COMPLETE UNITARY INVARIANT FOR SOME
SUBNORMAL OPERATOR

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If \( S \) is a subnormal operator with minimal normal extension \( N \) satisfying
the conditions that (i) \([S^*, S] \in L^1\), (ii) \( \sigma (S) \) is the unit disk and (iii) \( \sigma (N) = \{ z : |z| = 1 \text{ or } z = a_1, \ldots, a_k \} \) then

\[
\text{tr} \left( [S^*, (\lambda I - S)^{-1}][S^*, (\mu I - S)^{-1}] \right) = \frac{n}{\lambda^2 \mu^2} + \sum_{i,j=1}^{k} \frac{\gamma_{ij}}{\lambda \mu (\lambda - a_i)(\mu - a_j)}.
\]

where \( n = \text{index} (S^* - z I) \) for \( z \in \sigma (S) \backslash \sigma (N) \) and \( (\gamma_{ij}) \) is a real symmetric matrix. The set \( \{ n, \gamma_{ij}, i, j = 1, \ldots, k \} \) is a complete unitary invariant for an operator in the class of all irreducible subnormal operators satisfying conditions (i), (ii), (iii) and that there is at least one positive simple eigenvalue of \([S^*, S]\).

1. Let \( \mathcal{H} \) be a complex separable Hilbert space, \( \mathcal{L}(\mathcal{H}) \) be the algebra of linear bounded operators on \( \mathcal{H} \) and let \( \mathcal{L}^1(\mathcal{H}) \) be the trace ideal of \( \mathcal{L}(\mathcal{H}) \). Suppose \( \mathcal{F} \subset \mathcal{L}(\mathcal{H}) \) and for every \( T \in \mathcal{F} \) let \( CU(T) \) be a set of objects determined by \( T \). The set \( CU(\cdot) \) is said to be a complete unitary invariant for the operators in \( \mathcal{F} \) if for every pair of operators \( S \) and \( T \) in \( \mathcal{F} \)

\[
CU(T) = CU(S)
\]

is a necessary and sufficient condition for the existence of a unitary operator \( U \) satisfying \( S = UTU^{-1} \).

It is classical (for example [5]) that the family of measures associated with the spectral resolution is a complete unitary invariant for the normal operator. A unitary invariant for pairs of self-adjoint operators is given in [1]. The mosaic is a complete unitary invariant (cf.[2]) for the hyponormal operator. In [3], [6] there are

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some complete unitary invariants for the subnormal operator. In [4], some complete
unitary invariants for the operator in $B_n(\Omega)$ are given.

In the previous paper [7], the author studies the family $\mathcal{F}$ of all irreducible
subnormal operators $S$ with the minimal normal extension $N$ satisfying the conditions
that (i) $[S^*, S]^\frac{1}{2} \in L^1(\mathcal{H})$, (ii) there is at least a simple positive eigenvalue of $[S^*, S]$, and
(iii) the spectrum $\text{sp} (S)$ is the closure of a simply connected domain $\Sigma$ with boundary
$\text{sp} (N)$ which is a rectifiable Jordan curve. It is proved that $i(S) = \dim \ker (S^* - \overline{w}I)$
is a constant for $w \in \Sigma$. In [7], using the analytic model and the trace formulas for the
product of commutators, we proved that $CU(S) = \{ \text{sp} (S), i(S) \}$ for operator $S$ in $\mathcal{F}$.

For the family $\mathcal{F}$, the simple connectivity of $\Sigma$ plays an very important role.
If $\Sigma = \text{sp} (S) \setminus \text{sp} (N)$ is a multiply connected domain, then the situation becomes maybe
very complicated. In the present paper we study only a very special multiply connected
domain. Let $a_1, \ldots, a_k$ be $k$ distinct points in the unit circle. Let $\mathcal{F}_0(a_1, \ldots, a_k)$ be the
family of all subnormal operators $S$ with the minimal normal extension $N$ satisfying
the conditions that (i) $[S^*, S]^\frac{1}{2} \in L^1(\mathcal{H})$, (ii) $\text{sp} (S)$ is the unit disk and (iii) $\text{sp} (N) = \{ z : |z| = 1 \text{ or } z = a_1, \ldots, a_k \}$. In §3, using analytic model we prove that for $S \in \mathcal{F}_0(a_1, \cdots, a_k)$, there is a real symmetric $k \times k$ matrix $(\gamma_{m,j} (S))$ depending on $S$ such
that

\[
\text{tr} [S^*, (S - \lambda I)^{-1}] [S^*, (S - \mu I)^{-1}] = \frac{i(S)}{\lambda^2 \mu^2} + \sum_{m,j=1}^k \frac{\gamma_{m,j} (S)}{\lambda \mu (\lambda - a_m) (\mu - a_j)}
\]

for $|\lambda| > 1$, $|\mu| > 1$, where $i(S) = \text{index} (S^* - \overline{w}I)$ for $w \in \text{sp} (S) \setminus \text{sp} (N)$.

