectively. Let $J(x)$ be the operator-valued generalized function whose components are $J_j(x), j = 1, 2, \cdots, n$. From (1.2), the operators $\rho(f)$ and $J(g)$ satisfy the formal commutation relations

\[
[\rho(f_1), \rho(f_2)] = 0, \quad [\rho(f), J(g)] = i\rho(g^t \partial_t f),
\]

\[
\left[ J(g_1), J(g_2) \right] = iJ([g_1, g_2]).
\]

The local current algebra \{\( \rho(x), J(x) \)\} can be considered as an infinite dimensional Lie algebra with operations defined by (1.3). But usually we use (1.2) instead of (1.3), since $\rho(f)$ and $J(g)$ are unbounded operators.

Let $K^*_n$ be dual space of $K^*_n$, i.e. the space of generalized functions, and $\mathfrak{B}$ be the
smallest complete Borel field of subsets of $K_\ast$ such that every linear functional $(F, f)$, $F \in K_\ast$, $f \in K_\ast$ is measurable with respect to $\mathcal{B}$. Let $\mu$ be the $\sigma$-finite non-negative measure on $(K_\ast, \mathcal{B})$. For arbitrary $\varphi \in D_0(K_\ast)$, we define the mapping $\varphi^*$ from $K_\ast$ to $K_\ast$ by the equations

$$(\varphi^* F, f) = (F, f \circ \varphi), \quad F \in K_\ast, \quad f \in K_\ast.$$  

If for every $\varphi \in D_0(K_\ast)$, the measure $\mu(\varphi^*(\cdot))$ is equivalent to the measure $\mu(\cdot)$, then we say that $\mu$ is a quasi-invariant measure. In quantum theory of physics, the following unitary representation[6-4,6-4] of group $\mathfrak{G}$ is considered. Let $\mu$ be a quasi-invariant measure, $\mathcal{H} = L^2(K_\ast, \mathcal{B}, \mu)$. For any $(f, \varphi) \in \mathfrak{G}$, we construct the operator $U(f)$ and $V(\varphi)$ in the unitary representation of $\mathfrak{G}$ as follows:

$$U(f) = e^{itF_0}F, \quad V(\varphi) = \Psi(\varphi^* F)(d\mu(\varphi^* F)/d\mu(F))^{1/2}, \quad \Psi \in \mathcal{H},$$

where $d\mu(\varphi^* F)/d\mu(F)$ is the Radon-Nikodym derivative of measure $\mu(\varphi^*(\cdot))$ with respect to $\mu(\cdot)$. In order to investigate about unitary representation, we must investigate $d\mu(\varphi^* F)/d\mu(F)$.

In the representations (1.4---1.5), there is a vector $\Omega(F) \equiv 1$ in $\mathcal{H}$, which is called ground state and has a special physical significance. In physics, it is often needed that

$$\Omega \in D(J(g)), \quad g \in K_\ast,$$

where $D(J(g))$ denote the domain of self-adjoint operator $J(g)$. The condition (1.6) is equivalent to the existence of $\frac{d}{dt} V(\varphi^*_t)\Omega \bigg|_{t=0}$. From (1.5), we know that it is also equivalent to the existence of function

$$S(g, F) = \frac{d}{dt} \left( \frac{d\mu(\varphi^*_t F)}{d\mu(F)} \right)^{1/2} \bigg|_{t=0}$$

and $S(g, \cdot) \in L^2(K_\ast, \mathcal{B}, \mu)$ simultaneously. It is obvious that $S(g, \cdot) = iJ(g)\Omega$.

II

Now we shall investigate the relation between $S(g, \cdot)$ and the Radon-Nikodym derivative $d\mu(\varphi^* F)/d\mu(F)$ and its general form under some general assumption.

**Lemma 1.** Suppose that (1.6) holds and there is a subspace $S_\ast$ of $K_\ast$ with $\mu$ outer-measure 1 such that $S(g, F)$ is a continuous functional on $S_\ast$ for every $g \in K_\ast$. Then

$$\frac{d\mu(\varphi^*_t F)}{d\mu(F)} = \int_0^1 S(g, \varphi^*_t F)dr, \quad g \in K_\ast, \quad F \in S_\ast.$$

