TRACE FORMULAS AND COMPLETELY UNITARY INVARIANTS FOR SOME $k$-TUPLES OF COMMUTING OPERATORS

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ABSTRACT. Let $A = (A_1, \ldots, A_k)$ be a $k$-tuple of commuting operators. Let $A_1$ be a pure subnormal operator with minimal normal extension $N_1$. Assume that $\text{sp}(A_1) \setminus \text{sp}(N_1)$ is a simply-connected domain with Jordan curve boundary satisfying certain smooth condition. Assume also that $[A_i^*, A_j] \in \mathcal{C}^1$, $i, j = 1, \ldots, k$. Then $A$ is subnormal, and the set consisting of the $\text{sp}(A)$ and the function $Q(\lambda), \lambda \in \sigma_p(A^*)$ is a complete unitary invariant for $A$, where $Q(\lambda_1, \ldots, \lambda_k)$ is a parallel projection to $\ker(A_i^* - \lambda_i I) \cap \{f : (A_i^* - \lambda_i I)^{\ell}f = 0, \text{for some } \ell > 0, j = 2, \ldots, k\}$.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all linear operators on $\mathcal{H}$ and let $\mathcal{L}^1(\mathcal{H})$ be the trace ideal of $\mathcal{L}(\mathcal{H})$. Carey and Pincus [4], [5], [12] established a theory on the principal current and local index of certain class of $k$-tuples of commuting operators $A = (A_1, \ldots, A_k)$ satisfying conditions that (i) $[A_i^*, A_j] \in \mathcal{C}^1$ and (ii) the joint essential spectrum is a system of curves in $\mathbb{C}^n$. The one current $\ell$ is defined by setting

$$\ell(f dh) = \text{itr} [f(A), h(A)]$$

for functions $f, h$ in certain class. The $\ell$ is a real MC cycle in $\mathbb{C}^n$ which is the boundary in the sense of currents in $\mathbb{C}^n$ of a rectifiable current $\Gamma$, the so called principal current. With this $\Gamma$, they also defined the principal index for $A$ at some points in the Taylor $\text{sp}(A)$. Therefore it is worth to determine the explicit form of the current $\ell$ for some class of operator-tuples.

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A \( k \)-tuple \( S = (S_1, \ldots, S_k) \) of operators on \( \mathcal{H} \) is said to be subnormal ([7], [8], [13], [14], [15], [17]) if there is a \( k \)-tuple \( N = (N_1, \ldots, N_k) \) of commuting normal operators on a Hilbert space \( \mathcal{K} \supset \mathcal{H} \) such that

\[
N_j \mathcal{H} \subset \mathcal{H} \quad \text{and} \quad S_j = N_j|_{\mathcal{H}}, j = 1, 2, \ldots, k.
\]

This \( N \) is said to be a normal extension of \( S \). If there is no proper subspace \( \mathcal{H}_1 \subset \mathcal{K} \) satisfying conditions that \( \mathcal{H}_1 \perp \mathcal{H} \) and \( \mathcal{H}_1 \) reduces \( N_j, \quad j = 1, 2, \ldots, k \). Then \( N \) is said to be a m.n.e. (minimal normal extension) of \( S \).

**Theorem A.** (Pincus and Xia [13]). Let \( S = (S_1, \ldots, S_k) \) be a subnormal \( k \)-tuple of operators with minimal normal extension \( N \). If rank \( [S_i^*, S_j] < +\infty \) for \( i = 1, 2, \ldots, k \), then

\[
\text{itr} [f(S), h(S)] = \frac{1}{2\pi} \int_L m_f dh
\]

where \( L \) is the union of a finite collection of closed curves, and is also the union of a finite collection of algebraic arcs such that \( \text{sp}(N) \) is the union of \( L \) and a finite set. Furthermore, \( m(u) \) is an integer valued multiplicity function which is constant on the irreducible piece of \( L \), and \( f, h \in A(\text{sp}(S)) \).

Here, the algebra \( A(\sigma) \) is the algebra of functions on \( \sigma \) generated by all the analytic functions \( f \) and their conjugates \( \overline{f} \) defined on a neighborhood of \( \sigma \).

**Theorem B.** (Pincus and Zheng [14]). Under the condition of Theorem A,

\[
\text{itr} [f(A), g(A)] = \frac{1}{2\pi} \sum_i \int_{C_j} m_j f dh
\]

where \( \{C_j\} \) are the collection of cycles in \( \text{sp}_{\text{ess}}(S) \) and the weights \( m_j \) are the spectral multiplicities of the m.n.e. \( N \) at any regular point \( \zeta \) of \( C_j \).

In Lemma 2 of this paper, we also get the form \( \ell(f dh) \) in Theorem B for the case of a \( k \)-tuple of \( A = (A_1, \ldots, A_k) \) of commuting operators satisfying the conditions that (i) \( [A_i^*, A_j] \in L^1 \), \( i, j = 1, 2, \ldots, k \), and (ii) \( A_1 \) is a pure subnormal operator of which \( \text{sp}(N_1) \setminus \text{sp}(A_1) \) is a simply connected domain with Jorden curve boundary satisfying certain smooth condition, where \( N_1 \) is the m.n.e. of \( A_1 \).

The main part of this paper is studying a complete unitary invariant for certain operator tuples.

Let \( \mathcal{F} \) be a family of \( k \)-tuples \( A = (A_1, \ldots, A_k) \) of operators. Let \( CU(A) \) be a set of objects determined by \( A \in \mathcal{F} \). The object \( CU(A) \) is said to be a complete unitary invariant for the \( k \)-tuple \( A \) in \( \mathcal{F} \), if for \( A = (A_1, \ldots, A_k) \) and \( B = (B_1, \ldots, B_k) \) in \( \mathcal{F} \), \( CU(A) = CU(B) \) is a necessary and sufficient condition for the existence of a unitary operator \( U \) satisfying \( UA_jU^{-1} = B_j, \quad j = 1, 2, \ldots, k \).