Let $\mathcal{F}(a_1, \ldots, a_k)$ be the subfamily of all irreducible operators $S$ in $\mathcal{F}_0(a_1, \ldots, a_k)$
satisfying another condition (iv) that there is at least one positive simple eigenvalue
of $[S^*, S]$. Using some formula $A\phi_n = nb_x b_y \Theta_{n-1}$ (cf. [7]) in terms of cyclic cohomol-
ogy, in §4 and §5, we prove that the set $\{ i(S), (\gamma_{m,j} (S)) \}$ is a complete unitary invariant
for operator $S$ in $\mathcal{F}(a_1, \ldots, a_k)$. Therefore the function

\[
\text{tr} [S^*, (S - \lambda I)^{-1}] [S^*, (S - \mu I)^{-1}] \]

is a complete unitary invariant for $S$ in $\mathcal{F}(a_1, \ldots, a_k)$.

In [4], Douglas and Cowen proved that the function $\text{tr} (N_w^* N_w)$ for $w \in \Omega$ is a complete unitary invariant for operators in $B_1(\Omega)$. For some operators $S \in \mathcal{F}(a_1, \ldots, a_k)$.

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\[ F\{a_1, \ldots, a_k\} \] satisfying \( i(S) = 1 \), it can be proved that \( S^* \in B_1(\Omega) \), where \( \Omega = \{z : |z| < 1 \text{ and } z \neq \overline{a}_1, \ldots, \overline{a}_n\} \). The function \( \text{tr} (N_w^*N_w) \) is different from that in this paper, although the author still cannot get a very simple explicit form of the function \( \text{tr} (N_w^*N_w) \).

Using the method in the proof of (16) we may also obtain that

\[
\text{tr} \left( ([S^*, (S - wI)^{-1}])^*[S^*, (S - wI)^{-1}] \right) = \frac{i(S)}{|w|^2(|w|^2 - 1)} + \frac{1}{2} \sum_{m,n} \frac{r_{mn}(S)}{|w - \overline{a}_m|(|w - \overline{a}_n|)} (|w|^2 - 1)
\]

for \( |w| > 1 \). This is also a complete unitary invariant for \( S \). Like \( \text{tr} (N_w^*N_w) \), this is also the trace of the product of an analytic function with its adjoint, but this analytic function is defined on the resolvent set.

2. Let \( S \in F_0(a_1, \ldots, a_k) \). Suppose \( S \) is pure. Let \( M \) be the closure of the range of \([S^*, S]\), \( P_M \) be the orthogonal projection from \( K \) to \( M \), where \( K \) is the Hilbert space, containing \( \mathcal{H} \) as subspace, on which the minimal normal extension \( N \) of \( S \) is defined. Define a \( \mathcal{L}(M) \)-valued measure (cf. [6])

\[ e(\cdot) = P_M E(\cdot) \bigg|_M \]

on the spectrum \( \text{sp} (N) \), where \( E(\cdot) \) is the spectral measure of \( N \). Let \( \Lambda = (S^* M)^* \in \mathcal{L}(M) \). Define the \( \mathcal{L}(M) \)-valued analytic function

\[ \mu(z) = \int_{\text{sp} (N)} \frac{(uI - \Lambda)}{u - z} e(du), \quad z \in \rho(N), \]

where \( \rho(N) \) is the resolvent set of the operator \( N \). Denote

\[ V_j = (a_j I - \Lambda)e\{a_j\}, \quad j = 1, 2, \ldots, k. \]

Then \( \mu(z) = \sum_{j=1}^{k} (a_j - z)^{-1} V_j \) is analytic on \( |z| < 1 \). Since \( \mu(z) = 0 \) for \( z \in \rho(S) \) (cf. [6]), from [8] it is easy to prove that the complex measure

\[ \nu_{x,y}(F) = \left( \int_{F} (uI - \Lambda)e(du)x, y \right) \]
for fixed $x, y$ is absolutely continuous with respect to the Lebesgue measure on $\{u : |u| = 1\}$ and

$$\lim_{|u| \to 1} (\mu(x, y) = 2\pi i \frac{\nu_{x, y}(du)}{du}$$

which is denoted by $f_{x, y}(u)$. Then we have

$$\int \frac{(uI - \Lambda)e(du)x, y}{(\mu_0 - u)(\mu_1 - u)(\lambda - u)} = \frac{1}{2\pi i} \int \frac{f_{x, y}(u)udu}{(\mu_0 - u)(\mu_1 - u)(\lambda u - 1)}$$

$$+ \sum_{j=1}^{k} \frac{(V_jx, y)}{(-\pi a_j + (\mu_0 - a_1)(\mu_1 - a_1)(\lambda - a_1)}$$

for $|\lambda| > 1$ and $|\mu_j| > 1$, $j = 1, 2$. By the calculus of residues, (1) equals $(f(\lambda; \mu_0, \mu_1) x, y)$, where

$$f(\lambda; \mu_0, \mu_1) = \frac{\mu(\lambda^{-1})}{(\mu_0 \lambda - 1)(\mu_1 \lambda - 1)} + \sum_{j=1}^{k} \frac{V_j(\lambda^{-2} - 1)}{(\mu_0 - a_j)(\mu_1 - a_2)(\lambda a_j - 1)(\lambda - a_j).}$$