**Proof.** Since the function $f(t, F) = V(\varphi_t^*)\Omega = (d\mu(\varphi_t^* F)/d\mu(F))^{1/2}$ defined on $(-\infty, \infty)$ is a strongly continuous function with vector-values in $L^2(K_\ast, \mathcal{B}, \mu)$ and

$$f(t_1, \cdot) - f(t_2, \cdot) = \int_{t_1}^{t_2} V(\varphi_t^*)iJ(g)\Omega d\tau,$$
we have \( \| f(t_1, \cdot) - f(t_2, \cdot) \| = \| J(g) \| | t_2 - t_1 |. \) Thus there is a sequence of piecewise constant vector-valued functions \( \{ f_n(t, \cdot) \} \) defined on \( t \)-axis such that \( \| f(t, \cdot) - f_n(t, \cdot) \| \leq \frac{1}{n}. \) Let \( m \) be the Lebesgue measure of subsets of \( t \)-interval \( [a, b], \) \( \nu \) be the product measure \( m \times \mu. \) When \( [a, b] \) is a finite interval, \( \{ f_n(\cdot, \cdot) \} \) is obviously a Cauchy sequence in \( L^1(\nu) \) and hence is convergent to a function \( g(\cdot, \cdot) \) in \( L^1(\nu). \) We know that
\[
\int_a^b \int_{\mathbb{R}^n} |g(t, F) - f(t, F)|^2 \, d\mu(F) \, dt = \lim_{n \to \infty} \int_a^b \int_{\mathbb{R}^n} |f_n(t, F) - f(t, F)|^2 \, d\mu(F) \, dt = 0.
\]
Hence for almost all fixed \( t, \) \( g(t, F) \) is equal to \( f(t, F) \) for almost all \( F \) with respect to measure \( \mu. \) Thus \( f(t, F) \) as a function of two variables \( (t, F) \) is measurable and belongs to \( L^1(\nu). \)

From (2.2), we have
\[
f(t, F) - 1 = \int_0^t S(g, q_\tau^* F) f(\tau, F) \, d\tau \tag{2.3}
\]
for almost all \( F. \) We notice that for fixed \( g \) and \( F \in S^*, S(g, q_\tau^* F) \) is a continuous function of \( \tau. \) Thus, from (2.3), \( f(t, F) \) may be considered as an absolutely continuous function of \( t \) for almost all \( F \) and
\[
\frac{d}{dt} f(t, F) = S(g, q_\tau^* F) f(t, F).
\]
However, \( \exp \left\{ -\int_0^t S(g, q_\tau^* F) d\tau \right\} \) is also an absolutely continuous function. Hence
\[
h(t) = f(t, F) \exp \left\{ -\int_0^t S(g, q_\tau^* F) d\tau \right\}
\]
is absolutely continuous and \( h'(t) = 0 \) for almost all \( t. \) From \( h(0) = 1, \) we have \( h(t) = 1, \) i.e. (2.1) holds. The proof is completed.

For more general \( q \in D_0(R^*) \), there are \( g_i \in K^*_b, \) \( q_i \in D_0(R^*) \) such that
\[
\frac{d}{dt} q_i = g_i \circ q_i, \quad 0 < t < 1, \quad q_0 = I, \quad q_1 = q, \tag{2.4}
\]
where \( g_i \) is a piecewise continuous function of \( t, \) and \( q_i \) is a continuous function of \( t. \)

**Lemma 2.** Under the assumption of Lemma 1, if further \( S(g, q_\tau^* F) \) is a continuous functional of two variables \( g \) and \( q \) for every \( F \in S^* \), then for every \( q_i \in D_0(R^*), \) when \( q_i \) is a continuous function of \( t, t \in [0, 1] \) and \( q_i = q_i \circ q_i^{-1} \) is a piecewise continuous function of \( t \) we have
\[
\frac{d\mu(q_\tau^* F)}{d\mu(F)} = \exp \left\{ \frac{1}{2} \int_0^t S(g_\tau, q_\tau^* F) d\tau \right\}, \quad F \in S^*. \tag{2.5}
\]

**Proof.** We construct a sequence of functions \( \{ g^\tau \} \) such that \( g^\tau \in K^*_b, \) \( g^\tau \) is a

\[\hat{q}^{-1}\] is the inverse mapping of \( q_i. \)
piecewise constant function of \( t \in [0, 1] \) and \( \{ g^n \} \) is uniformly convergent to \( g \), for \( t \in [0, 1] \) with respect to the topology of \( K^n \). Let \( \varphi^n \) be a continuous function of \( t \), satisfying \( \frac{d}{dt} \varphi^n = g^n \cdot \varphi^n \) for all \( t \in [0, 1] \) except finite points of \( t \in [0, 1] \) and \( \varphi^n = I \). Then \( \varphi^n \in D_0(R^n) \). We can prove that \( \{ \varphi^n \} \) is uniformly convergent to \( \varphi \) (\( t \in [0, 1] \)) with respect to the topology of \( \text{Diff}(R^n) \). From Lemma 1 and (2.1), we have

\[
\frac{d\mu(\varphi^n \cdot F)}{d\mu(F)} = \exp \left\{ 2 \int_0^t S(g^n, \varphi^n \cdot F) dt \right\}, \quad F \in S^*,
\]

since \( g^n \) is a piecewise constant. Put \( n \to \infty \), thus (2.5) is obtained.

