ON FUNCTIONAL TRANSFORMATION OF HYPERSONAL OPERATORS

XIA DAOXING (夏道行) AND LI SHAOQIAN (李绍宽)

(Research Institute of Mathematics, Fudan University)

Received December 4, 1979.

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ (resp. $\mathcal{L}_+(\mathcal{H})$) be the set of all the linear bounded (resp. bounded self-adjoint) operators on $\mathcal{H}$. For $X, Y \in \mathcal{L}_+(\mathcal{H})$, let $\varphi, \psi$ be the bounded real valued Baire functions defined on $\sigma(X)$ and $\sigma(Y)$ respectively.

We consider the functional transformation $\tau_{\varphi, \psi}: X + iY \mapsto \varphi(X) + i\psi(Y)$, we also use $\tau_{\varphi, \psi}$ to denote the mapping $x + iy \mapsto \varphi(x) + i\psi(y)$ from the subset of the complex plane, where $x$ and $y$ are real.

Let $HN = \{ T | T \in \mathcal{L}(\mathcal{H}), D(T) = [ T^*, T ] \geq 0 \}$ be the set of all hyponormal operators. At the Second National Conference on Functional Analysis, Xia presented the following problems:

(i) What conditions $\varphi$ and $\psi$ must satisfy so that $T \in HN$ implies $\tau_{\varphi, \psi}(T) \in HN$.

(ii) When $\varphi$ and $\psi$ are strictly increasing continuous functions and $T \in HN$, whether the following equalities

$$\tau_{\varphi, \psi}(\sigma(T)) = \sigma(\tau_{\varphi, \psi}(T))$$

and

$$\|D(T)\| \leq \frac{1}{\pi} \int_{\sigma(\tau_{\varphi, \psi}(T))} d\varphi^{-1}(x)d\psi^{-1}(y)$$

are valid.

Obviously, from Putnam's inequality\(^{(1)}\) (2) is an immediate consequence of (1). This paper gives a partial solution of these problems. As to the case of semi-hyponormal operators, we shall discuss in another paper.

Let $E$ be a bounded closed set on the real line. If $K(\cdot, \cdot)$ is the integral kernel on $E \times E$, we denote the corresponding integral operator in $L^2(E)$ by

$$K: f \mapsto \int_E K(x, \cdot)f(x)dx.$$ 

Let $\varphi$ be a bounded Baire function on $E$. We define the kernel

$$K_{\varphi}(x_1, x_2) = \frac{\varphi(x_1) - \varphi(x_2)}{x_1 - x_2},$$

for $x_1 \neq x_2$ and $K_{\varphi}(x, x) = 0$. If $K_{\varphi}$ is a bounded linear operator in $L^2(E)$ and $(K_{\varphi}f)$,
If $\varphi$ is a function defined on $E$ such that

$$M_\varphi = \sup_{x_1, x_2 \in E} K_\varphi(x_1, x_2) < \infty, \quad m_\varphi = \inf_{x_1, x_2 \in E} K_\varphi(x_1, x_2) > 0,$$

then we say $\varphi \in L(E)$.

**Theorem 1.** Let $X + iY \in HN$, $\varphi \in S(\sigma(X))$, $\psi \in S(\sigma(Y))$, then

$$\varphi(X) + i\psi(Y) \in HN.$$  

**Proof.** We consider $X + iY$ as the singular integral model of the hyponormal operators\(^{1,2}\). We may suppose that $\mathcal{H} = L^2(\mathcal{O}, \mathcal{B}, P(\cdot))$, where $\mathcal{O} = (\sigma(X), \mathcal{B}, \mu)$ with $\mu = m + \nu$ and $\nu$ is concentrated on a Lebesgue null set $P \subset \sigma(X)$, $m$ is Lebesgue measure, $P(\cdot)$ is a strongly measurable function with values of projections in $\mathcal{B}$, and defined on $(\mathcal{O}, \mathcal{B})$. $X$ is the operator: $f(\cdot) \rightarrow (\cdot)f(\cdot)$ for $f \in \mathcal{H}$ and $(Yf)(\cdot) = \beta(\cdot)f(\cdot) + \alpha(\cdot)P_+(\sigma f)$, where $\alpha(\cdot), \beta(\cdot)$ are uniformly bounded strongly measurable self-adjoint operator-valued functions $\beta(x) = 0$ for $x \in \sigma(X)$, $\alpha(x) = 0$ for $x \notin \sigma(X) \setminus P$, they all commute with $P(\cdot)$ and