Let $S_\mu = (\mu I - S)^{-1}$. Define

$$\hat{\phi}_n(\lambda_0, \ldots, \lambda_n; \mu_0, \ldots, \mu_n) = \text{tr} ([S_{\lambda_0}^*, S_{\mu_0}] \cdots [S_{\lambda_n}^*, S_{\mu_n}]),$$

for $|\lambda_j| > 1$ and $|\mu_j| > 1$, $j = 0, 1, \ldots, n$. From Lemma 7 in [7], we have

**Lemma 1.** Let $S \in \mathcal{F}(a_1, \ldots, a_k)$, then

$$\hat{\phi}_n(\lambda_0, \lambda_1, \ldots, \lambda_n; \mu_0, \mu_1, \ldots, \mu_n)$$

$$= \text{tr} (f(\lambda_0; \mu_0, \mu_1)f(\lambda_1; \mu_1, \mu_2) \cdots f(\lambda_n; \mu_n, \mu_0))$$

$$= \text{tr} (f(\mu_0; \lambda_0, \lambda_n)^* f(\mu_1; \lambda_1, \lambda_0)^* \cdots f(\mu_n; \lambda_n, \lambda_{n-1})^*),$$

for $n \geq 1$.

Let $f(z_0, z_1) = \text{tr} (\mu(z_0) \mu(z_1))$ for $z_j \in \Sigma$, $j = 0, 1$. Define

$$f_j(z) = \text{tr} (V_j \mu(z)),$$ $z \in \Sigma$.

Let $c_{ij} = \text{tr} (V_i V_j)$. From $\mu(z)^2 = \mu(z)$, (cf. [6]), it is easy to verify that $V_j^2 = 0$. Thus $c_{jj} = 0$, $j = 1, 2, \ldots, k$. Besides

$$c_{ij} = c_{ji}, \quad i, j = 1, 2, \ldots, k.$$
Putting \( n = 1 \) in (3), we get

\[
\begin{align*}
1 & \sum_{k, l = 0}^{\infty} \sum_{i, j = 1}^{n} c_{ij} \left( a_{i}^{2} - 1 \right) a_{j} \left( 1 - z_{i} \right) (1 - a_{j} w_{j}) (1 - a_{j} z_{i}) (1 - a_{j} w_{l}) (1 - a_{j} z_{l}) \\
\sum_{i, j = 1}^{n} c_{ij} \left| \left( a_{i}^{2} - 1 \right) a_{j} \right| (1 - a_{j} w_{j}) (1 - a_{j} z_{i}) (1 - a_{j} w_{l}) (1 - a_{j} z_{l}) \\
\sum_{i, j = 1}^{n} c_{ij} \left( a_{i}^{2} - 1 \right) a_{j} \left( 1 - z_{i} \right) (1 - a_{j} w_{j}) (1 - a_{j} z_{i}) (1 - a_{j} w_{l}) (1 - a_{j} z_{l}) )
\end{align*}
\]

for \( z_{j}, w_{j} \in \mathbb{C} \), where \( w_{-1} = w_{1} \) and \( z_{-1} = z_{1} \).

3. In this section, we will find out the general form of functions \( f(\cdot, \cdot) \) and \( f_{j}(\cdot), j = 1, 2, \ldots, k \) from (5) for fixed constants \( c_{ij}, i, j = 1, \ldots, k \).

Putting \( z_{0} = z_{1} = z \) and \( w_{0} = w_{1} = w \), in (5) we get

\[
\begin{align*}
2 & \sum_{j = 1}^{k} f_{j}(w) \left( a_{j}^{2} - 1 \right) \left( 1 - a_{j} z \right) (1 - a_{j} w) (1 - a_{j} w) \\
+ & \sum_{i, k = 1}^{n} c_{ij} \left( a_{i}^{2} - 1 \right) \left( a_{j}^{2} - 1 \right) (1 - z w) w_{j} \left( 1 - a_{j} w \right) (1 - a_{i} w) (1 - a_{i} w) (1 - a_{i} w) \\
= & \sum_{j = 1}^{k} f_{j}(w) \left( a_{j}^{2} - 1 \right) \left( 1 - a_{j} w \right) (1 - a_{j} z) (1 - a_{j} z) \\
+ & \sum_{i, k = 1}^{n} c_{ij} \left( a_{i}^{2} - 1 \right) \left( a_{j}^{2} - 1 \right) (1 - z w) w_{j} \left( 1 - a_{j} w \right) (1 - a_{i} w) (1 - a_{i} w) (1 - a_{i} w) \\
\end{align*}
\]

since \( f(z, z) = \text{tr} (\mu(z)) = i(S) \) (cf. [9]). From (6), it is easy to see that \( f_{j}(w) a_{j} - 1 \)

has an analytic continuation which is a rational function with possible simple poles \( a_{1}, \ldots, a_{k} \) and double poles \( \overline{a}_{1}, \ldots, \overline{a}_{n} \). On the other hand,

\[
\begin{align*}
f_{j}(w) \sum_{m = 1}^{k} \frac{c_{jm}}{a_{jm} - w} = \text{tr} \left( V_{j}(\mu(w)) - \sum_{m = 1}^{k} \frac{V_{m}}{a_{m} - w} \right) 
\end{align*}
\]

is regular for \( |w| < 1 \). Therefore there are constants \( f_{jm}, d_{jm}, e_{j} \) with \( f_{jj} = 0 \) such that

\[
f_{j}(w) = \sum_{m = 1}^{k} \left( \frac{c_{jm}}{a_{m} - w} + \frac{f_{jm}}{1 - \overline{a}_{m} w} \right) + e_{j} \]