**Corollary.** Under the assumption of Lemma 2, the mapping \( g \mapsto S(g, F) \) is a linear continuous functional, for \( F \in S^* \).

**Proof.** Let \( (\varphi)^* \) be the \( n \)-fold composition of \( \varphi \) with itself, i.e. \( (\varphi)^* = \varphi \circ \varphi \cdot \cdots \circ \varphi \). By Trotter's Theorem\(^{[12]} \), it is easy to prove that

\[
\varphi^{g_1, g_2} = \lim_{n \to \infty} (\varphi^{g_1, g_2})^n,
\]

for \( g_1 \) and \( g_2 \in K^n \). Hence \( V(\varphi^{g_1, g_2}) \Omega = \lim_{n \to \infty} (V(\varphi^{g_1}) V(\varphi^{g_2}))^n \Omega \). From (2.5), we have

\[
\exp \left\{ \int_0^t S(g_1 + g_2, \varphi_t \cdot F) dt \right\} = \lim_{n \to \infty} \exp \left\{ \sum_{j=0}^{n-1} \int_0^t S(g_1, \varphi^{g_1 g_2} (\varphi^{g_1 g_2} \cdot F) dt \right\}
\]

\[
+ \sum_{j=0}^{n-1} \int_0^t S(g_2, \varphi^{g_2 g_2} (\varphi^{g_2 g_2} \cdot F) dt \right\}
\]

\[
= \exp \left\{ \int_0^t (S(g_1, \varphi_t \cdot F) + S(g_2, \varphi_t \cdot F) ) dt \right\}.
\]

We take the derivatives at \( t = 0 \) of terms in the two sides of the above equation, thus obtaining \( S(g_1 + g_2, F) = S(g_1, F) + S(g_2, F) \). Now this corollary follows easily.

III

In the following, we always suppose that the \( \mu \)-outer measure of \( S^* \) is 1, \( S^* \) is an invariant subspace of \( K_1^* \) with respect to \( D_0(R^n) \) and every non-empty open subset of \( S^* \) has positive \( \mu \)-measure. We also suppose that there is a space \( S_1 \) of test functions, i.e. continuous functions, \( S_1 \supset K_1^* \), and a topology \( T \) in \( S_1 \) such that \( (S_1, T) \) is a linear topological space, the relative topology of \( T \) in \( K_1^* \) is weaker than the topology of \( K_1^* \), \( S_1 \) is the completion of \( K_1^* \) with respect to the topology \( T \), \( K_1^* \) is the space of multiplier of \( S_1 \) and \( S_1 \) is invariant under the transformation group \( D_0(R^n) \). Further, the functional in \( S^* \) can be continued from \( K_1^* \) to \( S_1 \) and becomes linearly continuous functional on \( S_1 \), which has such property that function \( f(\cdot) \) locally belongs to \( S_1 \), if \( \partial f(\cdot) \in S_1 \) for all \( j \).

Let \( S_{1m}^*, m = 1, 2, \cdots \) be the symmetric tensor product of \( m \) copies of \( S_1 \) and \( S_{1m}^* = R^m \). We suppose that the pair of spaces \( S^* \) and \( S_1 \) has the following property
(U); for any sequence of functions \( \{a_m\}, \ a_m \in S_{\mathbb{R}^m}^* \), if
\[
\sum_{m=0}^{\infty} \left[ a_m(x_1, \cdots, x_m)F(x_1) \cdots F(x_m)dx_1 \cdots dx_m \right] = 0,
\]
(where the term corresponding to \( m = 0 \) is constant \( a_0 \)) for every \( F \in S^* \), then \( a_m = 0 \), \( m = 0, 1, 2, \cdots \). For example, if \( S^* \) is a linear space and there are sufficient functionals in \( S^* \) such that for any \( \varphi \in S_* \), the fact \( \langle F, \varphi \rangle = 0 \) for every \( F \in S^* \) implies \( \varphi = 0 \), then \( S^* \) and \( S_* \) have the property (U). The above assumptions about the structures of \( S^* \) and \( S_* \) are not very strong.