Even in the case of single operator \( (k = 1) \), to find a simpler useful complete unitary invariant is one of the basic problems in the operator theory. It is classical that the family of measures associated with the spectral resolution is
a complete unitary invariant for the normal operator (for example [11]), as well as the \(k\)-tuple of commuting normal operators. There are several directions in the theory of complete unitary invariants in the single operator case or \(k\)-tuple of commuting operators case, such as [3], [9], [10].

We are interested in finding some complete unitary invariants associated with some trace formulas or cyclic cohomology. It is well-known that the Pincus principal function is a complete unitary invariant for \(T\), if \(T\) is pure and rank \([T^*, T] = 1\). For the pure subnormal operator with m.n.e. \(N\), if \(\text{sp}(S) \setminus \text{sp}(N)\) is a simply connected domain with boundary \(\text{sp}(N)\) satisfying certain smooth condition and \([S^*, S] \in \mathcal{L}^1\) (in [19], [20] and [21], we assumed that \([S^*, S]^{1/2} \in \mathcal{L}^1\), but this restriction may be removed) then the principal function (index \((S^* - zI), z \in \rho_{\text{ess}}(S)\) is a complete unitary invariant for \(S\). A theorem of Abrahamse and Douglas [1] may be also interpreted in this way with different conditions.

If \(\text{sp}(S) \setminus \text{sp}(N)\) is not simply connected, then the principal function is no longer a complete unitary invariant. In [20] for some special case and then in [21] for the more general case, it proves that if \(S\) is a pure subnormal operator with m.n.e. \(N\) and satisfies conditions that (i) \(\text{sp}(S) \setminus \text{sp}(N)\) is a \(m\)-connected domain with boundary \(\text{sp}(N)\) consisting of \(m\) Jordan curves satisfying certain smooth condition or an external Jordan curve satisfying certain smooth condition and \(m - 1 \) points and (ii) \([S^*, S] \in \mathcal{L}^1\), then the function (which is related to cyclic cocycle)

\[
\text{tr} \left[ (S^* - \lambda_1 I)^{-1}, (S - \mu_1 I)^{-1} \right]((S^* - \lambda_0 I)^{-1}, (S - \mu_0 I)^{-1}]
\]

\(\lambda_i, \mu_j \in \rho(S)\) is a complete unitary invariant for \(S\). In the Remark of §3, we will show that \(\text{tr} \left( Q(\lambda, S)Q(\mu, S), \lambda, \mu \in \text{sp}(S) \setminus \text{sp}(N) \right)\) is a complete unitary invariant for \(S\), where \(Q(\lambda, S)\) is a certain parallel projection to the eigenspace \(\text{ker}(S^* - \lambda I)\).

Assume that \(S = (S_1, \ldots, S_k)\) is a pure subnormal tuple of operators with m.n.e. \(N = (N_1, \ldots, N_k)\) satisfying \([S_j^*, S_j] \in \mathcal{L}^1, j = 1, 2, \ldots, k\). In [22] it is proved that if \(\text{sp}(S_j) \setminus \text{sp}(N_j)\) is a simply connected domain with Jordan curve boundary \(\text{sp}(N_j)\) satisfying certain smooth condition for \(j = 1, 2, \ldots, k\), then

\[
\text{tr} \left[ f_0(A), g_0(A) \right] = \frac{1}{2\pi} \int_{\text{sp}(N)} m(u)f_0(u)dg_0(u)
\]

and

\[
\text{tr} \left[ f_0(A), g_0(A) \right] \ldots [f_n(A), g_n(A)] =
\]

\[
\left[ \frac{1}{(2\pi)^n+1} \int_{\text{sp}(N)^n+1} m(u^{(0)}, \ldots, u^{(n)}) \prod_{m=0}^n f_m(u^{(m)})(u^{(m)} - g_m(u^{(m-1)})) du_1^{(m)} \right]
\]

where \(u^{(j)} = (u_1^{(j)}, \ldots, u_k^{(j)}), u^{(-1)} = u^{(n)}\) for \(n \geq 1\) and \(f_j, g_j \in \mathcal{A}(\text{sp}(S))\). In [22], it is also proved that the sequence of functions \(m(u^{(0)}, \ldots, u^{(n)})\) (which are
related to cyclic n-cocyle), \( n = 0, 1, \ldots \) is a complete unitary invariant for \( S \).

In the present paper, we continue the study in [22]. We consider the class of \( k \)-tuples of commuting operators \( A = (A_1, \ldots, A_k) \) on \( \mathcal{H} \) satisfying the conditions that (i) \( [A_j^*, A_j] \in \mathcal{L}_1, j = 1, 2, \ldots, k \) and (ii) \( A_1 \) is a pure subnormal operator with m.n.e \( N_1 \) such that \( \text{sp}(N_1) \) is a Jordan curve satisfying certain smooth condition. In Theorem 1, it proves that the operator tuples \( A \) satisfying these conditions are subnormal, and the set consisting of the Taylor spectrum \( \text{sp}(A) \) and the sequence of \( n \)-points function

\[
(3) \quad \text{tr} \left( Q(\lambda^{(1)}; A) \cdots Q(\lambda^{(n)}, A) \right), \lambda^{(n)} \in \sigma_p(A^*)^*
\]

(which is related to cyclic \( n \)-1-cocycle) is a complete unitary invariant for \( A \) where \( Q(\lambda; A) \) is the parallel projection to the subspace \( \ker(A_1^* - \bar{\lambda} I) \cap \{ f : (A_j^* - \bar{\lambda}_j I)^{\ell_j} f = 0, \ell_j > 0, j = 2, \cdots, k \} \). Or, the set consisting of the \( \text{sp}(A) \) and the function \( Q(\cdot; A) \) is a complete unitary invariant.