(7)
Suppose \( a_m \neq 0 \). Multiplying (5) through by \((1 - \bar{a}_m w_0)^2\) and letting \( w_0 \to \bar{a}_m^{-1} \), we get

\[
\lim_{w_0 \to \bar{a}_m^{-1}} f(w_0, w_1)(1 - \bar{a}_m w_0)^2 + \sum_{j=1}^{k} f_{jm} \frac{|a_j|^2 - 1}{(a_j - w_1)(1 - a_j w_1)} \prod_{i=0}^{1} \frac{1 - w_1 z_i}{1 - a_j z_i} = 0, \tag{8}
\]

for \( w_1 \in \Sigma \) and \(|z_i| < 1\), \( i = 1, 2 \). From (8) it is easy to prove that

\[
f_{jm} = 0. \tag{9}
\]

If \( a_m = 0 \), then we change \( e_j \) to \( e_j + f_{jm} + d_{jm} \). Thus we always can assume that (9) holds and (7) reduces to

\[
f_j(w) = \sum_{m=1}^{k} \left( \frac{e_{jm}}{a_m - w} + \frac{d_{jm}}{1 - \bar{a}_m w} \right) + e_j \tag{10}
\]

where \( d_{jm} = 0 \) if \( a_m = 0 \).

Substituting the right hand side of (10) for \( f_j(\cdot) \) into (6), we may show that

\[
f_j(a_j) = 0, \tag{11}
\]

\[
e_{ij}(a_i - a_j)^{-2} = \bar{c}_{ij}(\bar{a}_i - \bar{a}_j)^{-2}, \tag{12}
\]

\[
\bar{a}_j d_{jj} = \bar{d}_{jj} a_j, \tag{13}
\]

and

\[
\bar{a}_j d_{jj} = -e_{ij}(a_i \bar{a}_j - 1)^2(a_i - a_j)^{-2}. \tag{14}
\]

From (10-14), we may prove that

\[
f_j(w) = \sum_{m=1}^{k} \frac{\gamma_{jm}(S) (w - a_j)^2}{2(1 - |a_j|^2)(w - a_m)(1 - \bar{a}_m w)} \tag{15}
\]

where \( \gamma_{ij}(S) = \gamma_{ij} = -2c_{ij}(a_i - a_j)^{-2}(1 - |a_i|^2)(1 - |a_j|^2) \) is real, \( \gamma_{ij} = \gamma_{ji} \), and \( \gamma_{jj} = 2d_{jj} \bar{a}_j \). Putting \( z_0 = z_1 = z \) in (5), then \( f(w_0, w_1) \) is a rational function and is determined by \( \gamma_{ij} \), \( i, j = 1, 2, \ldots, k \) and \( i(S) \), since \( f(z, z) = i(S) \). Thus we have the following

**Lemma 2.** Let \( S \in \mathcal{F}(a_1, \ldots, a_k) \), then the functions \( f(w_0, w_1) \) and \( f_j(w_0), j = 1, 2, \ldots, k \) are rational functions and are determined by the integer \( i(S) \) the real symmetric \( k \times k \) matrix \( (\gamma_{mn}(S)) \).
THEOREM 1. Let \( S \in \mathcal{F}_0(a_1, \cdots, a_k) \), then there is a real symmetric \( k \times k \) matrix \( (\gamma_{mn}(S)) \) such that

\[
\tr \left( [S^*, (S - \lambda I)^{-1}] [S^*, (S - \mu I)^{-1}] \right) = \frac{i(S)}{\lambda_2 \mu_2} + \sum_{m,n=1}^{k} \frac{\gamma_{mn}(S)}{\lambda \mu (\lambda - a_m) (\mu - a_n)} \tag{16}
\]

for \( |\lambda| > 1, |\mu| > 1 \), where \( i(S) = \text{index } (S^* - \bar{z}I) \) for \( |z| < 1 \) and \( z \neq a_1, \cdots, a_k \).

PROOF. Let \( F(w_0, w_1; z_0, z_1) \) be the left-hand side of (5), then

\[
\hat{\phi}_1(\lambda_0, \lambda_1; \mu_0, \mu_1) \prod_{m,n=0}^{1} (\lambda_m \mu_n - 1) = F \left( \frac{1}{\lambda_0}, \frac{1}{\lambda_1}; \frac{1}{\mu_0}, \frac{1}{\mu_1} \right).
\]

