Spectral Mapping of Hyponormal or Semi-Hyponormal Operators*

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Let $\mathcal{H}$ be a separable complex Hilbert space, $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators in $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called hyponormal [8] if $T^*T - TT^* \succeq 0$, semi-hyponormal [13] if

$$(T^*T)^{1/2} - (TT^*)^{1/2} \succeq 0.$$ 

For $T \in \mathcal{L}(\mathcal{H})$, we write $|T| = (T^*T)^{1/2}$. In this paper, when we consider a semi-hyponormal operator $T$, we always assume that the operator $U$ in the polar decomposition $T = U|T|$ is unitary, for the sake of simplicity.

Let $E$ be a bounded closed set in the real line $\mathbb{R}$, and $M(E)$ be the class of all strictly monotone increasing continuous function on $E$. Clearly every function $\phi \in M(E)$ can be continued to be a topological mapping from $\mathbb{R}$ onto itself, and we shall make such continuation when necessary. If $E_j, j = 1, 2,$ are two bounded closed sets in $\mathbb{R}$ and $\phi_j \in M(E_j)$, then we define the mapping

$$\tau_{\phi_1, \phi_2}: x_1 + ix_2 \rightarrow \phi_1(x_1) + i\phi_2(x_2), \quad \text{for} \quad x_j \in E_j, \quad j = 1, 2,$$

from $E_1 \times E_2 = \{x_1 + ix_2 \mid x_j \in E_j\}$ to $\phi_1(E_1) \times \phi_2(E_2)$ in the complex plane as well as the mapping in $\mathcal{L}(\mathcal{H})$,

$$\tau_{\phi_1, \phi_2}: X_1 + iX_2 \rightarrow \phi_1(X_1) + i\phi_2(X_2),$$

for all self-adjoint $X_j \in \mathcal{L}(\mathcal{H}), j = 1, 2,$ with $\sigma(X_j) \subset E_j$, where $\sigma(A)$ is the spectrum of $A$.

If $X \in \mathcal{L}(\mathcal{H})$, we write $X_1 = (X + X^*)/2$ and $X_2 = (X - X^*)/2i$.

Let $\sigma_a(A)$ and $\sigma_r(A)$ be the approximate point spectrum and the residue spectrum of an operator $A$, respectively. Let $\sigma_{ja}(A)$ be the joint approximate

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point spectrum of an operator $A = A_1 + iA_2$, i.e., the set of all such complex numbers $x_1 + ix_2$ for which there exists a sequence of unit vectors $\{f_n\} \subset \mathcal{H}$ satisfying

$$\lim_{n \to \infty} \|(A_j - x_j I)f_n\| = 0, \quad j = 1, 2.$$  

It is well known [8] that if $X$ is hyponormal then $\sigma_a(X) = \sigma_{ja}(X)$ and

$$\sigma_a(X) \subseteq \sigma(X_1) \times \sigma(X_2).$$

In this paper (Section 3), we shall consider the following problem. Under what condition of $E \in M(E_j)$ are the relations

$$\sigma_{ja}(\tau_{\phi_1, \phi_2}(X)) = \sigma_{ja}(\tau_{\phi_1, \phi_2}(X)), \quad (1)$$

$$\sigma_{ja}(\tau_{\phi_1, \phi_2}(X)) = \sigma_{ja}(\tau_{\phi_1, \phi_2}(X)), \quad (2)$$

$$\sigma_{ja}(\tau_{\phi_1, \phi_2}(X)) = \sigma_{ja}(\tau_{\phi_1, \phi_2}(X)), \quad (3)$$

and

$$\sigma_{ja}(\tau_{\phi_1, \phi_2}(X)) = \sigma_{ja}(\tau_{\phi_1, \phi_2}(X)). \quad (4)$$

true for every hyponormal operators $X = X_1 + iX_2 \in \mathcal{L}(\mathcal{H})$ with $\sigma(X_j) \subset E_j$.

We denote by $C_1 = \{z \mid |z| = 1\}$ the unit circle in the complex plane. If $E$ is a closed subset in $C_1$, let $M_0(E)$ be the class of all orientation preserving topological mappings $\phi : E \to \phi(E) \subset C_1$. We write $C_2 = [0, \infty)$. If $E$ is a bounded closed subset in $C_2$, then $M_0(E) = \{\phi \mid \phi \in M(E), \phi \geq 0 \text{ and } \phi(0) = 0\}$.