In §4 (cf. Corollary 1), we determine the form of the irreducible \( k \)-tuple of commuting operators \( H = (H_1, H_2, \cdots, H_k) \) satisfying the conditions that (i) \( \text{rank} [H^*_j, H_j] = 1 \), (ii) \( [H^*_j, H_j] \) is compact, \( j = 2, \ldots, k \) and (iii) the pairs \( (H_1, H_j), j = 2, \ldots, k \) are hyponormal. Under these conditions, either (i) \( H_1 \) is a linear combination of identity and unilateral shift with multiplicity one and \( \mathbb{H} \) is subnormal, or (ii) \( H_1 \) is a non-subnormal hyponormal operator and there are \( \alpha_j, \beta_j \in C \) such that \( H_j = \alpha_j H_1 + \beta_j I, j = 2, \cdots k \). This work is closely related to the Theorem 1 of [18].

2. A Lemma

Let \( H^p_D(\mathbb{T}), p \geq 1 \) be the Hardy space of \( D \)-valued analytic functions on the open unit disk with boundary function on \( \mathbb{T} \), where \( \mathbb{D} \) is an auxiliary Hilbert space. Let \( H^p(\mathbb{T}) = H^p_D(\mathbb{T}) \).

**Lemma 1.** Let \( T(\cdot) \in H^\infty_{L^1} D(\mathbb{T}) \). Let \( T \) be the operator

\[
(Tf)(\zeta) = T(\zeta)f(\zeta), \quad f \in H^2_D(\mathbb{T}).
\]

If \( [T^*, T] \in \mathcal{L}_1 \), then \( T(\zeta) \) is a normal operator on \( \mathbb{D} \) for almost every \( \zeta \in \mathbb{T} \).

**Proof.** There is an orthonormal sequence \( \{h_j\} \) in \( H^2_D(\mathbb{T}) \) and a sequence of real numbers \( \{\lambda_j\} \), satisfying \( \sum |\lambda_j| < \infty \), such that

\[
(4) \quad ([T^*, T]f)(z) = \sum \lambda_j(f, h_j)h_j(z)
\]

for \( |z| < 1 \). In (4), let \( f(z) = \alpha(z - \lambda)^{-1}, \alpha \in \mathbb{D}, \ |\lambda| > 1 \), then (4) implies that

\[
(F(z) - F(\lambda))\alpha - T(z)(H(z) - H(\lambda))\alpha = (\lambda - z)\lambda^{-1} \sum \lambda_j(\alpha, h_j(\lambda^{-1}))h_j(z)
\]

for \( |z| < 1 \) where

\[
F(z) = \frac{1}{2\pi i} \int \frac{T^*(\zeta)T(\zeta)d\zeta}{\zeta - z}, \quad H(z) = \frac{1}{2\pi i} \int \frac{T(\zeta)^*d\zeta}{\zeta - z},
\]
since $T^*f = P(T^*(\cdot)f(\cdot))$ where $P$ is the projection from $L^2$ to $H^2$.

For almost all $u \in T$, in (5) let $z \to u$ and $\lambda \to u$. Then

$$([T^*(u), T(u)]\alpha, \beta) = 0$$

for $\alpha, \beta \in D$ by Plemelj formula, since

$$\lim_{\lambda \to u} \sum \lambda_j(\alpha, h_j(\lambda^{-1}))(h_j(z), \beta) = \sum \lambda_j(\alpha, h_j(u))(h_j(u), \beta)$$

is finite for almost all $u \in T$.

### 3. Complete Unitary Invariant

For a $k$-tuple of operators $A = (A_1, \ldots, A_k)$, let $A^* = (A_1^*, \ldots, A_k^*)$ and $\sigma_p(A) = \{(\lambda_1, \ldots, \lambda_k) : \text{there is } f \neq 0 \text{ such that } A_jf = \lambda_jf\}$. For any set $\sigma \subset C^k$, let $\sigma^* = \{((\lambda_1, \ldots, \lambda_k) : (\lambda_1, \ldots, \lambda_k) \in \sigma\}$. Suppose $A = (A_1, \ldots, A_k)$ is a $k$-tuple of commuting operators on $H$, $\lambda = (\lambda_1, \ldots, \lambda_k) \in \sigma_p(A^*)^*$ and $\dim \ker(\lambda_1 - A_1^*) < +\infty$. Let $P_{D(\lambda_1)}$ be the orthonormal projection from $H$ to $D(\lambda_1) = \ker (\lambda_1 I - A_1^*)$. Let

$$Q(\lambda; A) = \prod_{j=2}^k \frac{1}{2\pi i} \int_{|\zeta - \lambda_j| = \delta} (\zeta I - A_j^*)^{-1} d\zeta P_{D(\lambda_1)},$$

where $\delta$ is a positive number such that there is no eigenvalue of $A_j^*$ in $0 < |\zeta - \lambda_j| \leq \delta$. Then $Q(\lambda; A)$ is a parallel projection from $H$ onto $D(\lambda_1) = \ker (\lambda_1 I - A_1^*)$. Let

$$Q(\lambda(1); A) \ldots Q(\lambda(n); A), \quad \lambda^{(n)} \in \sigma_p(A^*)^*$$

is a complete unitary invariant for $A$.

A Jordan curve $\gamma$ is said to satisfy the condition (CBI) if the univalent analytic mapping function $\phi(\cdot)$ from the interior domain of $\gamma$ onto the open unit disk satisfying the condition that $\phi'(\cdot)$ and $\phi'(\cdot)^{-1}$ are bounded.