$$(Pf)(\xi) = \lim_{t \to 0^+} \frac{1}{2\pi i} \int \frac{f(x)}{\xi - (x + i\varepsilon)} dx.$$  

It is easy to calculate that

$$(i[\varphi(X), Y]f, f) = \frac{1}{2\pi} \int_{E \times E} K_\varphi(x_1, x_2)(\alpha(x_1)f(x_2), \alpha(x_2)f(x_1))dx_1 dx_2.$$  

We have $i[\varphi(X), Y] \geq 0$, since $\varphi \in S(E)$. Thus $\varphi(X) + iY \in HN$.

If $Y_1 = -\varphi(X)$, $X_1 = Y$, then $X_1 + iY_1 \in HN$. From $\varphi \in S(\sigma(X_1))$, we have $\phi(X_1) + iY_1 \in HN$. It follows that $\varphi(X) + i\psi(Y) \in HN$.

**Lemma 1.** Let $T(t)$ be an operator-valued continuous function defined on $[0, 1]$. Suppose that

$$\sigma_\varphi(T(t)) = \varphi(\sigma_\varphi(T)),$$  

for all $t \in [0, 1]$, where $T = T(1)$. $\varphi_\lambda(\cdot) = \lambda$ and $\varphi_\lambda(\cdot)$ are bijective functions defined on the complex plane and for every fixed $\lambda$, $\varphi_\lambda(\cdot)$ is a continuous function of $t$. Then we have

$$\sigma_\varphi(T(t)) = \varphi_\lambda(\sigma_\varphi(T))$$  

and

$$\sigma(T(t)) = \varphi_\lambda(\sigma(T)).$$  

**Proof.** Without loss of generality we only prove $\sigma_\varphi(T(0)) = \varphi_\lambda(\sigma_\varphi(T))$. For $\lambda \in \sigma_\varphi(T)$, set

$$E_\lambda = \{t: \varphi_\lambda(\lambda) \in \sigma_\varphi(T_\lambda), \quad \forall \tau \in [t, 1]\}.$$  

Then $E_\lambda = (t_0, 1]$ or $[t_0, 1]$. If $E_\lambda = (t_0, 1]$, then $\varphi_\lambda(\lambda) \in \rho(T_\lambda)$ from (3). Since $\rho(T_\lambda)$ is an open set, from the continuity of $T(t)$ and $\varphi_\lambda(\lambda)$, there exists $\varepsilon > 0$ such that $\varphi_\lambda(\lambda) \in \rho(T(t))$ for $|t - t_0| < \varepsilon$. This is...
impossible, so that we must have $E_1 = [t_0, 1]$. If $E_1 = [t_0, 1]$ and $t_0 < 1$, then $\varphi_{t_0}(\lambda) \in \sigma_c(T_{t_0})$. By the same reason, there also exists $\varepsilon > 0$ such that $\varphi_{t_0}(\lambda) \in \sigma_c(T(t))$ for $|t - t_0| < \varepsilon$. This is also a contradiction. Hence, we have $E_1 = [0, 1]$, i.e. $\varphi_0(\lambda) \in \sigma_c(T(0))$.

Conversely, if $\varphi_0(\lambda) \in \sigma_c(T(0))$. By the same method, we can prove that the set $\tilde{E}_1 = \{t \mid \varphi_t(\lambda) \in \sigma_c(T_t) \forall t \in [0, t]\}$ is $[0, 1]$. Hence, we have $\lambda \in \sigma_c(T)$, i.e. $\sigma_c(T(0)) = \sigma_c(\sigma_c(T))$.