Now, we suppose that (1.6) holds and \( S(g, F) \) has the following properties. For every \( g \in K_*^n \) and every non-negative integer \( m \), there is \( S(g; x_1, \cdots, x_m) \in S_*^m \) such that
\[
S(g, F) = \sum_{m=0}^{\infty} \left[ S(g; x_1, \cdots, x_m)F(x_1) \cdots F(x_m)dx_1 \cdots dx_m \right], \tag{3.1}
\]
where the term corresponding to \( m = 0 \) is \( S(g) \). Hereinafter, we shall not give any explanation about the term corresponding to \( m = 0 \) again. Since \( S(g, F) \) is linear with respect to \( g \), we know from the property (U) of \( S^* \) and \( S_* \) and (3.1) that \( S(g; x_1, \cdots, x_m) \) is linear with respect to \( g \), for any \( m \). We say that \( S(g, F) \) is an analytic functional if \( g \mapsto S(g; x_1, \cdots, x_m) \) is linearly continuous functional on \( K_*^n \) for every \( x_1, \cdots, x_m; S(g; x_1, \cdots, x_m) \) and \( \partial_{x_1}S(g; x_1, \cdots, x_m) \) are continuous function of \( x_1, \cdots, x_m \) and \( g \) when \( g \) is a continuous function of \( g \); the mapping \( g \mapsto S(g, F) \) from \( [0, 1] \) to \( S_*^m \) is continuous for continuous function \( F \in D_0(R^*) \) of \( g \), and the series
\[
S(g, \varphi^*F) = \sum_{m=0}^{\infty} \left[ S(g, \varphi^*(x_1), \cdots, \varphi^*(x_m))F(x_1) \cdots F(x_m)dx_1 \cdots dx_m \right] \tag{3.2}
\]
is uniformly convergent for \( \tau \in [0, 1] \). Here the restriction about the analyticity of \( S(g, F) \) is somewhat strong in mathematics but is reasonable and acceptable in physics.

If we want to weaken the restriction, then we must establish a more complex theory, since it must use variation. Some other paper is prepared to discuss this problem.

We notice that for every \( \varphi \in D_0(R^*) \), there are many continuous \( \varphi_r \) and \( g_r \) satisfying (2.4) and corresponding to the same \( \varphi \). But the left-side of equation (2.5) only depends upon \( \varphi \) at \( t = 1 \). Since we suppose that every non-empty open set in \( S^* \) has positive measure, we know from the continuity of \( S(g, F) \) that for different \( \varphi_r \) satisfying (2.4), the value of the functional \( \int_0^1 S(g_r, \varphi^*_rF)dr \) at \( F \in S^* \) depends upon \( \varphi_r \) only. Hence
\[
\int_0^1 S(g_r, \varphi_r(x_1), \cdots, \varphi_r(x_m))dx_1 \cdots dx_m, \tag{3.3}
\]
only depends upon \( \varphi \) and is independent of concrete \( \varphi_r \) and \( g_r \) because of (3.2) and the property (U) of spaces \( S^* \) and \( S_* \). Take the variation of \( \varphi_r \) and keep \( \delta \varphi_r = \)
\( \delta \varphi_i = 0 \). We notice that \( \frac{d}{dr} \delta \varphi_i = \delta g^i \circ \varphi_r + \delta \varphi_i \circ (\partial_k g_r \circ \varphi_r) \). Let \( D_r(x) = \det (\partial_i \varphi^* \circ \varphi_r) \). Thus we have \( \frac{d}{dr} \ln D_r(x) = (\partial_i g^i \circ \varphi_r(x)) \) by Liouville's Theorem. If we denote

\[
S(g; x_1, \cdots, x_m) = \int S_f(x, x_1, \cdots, x_m) g^i dx,
\]

then we have

\[
D_r(x)[\partial_i \varphi^* \circ \varphi_r(x)] + g^i(x) \partial_i \varphi_r(x) + S_f \partial_i \varphi^* \circ \varphi_r(x) + S_f \partial_i \varphi^* \circ \varphi_r(x) g^i(x) + S_j \partial_i \varphi^* \circ \varphi_r(x) g^i(x) = 0, \tag{3.4}
\]

where \( S_f = S_f(x, x_1, \cdots, x_m) \), since the variation of (3.3) is zero. Putting \( \tau = 0 \) in (3.4), multiplying \( h^i(x) dx \) to (3.4), summing up \( j \) and integrating with \( x \), we obtain

\[
S([g, h], x_1, \cdots, x_m) + g^i(x) \partial_i S(h; x_1, \cdots, x_m) - h^i(x) \partial_i S(g; x_1, \cdots, x_m) = 0, \tag{3.5}
\]

where \( g^i, x^i_k \) is the \( l \)-th coordinate of \( x_k, g, h \in K^n \) and \( x_1, \cdots, x_m \in \mathbb{R}^n \).