**Theorem 1.** Let $A = (A_1, \ldots, A_k)$ be a $k$-tuple of commuting operators on $H$. Assume that $A_1$ is a pure subnormal operator with minimal normal extension $N_1$ on $K \supset H$. Suppose $\text{sp}(N_1)$ is a Jordan curve satisfying condition (CBI) and is the boundary of $\text{sp}(A_1)$. Assume also that $[A_i^*, A_i] \in L^1$, $i = 1, \ldots, k$. Then $A$ is subnormal, and the set consisting of the $\text{sp}(A)$ and the sequence of $n$-points function

$$\text{tr} (Q(\lambda^{(1)}; A) \ldots Q(\lambda^{(n)}; A)), \quad \lambda^{(n)} \in \sigma_p(A^*)^*$$

is a complete unitary invariant for $A$.

Or, the set consisting of the $\text{sp}(A)$ and the function $Q(\cdot; A)$ is a complete unitary invariant.

**Proof.** By the method in the proof of Theorem 3 in [19], where the condition $[A_i^*, A_i]^{1/2} \in L^1$ can be changed to $[A_i^*, A_i] \in L^1$, or by the Theorem 1 in [1], we may prove that there are a Hilbert space $D$ and a unitary operator $W$ from $K$ onto $L^2_{\mathcal{D}}(T)$ such that $W^{*} = H^2_{\mathcal{D}}(T)$ and

$$(WN_{1}W^{-1}f)(z) = A_1(z)f(z), \quad f \in L^2_{\mathcal{D}}(T)$$
where the function $A_1(z)$ is the conformal mapping from the open unit $D$ disk onto $\text{sp}(A_1) \setminus \text{sp}(N_1)$. For the simplicity of notation, we assume that $\mathcal{K} = \mathcal{L}_D(T)$, $\mathcal{H} = \mathcal{H}_D^2(T)$ and $W = I$.

Let $U_+$ be the unilateral shift, $(U_+ f)(z) = z f(z)$, $f \in \mathcal{H}_D^2(T)$. Then $A_1 = A_1(U_+)$. It is easy to see that

$$\text{index } (A_1^* - A_1(z)I) = \text{index } (U_+^* - zI), \text{ for } |z| < 1.$$ 

Since the mosaic (cf [16]) of $A_1$ is compact, we have

$$\text{index } (A_1^* - A_1(z)I) < +\infty, \text{ for } |z| < 1.$$ 

Therefore $\dim D = \text{index } (U_+ - zI) < +\infty$. There are functions $A_j(\cdot) \in \mathcal{H}_D^\infty(D(T))$ such that

$$(8) \quad (A_j f)(z) = A_j(z) f(z), \quad j = 2, \ldots, k.$$ 

By Lemma 1, $[A_j(z)^*, A_j(z)] = 0$ for a.e. $z \in T$. Define

$$(N_j f)(z) = A_j(z) f(z), \quad j = 2, \ldots, k, \quad f \in \mathcal{L}_D^2(T).$$

Then $N = (N_1, \ldots, N_k)$ is a normal extension of $A$ on $L_2^2(T)$. Thus $A$ is subnormal.

By the method of proving the Theorem 1 and the lemma 10 of [22], we may prove the following.

**Lemma 2.** Assume $A = (A_1, \ldots, A_k)$ satisfies the condition of this theorem and $f, g \in \mathcal{A}(\text{sp}(A))$. Then there is an integer valued measurable function $m(\cdot)$ which is the spectral multiplicity function of $N$ at the regular points of $\text{sp}(N)$ such that

$$(9) \quad \text{tr } [f(A), g(A)] = \frac{1}{2\pi i} \int_{\text{sp}(N)} m(u) f(u) dg(u).$$

For $n \geq 1$, if $f_j, g_j \in \mathcal{A}(\text{sp}(A))$ then there exists bounded measurable function $m(u^{(0)}, \ldots, u^{(n)})$ on $\text{sp}(N)^{n+1}$ satisfying

$$(10) \quad \frac{1}{(2\pi i)^{n+1}} \int_{\text{sp}(N)^{n+1}} m_n(u^{(0)}, \ldots, u^{(n)}) \prod_{j=0}^n f_j(u^{(j)}) g_j(u^{(j)} - u^{(j-1)}) \frac{u^{(j)} - u^{(j-1)}}{u^{(j)} - u^{(j-1)}} du^{(j)}$$

where $u^{(j)} = (u_1^{(j)}, \ldots, u_k^{(j)})$, $u^{(-1)} = u^{(n)}$.

Furthermore, $\text{tr } [f_0(A), g_0(A)][f_1(A), g_1(A)] \ldots [f_n(A), g_n(A)]$ is also an integral expressed by functions $f_j, g_j, j = 0, \ldots, n$ and $m_n(u^{(0)}, \ldots, u^{(n)})$.

The integral in (10) is a multiple singular integral of Cauchy's type. Let $A_j(\zeta) = \sum \lambda_{jn}(\zeta) P_{jn}(\zeta)$ be the spectral resolution of the normal operator $A_j(\zeta)$,
\[ \zeta \in T, \text{ where } \lambda_{j_m}(\zeta) \text{ and } P_{j_m}(\zeta) \text{ are eigenvalues and spectral projections of } A_j(\zeta) \text{ respectively. Then for almost all } (A_1(\zeta), \lambda_{2_m2}(\zeta), \ldots, \lambda_{km_k}(\zeta)) \in \text{sp}(\mathbb{N}), \text{ define} \]

\[ P(A_1(\zeta), \ldots, \lambda_{km_k}(\zeta)) = P_{2m_2}(\zeta) \ldots P_{km_k}(\zeta). \]

Then

\[ m_n(u^{(0)}, \ldots, u^{(n)}) = \text{tr} \left( P(u^{(0)}) \ldots P(u^{(n)}) \right). \]

By the method of proving Theorem 2 in [22], we may prove the following:

**Lemma 3.** Under the condition of the Theorem 1, \( \{\text{sp}(A), m_n, n = 0, 1, \ldots \} \) is a complete unitary invariant for the \( k \)-tuple \( A \) of operators.