If \( 0 \notin \{a_1, \cdots, a_k\} \), then

\[
\lim_{\lambda_j \to \infty} \frac{2 \lambda_1}{\lambda_0 \lambda_1} \hat{\phi}_1(\lambda_0, \lambda_1; \mu_0, \mu_1) = \frac{1}{\mu_0 \mu_1} \lim_{\lambda_j \to \infty} F \left( \frac{1}{\lambda_0}, \frac{1}{\lambda_1}; \frac{1}{\mu_0}, \frac{1}{\mu_1} \right)
= \frac{i(S)}{\mu_0 \mu_1} + \sum_{j=1}^{k} \frac{2 f_j(0)(|a_j|^2 - 1)}{(\mu_0 - a_j)(\mu_1 - a_j)a_j \mu_0 \mu_1}
+ \sum_{j,m=1}^{k} \frac{c_{jm}(|a_j|^2 - 1)(|a_m|^2 - 1)}{(\mu_0 - a_j)(\mu_1 - a_j)(\mu_0 - a_m)(\mu_1 - a_m)a_j a_m}
\]

which proves (16). If \( a_j = 0 \), then the factors \( f_j(0)/a_j \) and \( c_{jm}/a_j \) in (17) should be replaced by 0. Thus (16) also holds even if \( 0 \in \{a_1, \cdots, a_k\} \). Theorem 1 is proved.

4. Let \( A, B \) be two functions, if \( A - B \) is determined by \( \{a_1, \cdots, a_k\} \), \( i(S) \) and \( (\gamma_{mn}(S)) \), then we denote that \( A \approx B \). As in [7], let

\[
\hat{\psi}_n(\lambda_0, \cdots, \lambda_n; \mu_0, \cdots, \mu_n)
= \tr \left( S_{\lambda_0, \mu_0}^* [S_{\lambda_1}^*, S_{\mu_1}] \cdots [S_{\lambda_n}^*, S_{\mu_n}] \right)
\]

Define

\[
f_n(z_0, z_1, \cdots, z_n) = \tr \left( (\mu(z_0) \mu(z_1) \cdots \mu(z_n)) \right), z_j \in \Sigma, j = 0, 1, \cdots, n.
\]

LEMMA 3. Let \( S \in \mathcal{F}_0(a_1, \cdots, a_k) \). If \( f_n \) is a rational function and \( f_n \approx 0 \), then \( \hat{\psi}_n \) is a rational function of \( \lambda_0, \cdots, \lambda_n, \mu_0, \cdots, \mu_n \) and \( \hat{\psi}_n \approx 0 \).
PROOF. Assume $f_n \approx 0$. Then it is easy to see that $\hat{\phi}_j \approx 0$, $j = 0, \ldots, n$.

As in [7], let

$$\tilde{\psi}_n(\lambda_0, \ldots, \lambda_n; \mu_0, \ldots, \mu_n) = \frac{\text{tr} \left( e(du_1)(\bar{u}_1 I - \Lambda^*)e(du_2)(\bar{u}_2 I - \Lambda^*) \cdots e(du_n)(\bar{u}_n I - \Lambda^*) \right)}{(\lambda_0 - \bar{u}_1)(\mu_0 - u_1)(u_1 - v) \prod_{j=1}^{n} (\lambda_j - \bar{u}_j)(\lambda_j - u_{j+1}) \prod_{j=2}^{n}(\mu_j - u_j)}$$  \hspace{1cm} (18)

Then $\hat{\psi}_n \approx \tilde{\psi}_n$, since $\hat{\psi}_n - \tilde{\psi}_n$ is determined by $\hat{\phi}_n$ and $\hat{\phi}_{n-1}$, as it is shown in the proof of Theorem 3 in [7]. Thus

$$\tilde{\psi}_n(\lambda_0, \ldots, \lambda_n; \mu_0, \ldots, \mu_n) \approx \hat{\psi}_n(\mu_0, \mu_n, \ldots, \mu_1; \lambda_0, \lambda_n, \ldots, \lambda_1),$$  \hspace{1cm} (19)

since $\hat{\psi}_n(\lambda_0, \ldots, \lambda_n; \mu_0, \ldots, \mu_n) = \hat{\psi}_n(\mu_0, \mu_n, \ldots, \mu_1; \lambda_0, \lambda_n, \ldots, \lambda_1)$. By the method in §2, we may prove that

$$\int \frac{(uI - \Lambda)e(du)}{\prod_{j=0}^{2}(\mu_j - u)(\lambda - \bar{u})} = \frac{\mu(\lambda - 1)}{\prod_{j=0}^{2}(\lambda - 1)} - \sum_{j=1}^{k} \frac{V_j(1 - |a_j|^{2})}{(\lambda - a_j)(\lambda^{2} - 1) \prod_{j=0}^{2}(\mu_j - a_j)}$$

which is denoted by $f(\lambda; \mu_0, \mu_1, \mu_2)$.