Now, we have to determine the concrete form of \( S(g; x, \cdots, x_m) \). First of all, putting \( m = 0 \) in (3.5), we have \( S([g, h]) = 0 \) for all \( g, h \in K^n \). Let \( S_f(x) \) be the generalized function determined by the equation \( S(g) = \int S_f(x) g^i(x) dx \). The equality \( S([g, h]) = 0 \) implies

\[
S_f(x) \partial_k g^i(x) + \partial_i S_f(x) g^i(x) = 0.
\]

From this, we can prove that \( S_f(x) \equiv 0 \), i.e. \( S(g) = 0 \).

Next, consider the case \( m = 1 \). We shall prove that the support of the generalized function \( S_f(x, x_1) \) as function of \( x \) is the set \( \{x_1\} \) of single element \( x_1 \), where \( x_1 \) is considered as a parameter. If the functions \( h^i(x) \) and \( g^i(x) \) in (3.5) are equal to zero in the neighborhood of \( x_1 \), then (3.5) implies \( S([g, h], x_1) = 0 \). If \( g^i \) approximates \( (x^l - x^l) \partial^i \) for a fixed index \( l \leq n \), then from (3.5) we obtain that

\[
\int [S_f(x, x_1) h^i(x) - S_f(x, x_1) (x^l - x^l) \partial_i h^i(x)] dx = 0.
\]

Thus, when \( x \approx x_1 \), \( S_f(x, x_1) = 0 \), \( h^l \) and \( h^l \) are independent of \( l \approx j \). Hence the form of \( S_f(x, x_1) \) must be

\[
S_f(x, x_1) = p_{lk}(x_1) D^K \delta(x - x_1), \tag{3.6}
\]

where the dummy index \( K \) runs over finite values; \( K = (k_1, \cdots, k_n) \) and \( D^K = \partial_{k_1} \cdots \partial_{k_n} \). Under the assumption of \( S(g; x_1) \), we know that \( p_{lk}(x_1) \in S_\alpha \) and \( \partial_j p_{lk}(x_1) \in S_\alpha \). Now we have to determine \( p_{lk} \). Denote \( |K| = k_1 + \cdots + k_n \). Substituting (3.6) for \( S_f(x, x_1) \) in (3.5), we have

\[
\sum K |(-1)^K[\partial_{k_1} p_{lk}(x) D^K(h^i(x) \partial_i g^i(x) - g^i(x) \partial_i h^i(x))] + g^i(x) \partial_i [\partial_{k_1} p_{lk}(x) D^K h^i(x)] - h^i(x) \partial_i (p_{lk}(x) D^K g^i(x))] = 0. \tag{3.7}
\]
The coefficients of \( h^k(x) \) in (3.7) must be zero. We denote \( P_{jk} \) by \( P_{ij} \) for \( K = (0, 0, \ldots, 0) \). Thus we obtain

\[
g^l(x)(\partial_i P_{io}(x) - \partial_i P_{o0}(x)) - \sum_{|K| > 1} (-1)^{|K|} (\partial_i P_{jk}(x)) D^k g^l(x) = 0. \tag{3.8}
\]

Since \( g^l(x) \) is arbitrary, from (3.8) we obtain that (i) \( \partial_i P_{io}(x) - \partial_i P_{o0}(x) = 0 \), i.e. there is a function \( P(x) \) such that \( P_{io}(x) = \frac{1}{2} \partial_i P(x) \) and (ii) \( P_{jk}(x) = \text{constant} \) (we denote it by \( P_{jk} \)) for \( |K| > 1 \). Hence (3.7) reduces to

\[
\sum_{|K| > 1} (-1)^{|K|} P_{jk} \sum_{K_i + K_j = K, K_i \neq K} C^k_{K_i, K_j} (D^K \partial_i h^l)(D^K \partial_j h^l)
\]

\[
- \sum_{|K'| > 1} (-1)^{|K'|} P_{jk'} \sum_{K_i + K_j = K', K_i \neq K'} C^{k'}_{K_i, K_j} (D^{K'} \partial_i g^l)(D^{K'} \partial_j g^l) = 0, \tag{3.9}
\]

where \( C^k_{K_i, K_j} \) is the coefficient in the expansion. We can prove that the terms in (3.9) for \( |K| > 1 \) and \( |K'| > 1 \) which can be cancelled, are the terms for which \( D^{K_i} = D^{K_i} \partial_j, D^{K_j} = D^{K_j} \partial_i \) only. For fixed indexes \( i_0 \) and \( j_0 \) \((i_0 \neq j_0)\), if we take \( h^l = h^l \partial_i h^l \) and \( g^l = g^l \partial_j g^l \), then we can prove that \( P_{ij} = 0 \) for \( |K| > 1 \) and \( P_{ij} = 0 \) except \( D^K = \partial_m \) for \( |K| = 1 \). When \( D^K = \partial_m \), we denote the corresponding \( P_{ij} \) by \( P_{jm} \). Thus (3.9) reduces to

\[
P_{jm}((\partial_m h^l)(\partial_i h^l) - (\partial_m g^l)(\partial_i g^l)) = 0.
\]