Now, we only have to prove that the function (7) determines \( \{m, m_n : n = 1, 2, \ldots \} \).

For \( z_1 \in D \), let \( \lambda_1 = A_1(z_1) \) and \( \mathcal{D}(\lambda_1) = \ker(A_1^* - \bar{\lambda}_1 I). \) It is easy to see that

\[ \mathcal{D}(\lambda_1) = \ker(U_+^* - \bar{z}_1 I) = \left\{ \frac{\alpha}{1 - \bar{z}_1(\cdot)} ; \alpha \in \mathcal{D} \right\} \]

is of finite dimension, since \( A_1 = A_1(U_+). \)

On the other hand

\[ A_j^* \frac{\alpha}{1 - \bar{z}_1(\cdot)} = \frac{1}{2\pi i} \int \frac{A_j(\zeta)^* d\zeta}{(1 - \bar{z}_1(\zeta)(\zeta - (\cdot)))^\alpha} = A_j(z_1)^* \frac{\alpha}{1 - \bar{z}_1(\cdot)}. \]

Let \( W(z_1) \) be the operator from \( \mathcal{D}(\lambda_1) \) to \( \mathcal{D} \) defined by

\[ W(z_1) \left( \frac{1 - |z_1|^2}{1 - \bar{z}_1(\cdot)} \right)^{1/2} = \alpha \]

Then \( W(z_1) \) is a unitary operator from \( \mathcal{D}(\lambda_1) \) onto \( \mathcal{D} \) and

\[ W(z_1)A_j^*W(z_1)^{-1} = A_j(z_1)^*. \]

For \( \lambda = (\lambda_1, \ldots, \lambda_n) \), define an operator

\[ E(\lambda) = \prod_{j=2}^{k} \frac{1}{2\pi i} \int_{|\zeta - \bar{\lambda}_j| = \delta} \left( \zeta I - A_j(z_1)^* \right)^{-1} d\zeta, \]

where \( \delta \) is a small positive number satisfying the condition that there is no eigenvalue of \( A_j(z_1)^* \) in \( \{\zeta \in \mathbb{C} : 0 < |\zeta - \bar{\lambda}_j| \leq \delta\} \). Then \( E(\lambda) \) is the parallel projection from \( \mathcal{D} \) to the intersection of root spaces

\[ \{x \in \mathcal{D} : (A_j(z_1)^* - \bar{\lambda}_j I)^{\ell_j} x = 0, \text{ for some } \ell_j > 0, j = 2, \ldots, k\}. \]

Thus

\[ Q(\lambda) = W(z_1)^{-1}E(\lambda)W(z_1)P_{\mathcal{D}(\lambda_1)} \]

where \( Q(\lambda) = Q(\lambda; A) \) and \( P_{\mathcal{D}(\lambda_1)} \) is the orthogonal projection from \( \mathcal{H}^2_2(T) \) to \( \mathcal{D}(\lambda_1) \).
LEMMA 4. Let $\lambda^{(j)} = (A_1(z^{(j)}), \lambda_2^{(j)}, \ldots, \lambda_k^{(j)})$, then

$$\text{tr} \left( Q(\lambda^{(m)}) \cdots Q(\lambda^{(1)}) \right) = \prod_{j=1}^{m} \frac{(1 - |z^{(j)}|^2)^{1/2}}{1 - |z^{(1)}|z^{(2)}} \text{tr}_{D}(E(\lambda^{(m)}) \cdots E(\lambda^{(1)}))$$

where $z^{(m+1)} = z^{(1)}$ and $z^{(j)} \in D$.

PROOF. Let $\{e_j\}$ be an orthonormal basis for $D$. It is easy to see that

$$P_{\mu^{(2)}} W(z^{(1)})^{-1} a = \left(1 - |z^{(1)}|^2\right)^{1/2} W(z^{(2)})^{-1} a, \ a \in D,$$

where $\mu^{(2)} = A_1(z^{(2)})$. Thus by (14) we have

$$\text{tr} \left( Q(\lambda^{(m)}) \cdots Q(\lambda^{(1)}) \right) = \sum_{j} (Q(\lambda^{(1)})W(z^{(1)})^{-1} e_j, Q(\lambda^{(2)}) \cdots Q(\lambda^{(m)}) W(z^{(1)})^{-1} e_j)$$

$$= \sum_{j} (Q(\lambda^{(2)})W(z^{(1)})^{-1} E(\lambda^{(1)}) e_j, Q(\lambda^{(3)}) \cdots Q(\lambda^{(m)}) W(z^{(1)})^{-1} e_j)$$

$$= \frac{(1 - |z^{(1)}|^2)^{1/2}}{1 - |z^{(1)}|z^{(2)}} \sum_{j} (Q(\lambda^{(3)}) W(z^{(2)})^{-1} E(\lambda^{(2)}) E(\lambda^{(1)}) e_j, Q(\lambda^{(4)}) \cdots Q(\lambda^{(m)}) W(z^{(1)})^{-1} e_j).$$

continuing this process, we may prove Lemma 4.