From (18) and (19) we have

$$\text{tr} \left( f(\mu_0; \lambda_0, \lambda_1, \lambda_2) \cdots f(\mu_n; \lambda_{n-1}, \lambda_n) \cdots \right) \approx \text{tr} \left( f(\lambda_1; \mu_2, \mu_1) \cdots f(\lambda_{n-1}; \mu_n, \mu_{n-1}) \right) \int \frac{e(du)}{\lambda_n - \bar{u}} f(\lambda_0; \mu_0, \mu_1, \mu_2) \right).$$  \hspace{1cm} (20)

Multiplying (20) through by $\mu_0$ and then letting $\mu_0 \rightarrow \infty$, we get

$$\text{tr} \left( f(\lambda_1; \mu_2, \mu_1) \cdots f(\lambda_{n-1}; \mu_n, \mu_{n-1}) \right) \int \frac{e(du)}{\lambda_n - \bar{u}} f(\lambda_0; \mu_0, \mu_1, \mu_2) \right) \approx 0.$$  \hspace{1cm} (21)

From (3) it is easy to see that $\hat{\phi}_n(\lambda_0, \ldots, \lambda_n; \mu_0, \ldots, \mu_n)$ is a rational function of $\bar{\lambda}_0, \ldots, \bar{\lambda}_n$ and $\mu_0, \ldots, \mu_n$. Therefore the left hand side of (21) is also a rational function of $\bar{\lambda}_0, \ldots, \bar{\lambda}_n$ and $\mu_0, \ldots, \mu_n$. The simple fractional expansion of the left hand side of (21) as a rational function of $\mu_0, \ldots, \mu_n$ contains a term of

$$\text{tr} \left[ \int e(du) (\bar{\lambda}_n - v)^{-1} \mu(\bar{\lambda}_0^{-1}) \cdots \mu(\bar{\lambda}_{n-1}^{-1}) \right] \prod_{j=0}^{n-1} (\lambda_j - \bar{\lambda}_j)$$

\hspace{1cm} $\prod_{j=1}^{n}(\mu_j \bar{\lambda}_j - 1) \prod_{j=1}^{n}(\lambda_j - \bar{\lambda}_j)$
Therefore \( \text{tr} \left( \int e(dv) (\overline{\lambda}_n - \overline{v})^{-1} \mu(\overline{\lambda}_0^{-1}) \cdots \mu(\overline{\lambda}_{n-1}^{-1}) \right) \) is a rational function of \( \overline{\lambda}_0, \ldots, \overline{\lambda}_n \) and

\[
\text{tr} \left( \int \frac{e(dv)}{\overline{\lambda}_n - \overline{v}} \mu(\frac{1}{\overline{\lambda}_0}) \cdots \mu(\frac{1}{\overline{\lambda}_{n-1}}) \right) \approx 0,
\]

which proves Lemma 3.

5. In this section, we use formulas of cyclic cohomology in [7]. Let

\[
\psi_n(x_0, \ldots, x_n; y_0, \ldots, y_n) = \text{tr} (e^{x_0 S_0} e^{y_0 S_0} [e^{x_1 S_1}, e^{y_1 S_1}] \cdots [e^{x_n S_n}, e^{y_n S_n}])
\]

and

\[
\phi_n(x_0, \ldots, x_n; y_0, \ldots, y_n) = \psi_{n+1}(0, x_0, \ldots, x_n; 0, y_0, \ldots, y_n).
\]

The function \( \Theta_{n-1} \) in [7] is determined by \( \psi_1, \ldots, \psi_{n-1} \) and

\[
A\phi_n = nb_x b_y \Theta_{n-1}, \quad n = 1, 2, \ldots
\]

(cf. Corollary 1 of [7]).

**Lemma 4.** Let \( S \in \mathcal{F}_0(a_0, \ldots, a_k) \). If \( f_n \) is a rational function and \( f_n \approx 0 \), then \( A\hat{\phi}_n \) is a rational function of \( \overline{\lambda}_0, \ldots, \overline{\lambda}_n, \mu_0, \ldots, \mu_n \) and \( A\hat{\phi}_n \approx 0 \). where

\[
(A\hat{\phi}_n)(\lambda_0, \ldots, \lambda_n; \mu_0, \ldots, \mu_n) = \sum_{j=0}^{n} (-1)^{n-j} \hat{\phi}_n(\lambda_0, \ldots, \lambda_n; \mu_j, \ldots, \mu_{j+n})
\]

and \( \mu_j = \mu_{j-n-1} \) for \( j > n \).

**Proof.** The Laplace transform of \( \hat{\Theta}_{n-1} \) is determined by \( \hat{\psi}_1, \ldots, \hat{\psi}_{n-1} \). Thus \( \hat{\Theta}_{n-1} \) is a rational function of \( \overline{\lambda}_0, \ldots, \overline{\lambda}_{n-1} \) and \( \mu_0, \ldots, \mu_{n-1} \) and \( \hat{\Theta}_{n-1} \approx 0 \). Besides \( A\hat{\phi}_n \) is the Laplace transform of \( A\phi_n \). Thus Lemma 4 follows from (22).

**Lemma 5.** If \( n > 2 \), \( f_{n-1} \) is a rational function, and \( f_{n-1} \approx 0 \), then \( f_n \) is a rational function and \( f_n \approx 0 \).