Theorem 2. Let $X + iy \in HN$, and $\varphi$ be bijective real Baire function defined on the real line. If $\varphi(X) + iy \in HN$. Then we have

$$\sigma(\varphi(X) + iy) = \tau_{\varphi}(\sigma(X + iy)).$$

Proof. We define $T(t) = [Xt + \varphi(X)(1 - t)] + iy$. It is obvious that $T(t)$ is an operator-valued function defined on $[0, 1]$, and the operators $T(t)$ are all hyponormal. We notice that for the hyponormal operator $T = X + iy$ and $z = x + iy$, there is an equality

$$(T - z)\ast(T - z) = (X - x)^2 + (Y - y)^2 + i[X, Y].$$

Hence for $(x + iy) \in \sigma_c(T)$, there exist unit vectors $f_n \in H$ such that

$$\|(T - z)f_n\| \to 0, \quad \|(X - x)f_n\| \to 0, \quad \|(Y - y)f_n\| \to 0.$$ 

From this we can prove

$$\sigma_c(T(t)) = \varphi_t(\sigma_c(T)),$$

where $\varphi_t(x + iy) = [xt + \varphi(x)(1 - t)] + iy$. By Lemma 1, we have

$$\sigma(T(t)) = \varphi_t(\sigma_c(T)).$$

If we take $t = 1$ in the above equality, then (5) is followed.

From Theorems 1, 2 and Putnam's inequality we have

Theorem 3. Let $T = X + iy$ be hyponormal operator and $\varphi, \phi$ be bijective functions defined on the real line and $\varphi \in S(\sigma(X))$, $\phi \in S(\sigma(Y))$. Then we have (1) and (2).

Theorem 4. $T = X + iy \in HN$, $\varphi \in L(\sigma(X))$, $\varphi(\xi) = \xi$, then (1) and (2) hold.

Proof. Without loss of generality, we can suppose that $T$ is completely non-normal. Hence $\sigma_p(T) = \phi$. We denote $\tilde{T} = r_{\varphi}(T)$ and $\tilde{z} = r_{\varphi}(z)$, for $z = x + iy$. We have $(\tilde{T} - \tilde{z}) = (I + B)(T - z)$, where $B = [(\varphi(X) - X) - (\varphi(x) - x)] \cdot (T - z)^{-1}$. Firstly we suppose $M_{\varphi} + m_{\varphi} = 2$ and then it is necessary to prove $\|B\| < 1$. For this, let $f \in \mathcal{D}(B)$. There exists $g$ such that

$$f = (T - z)g, \quad Bf = [(\varphi(X) - X) - (\varphi(x) - x)]g.$$ 

Since $\|f\| = \|(T - z)g\| \geq \|(X - x)g\|$, if $X = \int \lambda dE_1$ is the spectral decomposition of $X$, then we have

$$\|Bf\|^2 = \int (\lambda - x)^2(K_{\phi}(\lambda, x) - 1)d(E_1g, g) \leq 2\|f\|,$$
where \( s = \frac{M - m}{2} \), i.e. \( \|B\| < 1 \), so \( (I + B) \) is invertible. Hence (1) holds. For the general case, we take \( K = \frac{2}{M + m} \), \( \varphi_1(t) = \varphi(Kt) \), and \( T_1 = \frac{1}{K} X + iY \). We have \( \sigma(T_1) = \left\{ \frac{1}{K} x + iy \mid x + iy \in \sigma(T) \right\} \). From Theorem 2 and the above for \( T_1 \), \( \varphi_1 \) we have \( \sigma(T) = r_{\varphi_1, \varphi}(\sigma(T_1)) \). Hence (1) is followed. Eq. (2) is the consequence of (1) by Putnam's inequality.

From Theorems 1, 2 and 4, we have

**Theorem 5.** If \( X + iY \in \mathcal{H}N, \varphi \in S(\sigma(X)) \), and \( \varphi \in L(\sigma(Y)) \). then (1) and (2) hold.

**References**