Let $B_n = (B_{1n}, \ldots, B_{mn}), n = 1, 2, \ldots$ be a sequence of $m$-tuples of commuting operators on a finite dimensional inner product space $D$ satisfying $\lim_{n \to \infty} B_{jn} = B_j$, $j = 1, 2, \ldots, m$, where $B_j, j = 1, 2, \ldots, m$ are commuting normal operators with common eigenvalue 0. For every $\lambda = (\lambda_1, \ldots, \lambda_m)$, let

$$E_n(\lambda) = \prod_{j=1}^{m} \frac{1}{2\pi i} \int_{|\zeta - \lambda_j| \leq \delta} (\lambda_I - B_{jn})^{-1} d\lambda$$

be the parallel projection to $\{x \in D : (B_{jn} - \lambda_j I) e_{j'}x = 0, j' > 0, j = 1, \ldots, m\}$, where $\delta$ is a small positive number such that there is no eigenvalue of $B_{jn}$ in $\{\zeta : 0 < |\zeta - \lambda_j| \leq \delta\}$. Let $E$ be the orthogonal projection from $D$ to $\bigcap_{j=1}^{m} \ker B_j$.

LEMMA 5. There is a $\epsilon > 0$ such that

$$\lim_{n \to \infty} \sum_{|\lambda_j| < \epsilon} E_n(\lambda_1, \ldots, \lambda_m) = E.$$

PROOF. Let $\epsilon$ be a positive number satisfying the conditions that (i) $\{\zeta \in C : 0 < |\zeta| < \epsilon\} \cap \sigma_p(B_j) = \emptyset$, $j = 1, \ldots, m$ and (ii) $\{\zeta \in C : |\zeta| = \epsilon\} \cap \sigma_p(B_{jn}) = \emptyset$, $j = 1, 2, \ldots, m; n = 1, 2, \ldots$.

It is obvious that this $\epsilon$ exists. It is easy to calculate that

$$\sum_{|\lambda_j| < \epsilon} E_n(\lambda_1, \ldots, \lambda_m) = \prod_{j=1}^{m} \frac{1}{2\pi i} \int_{|\zeta| = \epsilon} (\zeta I - B_{jn})^{-1} d\zeta.$$
Letting $n \to \infty$ in (16), we have

$$\lim_{n \to \infty} \sum_{|\lambda_j| < \epsilon} E_n(\lambda_1, \ldots, \lambda_m) = \prod_{j=1}^{m} \frac{1}{2\pi i} \int_{|\zeta| = \epsilon} (\zeta I - B_j)^{-1} d\zeta$$

which proves (18).

From Lemma 4 and 5, we obtain the following.

**Lemma 6.** Under the conditions of the Theorem 1, the function

$$m(u(0), u(1), \ldots, u(p)) = \lim_{\epsilon \to 0} \lim_{r \to 1^-} \prod_{j=0}^{p} \frac{(1 - r^2 z(j)) z(j+1)}{(1 - r^2)^{p+1}} \sum_{\eta^{(\epsilon)} \in V(\epsilon, r, u^{(\epsilon)})} \text{tr} (Q(\eta^{(0)}) \ldots Q(\eta^{(p)})).$$

where $z^{(j)} = A_1^{-1}(u^{(j)}_1), u^{(p+1)} = u^{(0)}$ and

$$V(\epsilon, r, u) = \{A_1(r A_1^{-1}(u_1)), \eta_2, \ldots, \eta_k) \in \sigma_p(A^*)^* : |\eta_j - u_j| < \epsilon, j = 2, \ldots, k\}.$$

From Lemma 3 and 6, it proves Theorem 1.

**Remark.** In [20] and [21], it studies the complete unitary invariant for the pure subnormal operator $S$ on $\mathcal{H}$ with the m.n.e. $N$ on $\mathcal{K} \supset \mathcal{H}$ satisfying the conditions that (i) $\text{sp}(S) \setminus \text{sp}(N)$ is a $k$-connected domain with boundary $\text{sp}(N)$ consisting of $k$ Jordan curves satisfying smooth condition (CBI) or an external Jordan curve satisfying smooth condition (CBI) and $k - 1$ points and (ii) $[S^*, S]^{1/2} \in \mathcal{L}^1$. We may make small changes in the proofs of the theorems in [20] and [21], such that the most argument in [20] and [21] remain true if we release the condition $[S^*, S]^{1/2} \in \mathcal{L}^1$ to $[S^*, S] \in \mathcal{L}$. Besides, it is easy to see that the function

$$\text{tr} (\mu(z)^* \mu(w)^*), \quad z, w \in \text{sp}(S) \setminus \text{sp}(N)$$

is a complete unitary invariant for this $S$, where

$$\mu(z)^* = P_M (N^* - zI)^{-1} (N^* - S^*) |_{M}$$

$M = \text{closure of } [S^*, S] \mathcal{H}$ and $P_M$ is the projection from $\mathcal{K}$ to $M$. By means of the analytic model of subnormal operator, we may prove that

$$\text{tr} (\mu(z)^* \mu(w)^*) = \text{tr} (Q(z; S)Q(w; S))$$

where $Q(z, S)$ is the parallel projection from $\mathcal{H}$ to the eigenspace $\text{ker}(S^* - zI)$ satisfying the condition that

$$\text{ker} Q(z, S) = \text{ker} P_M (N^* - zI)^{-1} (N^* - S^*) P_M.$$

As a matter of fact,

$$Q(z, S) = [(N^* - zI)^{-1} (N^* - S^*) P_M]^2.$$
Therefore for this family of pure subnormal operators $S$ satisfying the conditions (i) and $[S^*,S] \in \mathcal{L}^1$, the function
\[ \text{tr} \left( Q(\lambda; S)Q(\mu; S) \right), \quad \lambda, \mu \in \text{sp}(S) \setminus \text{sp}(N) \]
is a complete unitary invariant for $S$.