**Proof.** Let \( \lambda_j = \overline{z}_j^{-1}, \ j = 0, \ldots, n \) and

\[
g(z; \mu_0, \mu_1) = \frac{\mu(z)}{(\mu_0 - z)(\mu_1 - z)} + \sum_{j=1}^{k} \frac{V_j(|a_j|^2 - 1)}{(a_j - \mu_0)(a_j - \mu_1)(a_j - z)(1 - \overline{a}_j z)}.
\]
From Lemma 1 and 4, it is easy to see that the function
\[ g_n(\mu_0, \cdots, \mu_n; z_0, \cdots, z_n) = \prod_{j=0}^{n-1} z_j^2 A\phi_n(z_0^{-1}, \cdots, z_n^{-1}; \mu_0, \cdots, \mu_n) \]
\[ = A\text{tr} \left( g(z_0; \mu_1, \mu_1) g(z_1; \mu_1, \mu_2) \cdots g(z_n; \mu_n, \mu_0) \right) \]
is a rational function of \( z_0, \cdots, z_n, \mu_0, \cdots, \mu_n \) and \( g_n \approx 0 \). Consider the simple fractional expansion of \( g_n \) as a function of variable \( \mu_0, \cdots, \mu_n \). The coefficient of
\[ (\mu_0 - z_0)^{-1}(\mu_1 - z_0)^{-1} \prod_{j=2}^{n} (\mu_j - z_j)^{-1} \]
is
\[ - \frac{f_n(z_0, \cdots, z_n)}{\prod_{m=0}^{n} (z_m - z_{m-1})} + \sum_{j=1}^{n} \frac{\text{tr} \left( V_j \mu(z_2) \cdots \mu(z_n) \mu(z_0) \right)(1 - |a_j|^2)}{\prod_{m=0}^{n} (z_m - a_j)(1 - \overline{a_j} z_1)} \]
where \( z_{-1} = z_n \). Therefore the function (23) is a rational function of \( z_0, \cdots, z_n \) and it is determined by \( \{a_1, \cdots, a_k\}, i(S) \) and \( (\gamma_{pq}(S)) \). Thus
\[ f_n(z_0, \cdots, z_n) \approx \sum_{j=1}^{k} \frac{\text{tr} \left( V_j \mu(z_2) \cdots \mu(z_n) \mu(z_0) \right)(z_0 - z_1)(z_1 - z_2)(1 - |a_j|^2)}{\prod_{m=0}^{n} (z_m - a_j)(1 - \overline{a_j} z_1)} \]
(24)
Therefore
\[ \sum_{j=1}^{k} \frac{\text{tr} \left( V_j \mu(z_2) \cdots \mu(z_n) \mu(z_0) \right)(z_0 - z_1)(z_1 - z_2)(1 - |a_j|^2)}{\prod_{m=0}^{n} (z_m - a_j)(1 - \overline{a_j} z_1)} \]
\[ - \sum_{j=1}^{k} \frac{\text{tr} \left( V_j \mu(z_1) \cdots \mu(z_n) \mu(z_0) \right)(z_n - z_0)(z_0 - z_1)(1 - |a_j|^2)}{\prod_{m=1}^{n} (z_m - a_j)(1 - \overline{a_j} z_0)} \approx 0, \]
(25)
where the left hand side is a rational function of \( z_0, \cdots, z_n \). Consider the residue at \( z_0 = a_j \) of the left hand side of (25) as a function of \( z_0 \). Then
\[ \text{tr} \left( V_j \mu(z_1) \cdots \mu(z_n) \right) \approx \frac{\text{tr} \left( V_j \mu(z_2) \cdots \mu(z_n) \mu(a_j) \right)(z_2 - z_1)(1 - |a_j|^2)}{(a_j - z_2)(1 - \overline{a_j} z_1)} \]
(26)
where \( \mu_j(z) = \mu(z) - V_j(a_j - z)^{-1} \), and the difference of both sides of (26) is a rational function. Multiplying both sides of (25) by \( z_1 - a_j \) and letting \( z_1 \rightarrow a_j \), we get
\[ \text{tr} \left( V_j \mu(z_2) \cdots \mu(z_n) \mu(z_0) \right) \approx \frac{\text{tr} \left( V_j \mu_j(a_j) \mu(z_2) \cdots \mu(z_n) \right)(z_n - z_0)(1 - |a_j|^2)}{(a_j - z_n)(1 - \overline{a_j} z_0)}. \]
Thus
\[ \text{tr} \left( V_j \mu(z_1) \cdots \mu(z_n) \right) \approx \text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_1) \cdots \mu(z_{n-1}) \right) \left( z_{n-1} - z_n \right) \left( 1 - |a_j|^2 \right) / \left( a_j - z_{n-1} \right) \left( 1 - \bar{a}_j z_n \right). \] \quad (27)

In (27), letting \( z_n \to a_j \), we get
\[ \text{tr} \left( V_j \mu(z_1) \cdots \mu(z_{n-1}) \hat{\mu}_j(a_j) \right) \approx - \text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_1) \cdots \mu(z_{n-1}) \right). \] \quad (28)

From (26), (27) and (28), we have
\[ \text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_1) \cdots \mu(z_{n-1}) \right) \approx -\text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_1) \cdots \mu(z_{n-1}) \right) \frac{(z_2 - z_1)(a_j - z_{n-1})(1 - \bar{a}_j z_n)}{(z_{n-1} - z_n)(a_j - z_2)(1 - \bar{a}_j z_1)}. \] \quad (29)