From this equation we can prove \( P_{jm} = -\frac{1}{2} Q_j \delta_{jm} \), where \( Q_j \) is a constant. Thus

\[
S_j(x; z_j) = \frac{1}{2} \delta(x - x_j) \partial_i P(x_i) - \frac{1}{2} Q_j \partial_j \delta(x - x_j).
\]

By the same method, we can prove in general that there are a function \( P(x_1, \ldots, x_m) \) which is symmetric with respect to arguments \( x_1, \ldots, x_m \), and a constant \( Q_m \) such that

\[
S_j(x; x_1, \ldots, x_m) = \frac{1}{2} \delta(x - x_j) \partial_{x_j} P(x_1, \ldots, x_m)
\]

\[
- \frac{1}{2} Q_m \partial_j \sum_{i=1}^{m} \delta(x - x_i).
\]

Thus, we obtain the following expansion

\[
S(g, F) = \frac{1}{2} \sum_{m=1}^{\infty} \left[ g^l(x_i) \partial_{x_i} P(x_1, \ldots, x_m) + Q_m \partial_j \sum_{l=1}^{m} g^l(x_i) \right] F(x_i)
\]

\[
\cdots F(x_m)dx_1 \cdots dx_m. \tag{3.10}
\]

Let \( C_n(\lambda) \) be the partial sum of the power series \( C(\lambda) = \sum_{m=1}^{\infty} Q_m \lambda^{m-1} \) and \( P_N(F) \) be the partial sum of the functional series.
\[ P(F) = \sum_{m=1}^{\infty} \int P(x_1, \ldots, x_m) F(x_1) \cdots F(x_m) dx_1 \cdots dx_m. \]  

(3.11)

Hence
\[ \frac{\delta P_N(F)}{\delta F(x)} = \sum_{m=1}^{N} \left[ \sum_{j=1}^{m} P(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_m) F(x_1) \cdots F(x_m) dx_1 \cdots dx_m, \right. \]

and (3.10) may be rewritten as
\[ S(g, F) = \lim_{N \to \infty} \left\{ \frac{1}{2} C_N \left( \int F dx \right) (\partial_1 F, g') + \frac{1}{2} \left( F \partial_1 \frac{\delta P_N(F)}{\delta F(\cdot)}, g' \right) \right\}. \]  

(3.12)

From (3.5) and (3.12), we obtain
\[ \frac{d\mu(q^*F)}{d\mu(F)} = \lim_{N \to \infty} \exp \left\{ C_N \left( \int F dx \right) (F, \ln D(x, q)) + P_N(q^*F) - P_N(F) \right\}, \]  

(3.13)

where \( D(x, q) = \det (\partial_1 q^p) \). In (3.13), if \( \int F dx \) does not exist, then the function \( C_N(\cdot) \) must be a constant \( C \). If the power series \( C(\lambda) \) is convergent in the range of \( \lambda = \int F dx \), then (3.12) and (3.13) can be rewritten as
\[ \frac{d\mu(q^*F)}{d\mu(F)} = \exp \left\{ C \left( \int F dx \right) (F, \ln D(x, q)) + P(q^*F) - P(F) \right\}, \]  

(3.14)

where
\[ P(q^*F) - P(F) = \sum_{m=1}^{\infty} \int (P(q(x_1), \ldots, q(x_m)) - P(x_1, \ldots, x_m)) F(x_1) \cdots F(x_m) dx_1 \cdots dx_m. \]  

(3.15)

is a convergent series.

We must notice that the integral \( \int P(x_1, \ldots, x_m) F(x_1) \cdots F(x_m) dx_1 \cdots dx_m \) in (3.11) may be divergent, but the integral \( J_m = \int \left( P(q(x_1), \ldots, q(x_m)) - P(x_1, \ldots, x_m) \right) dx_1 \cdots dx_m \) in (3.15) is convergent and the functional \( P_N(q^*F) - P_N(F) = \sum_{m=1}^{N} J_m \) is also well-defined. Thus we obtain the following Theorem.