4. Form of Some Hyponormal Tuples

A $k$-tuple of operators $A = (A_1, \ldots, A_k)$ on $\mathcal{H}$ is said to be irreducible, if the only subspaces of $\mathcal{H}$ reducing $A_j$, $j = 1, 2, \ldots, k$ are trivial.

**Theorem 2.** Let $\mathbb{H} = (H_1, \ldots, H_k)$ be an irreducible $k$-tuple of commuting operators on $\mathcal{H}$. (a) Suppose that $H_1$ is hyponormal, $\text{rank}[H_1^*, H_1] = 1$, $\text{ran}[H_1^*, H_1] \subset \text{ran}[H_j^*, H_j]$, and $[H_1^*, H_j]$ is compact, for $j = 2, \ldots, k$. Then either (i) $H_1$ is a linear combination of identity and unilateral shift with multiplicity one and $\mathbb{H}$ is subnormal, or (ii) $H_1$ is a non-subnormal and there are $\alpha_j, \beta_j \in \mathbb{C}$ such that
\[ H_j = \alpha_j H_1 + \beta_j I, \quad j = 2, \ldots, k. \]

(b) suppose that $H_1$ is cohyponormal, $\text{rank}[H_1^*, H_1] = 1$ and $\text{ran}[H_1^*, H_1] \subset \text{ran}[H_j^*, H_j]$, $j = 1, \ldots, k$. Then there are $\alpha_j, \beta_j \in \mathbb{C}$ such that (18) holds.

This theorem is an improvement of Theorem 1 in [18]. The proof of this theorem is closely related to the proof of Theorem 1 in [18]. In order to make the proof of this theorem readable, we have to copy some parts of the proof of Theorem 1 in [18].

**Proof.** Case (a): (1) First let us study the pair $(H_1, H_2)$. There is a vector $e_1 \in \mathcal{H}$, $e_1 \neq 0$ such that
\[ [H_1^*, H_1]x = (x, e_1)e_1, \quad x \in \mathcal{H}. \]
Therefore there exists a vector $e_2$ such that
\[ [H_2^*, H_1]x = (x, e_2)e_1, \]

since $\text{ran}[H_2^*, H_1] \subset \text{ran}[H_1, H_1]$. Let $\mathcal{H}_1$ be the smallest subspace containing $e_1$ and reducing $H_1$. From (19) and (20) it is obvious that
\[ \|e_1\|^2 H_2^* H_1 e_1 = (e_1, e_2) H_1^* e_1 + (H_2^* e_1, e_1) e_1 - (H_1^* e_1, e_2) e_1 \in \mathcal{H}_1. \]

Also we have that
\[ H_2^* H_1^{*m} H_1^n e_1 = \sum_{j=0}^{n-1} H_1^{*m} H_1^{n-j-1} [H_2^*, H_1] H_1^j e_1 + H_1^{*m} H_1^n H_2^* e_1. \]

Therefore $H_2^* H_1^{*m} H_1^n e_1 \in \mathcal{H}_1$ for $m, n = 0, 1, 2, \ldots$. Thus $\mathcal{H}_1$ is invariant with respect to $H_2^*$. 

(ii) According to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_i^{\perp}$, $H_j$ may be written as

$$H_j = \begin{pmatrix} A_j & 0 \\ B_j & C_j \end{pmatrix}, \quad j = 1, 2,$$

where $A_j^* = H_j^* \mid _{\mathcal{H}_1}$, $B_1 = 0$ and $C_1$ is normal. Let $e_2^1$ be the projection of $e_2$ on $\mathcal{H}_1$. Let $e_2^2 = e_2 - e_2^1$. We have to prove that $e_2^2 = 0$. Suppose on contrary that $e_2^2 \neq 0$.

**Lemma 7.** If $e_2^2 \neq 0$, then $\mathcal{H}_1$ is the closure of the

$$\text{span}\{(zI - H_1)^{-1}e_1, \ z \in \rho(H_1)\}$$

and

$$(22) \quad ((zI - H_1)^{-1}e_1, (wI - H_1)^{-1}e_1) = \|e_1\|^2((w - \beta)(z - \beta)I - \|e_1\|^2)^{-1},$$

where $\beta \in \mathbb{C}$.

The proof of this lemma is in [18]. See the part of [18] from lemma 1 of p. 427 to line 13 of p. 429. Especially notice the line 5 of p. 429.

Thus up to a unitary equivalence, we may assume that $\mathcal{H}_1 = H^2(\mathbb{T})$,

$$(A_1f)(z) = (\alpha z + \beta)f(z) \quad \text{for } f \in H^2(\mathbb{T}),$$

and

$$(A_2f)(z) = M(z)f(z), \quad \text{for } f \in H^2(\mathbb{T})$$

where $M(\cdot) \in H^2(\mathbb{T})$, since $[A_1, A_2] = 0$.

Let $\mathcal{H}_2$ be closure of the set $\{B_2p(\cdot) + q(C_1^*)e_2^2 : p$ and $q$ are polynomials $\}$. Follow the steps from (20) through (24) in [18], we have the following.