Let \( z_n \to a_j \) in (29), then
\[ \text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_1) \cdots \mu(z_{n-1}) \right) \approx \text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_2) \cdots \mu(z_{n-1}) \hat{\mu}_j(a_j) \right) \frac{(z_2 - z_1)(a_j - z_{n-1})(1 - |a_j|^2)}{(a_j - z_2)(1 - \bar{a}_j z_1)}. \]

By mathematical induction we may prove that
\[ \text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_2) \cdots \mu(z_{n-\ell-1}) V_j \hat{\mu}_j(a_j) \ell \right) \approx 0, \]
and
\[ \text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_1) \cdots \mu(z_{n-\ell-1}) \hat{\mu}_j(a_j) \ell \right) \approx \text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_2) \cdots \mu(z_{n-\ell-1}) \hat{\mu}_j(a_j) \ell+1 \right) \frac{(z_2 - z_1)}{(a_j - z_2)(1 - \bar{a}_j z_1)}. \] \quad (30)

for \( \ell = 1, 2, \ldots n - 2 \). Using (30), it is easy to show that there is a constant \( \hat{c}_j \) such that
\[ \text{tr} \left( V_j \hat{\mu}_j(a_j) \mu(z_1) \cdots \mu(z_{n-1}) \right) \approx \hat{c}_j \prod_{\ell=1}^{n-2} \frac{z_{\ell+1} - z_{\ell}}{(a_j - z_{\ell+1})(1 - \bar{a}_j z_{\ell})} \cdot \frac{z_{n-1} - a_j}{1 - \bar{a}_j z_{n-1}}. \] \quad (31)

Using (24) and (31) we may show that there are constants \( c_j, j = 1, 2, \ldots, k \) such that
\[ f_n(z_0, \ldots, z_n) \approx \sum_{j=1}^{k} c_j \prod_{\ell=0}^{n-1} \frac{(z_{\ell+1} - z_{\ell})}{(a_j - z_{\ell})(1 - \bar{a}_j z_{\ell})}. \] \quad (32)

where \( z_{n+1} = z_0 \). Thus
\[ \text{tr} \left( V_j \mu(z_1) \cdots \mu(z_n) \right) \approx \sum_{j=1}^{k} c_j \frac{(z_n - z_j)(z_1 - z_j) \prod_{\ell=1}^{n-1} (z_{\ell+1} - z_{\ell})}{\prod_{\ell=1}^{n} (a_j - z_{\ell})(1 - \bar{a}_j z_{\ell})}. \] \quad (33)
Multiplying (33) through by \((a_\ell - z_m)\) for \(\ell \neq j\), \(m \in \{1, \cdots, n\}\) and letting \(z_m \rightarrow a_\ell\), we get

\[
\text{tr} (V_j \mu(z_1) \cdots \mu(z_{m-1}) V_k \mu(z_{m+1}) \cdots \mu(z_n)) \approx 0
\]

(34)

Therefore

\[
g_n(\mu_0, \cdots, \mu_n; z_0, \cdots, z_n) \approx \sum_{j=1}^{k} c_j A_{\mu} \left\{ \prod_{\ell=0}^{n} \frac{(z_{\ell+1} - z_\ell)}{(\mu_\ell - z_\ell)(\mu_{\ell+1} - z_\ell)} - \prod_{\ell=1}^{m-1} (z_{\ell+1} - z_\ell) \cdot (a_j - z_n)(a_j - z_1) \frac{1}{(\mu_0 - a_j)(\mu_1 - a_j) \prod_{\ell=1}^{n} (\mu_\ell - z_\ell)(\mu_{\ell+1} - z_\ell)} \right. \\
- \left. \prod_{\ell \neq 0, 1} (z_{\ell+1} - z_\ell) \cdot (a_j - z_0)(a_j - z_2) \frac{1}{(\mu_\ell - z_\ell)(\mu_{\ell+1} - z_\ell)(\mu_1 - a_j)(\mu_2 - a_j)} \cdots \prod_{\ell=0}^{n} (z_{\ell+1} - z_\ell) (1 - \bar{a}_j z_\ell)^{-1}, \right.
\]

(35)

where \(A_{\mu}\) means that the operation \(A\) is with respect to the variables \(\mu_0, \cdots, \mu_n\). Multiplying (35) through by \((\mu_0 - a_j)(\mu_1 - a_j)\), and letting \(\mu_0 \rightarrow a_j, \mu_1 \rightarrow a_j\), we get

\[
c_j X \approx 0
\]

where \(X\) is a function which is not identically zero, but is determined by \(a_1, \cdots, a_k\) if \(n > 2\). Therefore \(c_j \approx 0\), which proves Lemma 5.

**THEOREM 2.** The set \(\{i(S), r_{mn}(S)\}\) is a complete unitary invariant for the operator in the class \(\mathcal{F}(a_0, \cdots, a_n)\).

**PROOF.** From Lemma 2, 3 and 5, we have \(f_n \approx 0\) for all \(n\). Thus \(\hat{\psi}_n\) and \(\hat{\phi}_n\) are determined by \(i(S)\) and \((\gamma_{mn}(S))\). By the step 3 in the proof of Theorem 3 in [7], Theorem 2 follows.

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