**Theorem.** If \( S(g, F) \) is an analytical functional, then the Radon-Nikodym derivative \( d\mu(q^*F)/d\mu(F) \) has the expression (3.13). In particular, if the power series \( C(\lambda) \) is convergent, then \( d\mu(q^*F)/d\mu(F) \) has the expression (3.14).

**Example.** Consider the Poisson measure\(^7\). Let \( f(x) \) be a fixed smooth function on \( R^n \), \( 0 < f(x) < \infty \) and \( m \) be a measure on \( R^n \), \( m(U) = \int_U f(x) dx \). If \( \{x_n\} \) is a sequence of points in \( R^n \) without limit point and \( x_m \approx x_{m'} \) for \( m \neq m' \), then we construct the corresponding generalized function \( \sum_{m=1}^{\infty} \delta(x - x_m) \). Let \( S^* \) be the set of all
such generalized functions. If \( \mu_0 \) is the Poisson measure on \( S^* \),

\[
\mu_0 \left( \sum_{n=1}^{\infty} \delta(x - x_n) \in S^* \mid \text{there are } n \text{ points of } \{x_n\} \text{ in } U \right) = \frac{(\lambda m(U))^n}{n!} e^{-\lambda m(U)},
\]

then we have

\[
\frac{d\mu_0(q^*F)}{d\mu_0(F)} = \exp \left( F, \ln \left( D(x, \varphi) \frac{f(q(x))}{f(x)} \right) \right).
\]

If we take some measurable functional \( P(F) \) such that \( 0 < \exp(P(F) - (F, \ln f)) < \infty \) for almost all \( F \) and define a measure \( \mu \) as 

\[
\mu(E) = \int_E \exp(P(F) - (F, \ln f)) d\mu_0(F),
\]

then

\[
\frac{d\mu(q^*F)}{d\mu(F)} = \exp((F, \ln D(x, \varphi)) + P(q^*F) - P(F)) \tag{3.16}
\]

and

\[
S(g, F) = \frac{1}{2} \left( \partial_j F, g^j \right) + \frac{1}{2} \left( \int \partial_j \frac{\delta P(F)}{\delta F(x)}, g^j \right). \tag{3.17}
\]

IV

In [6], Menikoff considered the case when the Hamiltonian \( H \) for a system of particles is of the form \( H = H_K + H_p \), where \( H_K \) and \( H_p \) are the kinetic energy term and potential energy term respectively,

\[
H_K = \frac{1}{8} \int dxK(x) \frac{1}{\rho(x)} K(x),
\]

\[
H_p = \frac{1}{2} \int dy \rho(x)(\rho(y) - \delta(x - y))V(x - y). \tag{4.1}
\]

In (4.1), \( K(x) = \nabla \rho(x) + 2iJ(x) \) and the \( j \)-th component of \( \nabla \) is \( \partial_j \). In the papers [6—8], Menikoff investigated the problem to find the representation space, but he did not solve it. Now we investigate this problem by means of the results in the above sections. We have to find out a quasi-invariant measure \( \mu \) such that in the representation of the current algebra determined by the measure \( \mu \), we can give a definite interpretation of \( H \) such that \( H \) becomes a non-negative self-adjoint operator with ground state as eigenvector of \( H \) corresponding to eigenvalue zero. Here \( H_p \) is not necessary to be the concrete form of (4.1), but \( H_p \) may be a general functional operator of \( \rho(\cdot) \).

Suppose that \( \mu \) satisfies the condition in our Theorem. We use some notations in [3], but we do not use the boldfaced letter for vectors. We consider the case of \( R^* \) instead of \( R^3 \). We also notice that \( \rho(x) = F(x) \). Let \( e_j \) be the unit vector whose \( j \)-th component is 1 and other components are zero. Let

\[
R_j(x, F) = S(e_j \delta(-x), F) + \frac{1}{2} \partial_j F(x). \tag{4.2}
\]

Then from [6], we have
\[ \langle f_1 | H \kappa | f_2 \rangle = \frac{1}{2} \int e^{i(\mathcal{F},_1, \mathcal{F},_1)} \left[ \sum_{i} \frac{R_i(x, F)^2}{F(x)} ight. \\
+ i \partial_j(f_t(x) - f_i(x))(R_j(x, F) + F(x) \partial_j f_i(x) \partial_j f_i(x)) \bigg] dx d\mu(F), \]
and
\[ \langle f_1 | H | f_2 \rangle = \frac{1}{2} \int e^{i(\mathcal{F},_1, \mathcal{F},_1)} \left[ F(x) \partial_j f_i(x) \partial_j f_i(x) dx d\mu(F). \right] \]