**Lemma 8.** If $e_2^2 \neq 0$, then $\mathcal{H}_2$ reduces $C_1$, and up to a unitary equivalence, we may assume that $\mathcal{H}_2$ is the Hilbert space of all Borel measurable function $f(\cdot)$ on $\mathbb{T}$ satisfying

$$(f, f)_{\mathcal{H}_2} = \frac{1}{2\pi} \int |f(e^{i\theta})|^2 F(e^{i\theta})d\theta < +\infty$$

where $F(\cdot)$ is a bounded measurable function,

$$(B_2f)(e^{i\theta}) = f(e^{i\theta}), \quad f \in \mathcal{H}_1 = H^2(\mathbb{T}).$$

Thus, by (27) and (28) in [18], we have

$$([H_2^*, H_2]f, f) = ((A_2^*A_2 - A_2A_2^* + B_2^*B_2)f, f) = \|(I - P)Mf\|^2 + \|f\|_{\mathcal{H}_2}^2,$$

for $f \in \mathcal{H}_1$. If $F$ is not zero almost everywhere, then $[H_2^*, H_2]$ cannot be compact, which contradicts the assumption of the theorem. Thus $e_2^2 = 0$. Then follow the step (vi) of the proof of Theorem 1 in [18] we conclude that $\mathcal{H}_1 = \mathcal{H}$ and $H_1$ is pure.
As shown in (35) and (37) of [18], we have

\[(H_2 - k_2 I)e_1 = (H_1 - k_1 I)e_2\]
\[\{(H_2 - k_2 I)x, e_1\}e_1 = \{(H_1 - k_1 I)x, e_1\}e_2\]

where \(k_j = (H_j e_1, e_1)\|e_1\|^{-1}, j = 1, 2.\)

Case 1. If \((H_1^* - \overline{k_1} I)e_1 = 0\), then define

\[T(z, w) = (\overline{w} I - H_1^*)^{-1}e_1, (\overline{w} I - H_1^*)^{-1}e_1)\]
\[= (\overline{w} - \overline{k_1})^{-1}(z - k_1)^{-1}\|e_1\|^2\]

By the commutator property

\[((zI - H_1)^{-1}e_1, (wI - H_1)^{-1}e_1) = T(z, w)(1 - T(z, w))^{-1}\]
\[= \|e_1\|^2((\overline{w} - \overline{k_1})(z - k_1) - \|e_1\|^2)^{-1}\]

Thus \(H_1\) is a linear combination of the identical operator and the unilateral shift with multiplicity one. In that case, it is easy to see that \(H\) is subnormal, since \(H\) is a \(k\)-tuple of commuting operators.

Case 2. If \((H_1^* - \overline{k_1} I)e_1 \neq 0\), then from (24), there is a \(\alpha \in \mathbb{C}\) such that \(\alpha^2 = \alpha e_1\). From (23), there is a \(\beta \in \mathbb{C}\) such that

\[H_2 e_1 = (\alpha H_1 + \beta I)e_1.\]

Follow the step (viii) of [18], we may prove that \(H_2 = \alpha H_1 + \beta I\). Case (a) is proved.

Case (b). In this case (19) and (20) become

\[(zI - H_1)^{-1}e_1, (wI - H_1)^{-1}e_1) = ((zI - H_1)^{-1}e_1, (wI - H_1)^{-1}e_1)\]
\[= ((zI - H_1)^{-1}e_1, (wI - H_1)^{-1}e_1)\]
\[= ((zI - H_1)^{-1}e_1, (wI - H_1)^{-1}e_1)\]

The left-hand sides of (14), (15), (16) of p. 428 in [18] have to be multiplied by \((-1)\). But Corollary 1 in [18] is still true. The formula (21) becomes

\[((zI - H_1)^{-1}e_1, (wI - H_1)^{-1}e_1) = \|e_1\|^2((\overline{w} - \overline{\beta})(z - \beta) + \|e_1\|^2)^{-1}.\]

It leads to a contradiction that \(\|(H_1 - \beta I)e_1\|^2 = -\|e_1\|^4\). Therefore \(e_2^2 = 0\). By (vi) in the proof of Theorem 1 in [18]. We may prove that \(\mathcal{H}_1 = \mathcal{H}\). The left-hand sides of (35) and (37) in [18] must be multiplied by \(-1\) in this case. If \(H_1^* e_1 = \overline{k_1} e_j\), then we have

\[((zI - H_1)^{-1}e_1, (wI - H_1)^{-1}e_1) = (zw - 1)^{-1},\]

if we change \(H_j\) to \(\alpha_j H_j + \beta_j\), so that \(k_1 = k_2 = 0\) and \(\|e_1\| = 1\). However (26) contradicts to (25). Thus \((H_1^* - \overline{k_1} I)e_1 \neq 0\) and there is a \(\alpha \in \mathbb{C}\) such that \(e_2 = \alpha e_1\). By (viii) of [18], we prove that \(H_2 = \alpha H_1 + \beta\). Theorem 3 is proved.
A $k$-tuple of operators $\mathcal{H} = (H_1, \ldots, H_k)$, $H_j \in \mathcal{L}(\mathcal{H})$ is said to be hyponormal (or jointly hyponormal [2]), if
\[
\sum_{i,j} ([H_i^*, H_j] x_i, x_j) \geq 0, \quad x_i \in \mathcal{H}.
\]

**COROLLARY 1.** Let $\mathcal{H} = (H_1, \ldots, H_k)$ be an irreducible $k$-tuple of commuting operators. Suppose that $\operatorname{rank}[H_1^*, H_1] = 1$, $(H_1, H_j)$ is hyponormal and $[H_j^*, H_j]$ is compact for $j = 2, \ldots, k$. Then the conclusion (a) of Theorem 2 holds.

**PROOF.** From the hyponormality (27) of every pair $(H_1, H_j)$ we have
\[
([H_2^*, H_1] x_2, x_1)^2 \leq ([H_1^*, H_1] x_1, x_1)([H_2^*, H_2] x_2, x_2).
\]
for $x_1, x_2 \in \mathcal{H}$. This implies that $\operatorname{ran} [H_2^*, H_1] \subset \operatorname{ran} [H_1^*, H_1]$ which returns to the case (a) of Theorem 2.

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