Thus the equation \( H \Omega = 0 \) is equivalent to
\[ \left\{ e^{i(\mathcal{F},_1)} \left( \left[ \frac{1}{2} \sum_{i} \frac{R_i(x, F)^2}{F(x)} + \frac{i}{2} R_i(x, F) \partial_j f_i \right] dx + H_p(F) \right) d\mu(F) = 0, \right(4.3 \right) \]

where \( H_p(F) \) is \( H_p \). Now, we consider such case that \( S(g, F) \) has the expression 
\[ (3.15) \] but \( C \left( \int F(x) dx \right) = 1 \). Then \( R_i(x, F) = \frac{1}{2} F(x) \partial_j \frac{\delta P(F)}{\delta F(x)} \). Hence \( (4.3) \) becomes
\[ \left\{ e^{i(\mathcal{F},_1)} \left( \left[ \frac{1}{8} F(x) \sum_{i} \left( \partial_j \frac{\delta P(F)}{\delta F(x)} \right)^2 + \frac{i}{4} F(x) \partial_j f_i (x) \cdot \partial_j \frac{\delta P(F)}{\delta F(x)} \right] dx \right. \right. \]
\[ + \left. \left. H_p(F) \right) d\mu(F) = 0. \right(4.4 \right) \]

For a given \( H_p(F) \), our aim is to find out the functional \( P(F) \) satisfying \( (4.4) \).

We construct a one-parameter family of measurable transformations \( \beta_t \), such that 
\[ \beta_t S^* = S^*, \ beta_p^* = F \] and
\[ \frac{d}{dt} \beta_t F \bigg|_{t=0} = - \partial_j(F R_j(\cdot, F)). \]

From
\[ \frac{d}{dt} \left( e^{i(\mathcal{F},_1)} d\mu(\beta_t F) \right) \bigg|_{t=0} = 0, \]
we have
\[ \left\{ e^{i(\mathcal{F},_1)} \left[ -i(\partial_j(F R_j(\cdot, F)), f) + \frac{d}{dt} \frac{d\mu(\beta_t F)}{d\mu(F)} \right] d\mu(F) = 0. \right(4.5 \right) \]

Thus
\[ \frac{d}{dt} \frac{d\mu(\beta_t F)}{d\mu(F)} \bigg|_{t=0} = 2(S_R(\cdot, F), R_j(\cdot, F)), \] since \( \frac{d}{dt} q_t^* F = - \partial_j(q^t F) \). Hence from 
\[ (4.14-15) \], we obtain
\[ H_p(F) = \left[ \frac{1}{8} F(x) \sum_{i} \left( \partial_j \frac{\delta P(F)}{\delta F(x)} \right)^2 + \frac{1}{2} \frac{\delta P(F)}{\delta F(x)} \nabla^2 F \right] dx, \]
where $\nabla^2 = \sum_{j=1}^{n} \partial^2_j$. For a given $H_p(F)$, we must take $P(F)$ to satisfy the equation (4.6).

Suppose that

$$H_p(F) = V_0 + \sum_{m=1}^{\infty} \int V(x_1, \cdots, x_m) F(x_1) \cdots F(x_m) dx_1 \cdots dx_m,$$

(4.7)

where $V_0$ is a constant, and $V(x_1, \cdots, x_m)$ is a symmetric function. For example, the Hamiltonian in (4.1) is of the form

$$H_p(F) = -\frac{1}{2} V(0) \int P(x) dx + \frac{1}{2} \int V(x_1 - x_2) F(x_1) F(x_2) dx_1 dx_2,$$

where the corresponding $V_0 = 0$, $V(x_1) = -\frac{1}{2} V(0) V(x_1, x_1) = \frac{1}{4} (V(x_1 + x_2) + V(x_1 - x_2))$ and $V(x_1, \cdots, x_m) = 0$ for $m \geq 3$. From (3.11) and (4.6—4.7), we obtain a relation between $P(x_1, \cdots, x_m)$ and $V(x_1, \cdots, x_m)$. Denoting the symmetrization of the function with respect to the arguments by the symbol $S$, we deduce from (4.6) the following equation,

$$\frac{1}{8} S \sum_{m_1 + m_2 = m} \frac{m!}{(m_1 - 1)! (m_2 - 1)!} \partial^2_{m_1} P(x_1, \cdots, x_m) \partial^2_{m_2} P(x_m, x_{m+1}, \cdots, x_m)

+ \frac{1}{2} \sum_{i=1}^{n} \nabla^2 P(x_1, \cdots, x_m) = V(x_1, \cdots, x_m),$$

(4.8)

and $V_0 = 0$. Hence the solution of the problem of representation is determined by the equation (4.8). In general, the solution $\{P(x_1, \cdots, x_m)\}$ is not unique.

REFERENCES