If $E_j \subset C_j, j = 1, 2$, are bounded closed sets and $\phi_j \in M_0(E_j)$, we define the mapping

$$\tau_{\phi_1, \phi_2}(e^{i\theta} \rho) = \phi_1(e^{i\theta})\phi_2(\rho), \quad e^{i\theta} \in E_1, \quad \rho \in E_2,$$

which will be extended to a topological mapping from the complex plane onto itself. If $X = U|X|$ is the polar decomposition of an operator $X \in \mathcal{L}(\mathcal{H})$, $U$ is unitary, $\sigma(U) \subset E_1$ and $\sigma(|X|) \subset E_2$, then we define

$$\tau_{\phi_1, \phi_2}(X) = \phi_1(U)\phi_2(|X|).$$

If $A = U|A| \in \mathcal{L}(\mathcal{H})$ and $U$ is unitary then we define the polar joint approximate point spectrum $\sigma_{ja}(A)$ as the set of all complex numbers $\rho e^{i\theta}, |e^{i\theta}| = 1, \rho > 0$, for which there exists a sequence of unit vectors $f_n \subset \mathcal{H}$ satisfying

$$\lim_{n \to \infty} \|(U - e^{i\theta} I)f_n\| = \lim_{n \to \infty} \|(|A| - \rho I)f_n\| = 0.$$
If $0 \in \sigma_s(A)$ then we also define that $0 \in \sigma_{pia}(A)$ for convenience. If $A$ is a self-adjoint operator in $\mathcal{L}(\mathcal{H})$, we denote

$$m(A) = \inf_{\|f\| = 1} (Af, f) \quad \text{and} \quad M(A) = \inf_{\|f\| = 1} (Af, f).$$

If $X = U|X|$ is semi-hyponormal and $U$ is unitary, we proved that [13]

$$\sigma_s(X) = \sigma_{pia}(X),$$

and

$$\sigma_s(X) \subset \{ \rho e^{i\theta} \mid e^{i\theta} \in \sigma(U), \rho \in \{ m(|X|), M(|X|) \} \},$$

and Li [4] proved that

$$\sigma_s(X) \subset \{ \rho e^{i\theta} \mid e^{i\theta} \in \sigma(U), \rho \in \sigma(|X|) \}$$

if $\sigma(U) \neq C_1$.

The second problem which we shall consider (in Section 4) is under what condition of $\phi \in M_0(E_j)$ with $E_j \subset C_j$ will (1)–(4) hold for every semi-hyponormal operator $X = U|X|$ with $\sigma(U) \subset E_j$, $\{ m(|X|), M(|X|) \} \subset E_2$ or $\sigma(|X|) \subset E_2$ when $\sigma(U) \neq C_1$.

The author and Li have considered these two problems in some simpler cases in the previous papers [5, 6] (cf. Section 2). For the convenience of the reader we shall give a brief account of the related results and preliminaries of the theory of hyponormal or semi-hyponormal operators in Section 2. By means of (3), we shall also consider the generalization of Putnam's inequality in Section 5.

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First, we have to consider the singular integral models of the hyponormal or the semi-hyponormal operators.

Let $\mathbb{R}$ be the real line $R_1$ or $C_1$, $E$ be a closed bounded set in $\mathbb{R}$, $\mathcal{B}_E$ be the $\sigma$-algebra of all Borel subsets in $E$, and $m$ be the Lebesgue measure on $(E, \mathcal{B}_E)$. When $M \subset C_1$,

$$dm(e^{i\theta}) = \frac{1}{2\pi} d\theta.$$

Let $\nu$ be a singular finite measure on $(E, \mathcal{B}_E)$, which is concentrated in $F \in \mathcal{B}_E$ with $m(F) = 0$. We denote $\mu = m + \nu$, $\Omega = (E, \mathcal{B}_E, \mu)$. Let $\mathcal{D}$ be an auxiliary complex separable Hilbert space. Let $L^2(\Omega, \mathcal{D})$ be the Hilbert space.
of all strongly measurable and square integrable $\mathcal{D}$-valued functions on $\Omega$ with inner product

$$(f, g) = \int_{E} (f(x), g(x))_{\mathcal{D}} d\mu(x).$$

For $f \in L^2(\Omega, \mathcal{D})$, we define $f(x) = 0$ for $x \in M - E$. In the following, we shall often use singular integral operators $P_+$. If $E \subset R_1$ and $f \in L^2(\Omega, \mathcal{D})$ then

$$(P_+ f)(x) = st - \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{R_1} \frac{f(s) ds}{x - (s + i\epsilon)}, \quad x \in R_1,$$

and if $E \subset C_1, f \in L^2(\Omega, \mathcal{D})$, then

$$(P_+ f)(x) = st - \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{C_1} \frac{f(s) ds}{s - x(1 - \epsilon)}, \quad x \in C_1.$$

Let $\mathcal{D}_1 \subset \cdots \subset \mathcal{D}_n \subset \cdots \subset \mathcal{D}$ be a sequence of subspaces and $\emptyset = F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots$ be a sequence of subsets in $\mathcal{B}_E$. We define the projection-valued function $P(\cdot)$ on $E$ by

$$P(x) \mathcal{D} = \mathcal{D}_n, \quad \text{for} \ x \in F_n - F_{n-1},$$

and $P(x) = I$ for $x \in M - \bigcup_{n=0}^{\infty} F_n$. Let

$$\mathcal{H} = \{ f \mid f \in L^2(\Omega, \mathcal{D}), P(x)f(x) = f(x) \text{ for } x \in E \}$$

be a subspace of $L^2(\Omega, \mathcal{D})$, let $\alpha(\cdot)$ and $\beta(\cdot)$ be strongly measurable and uniformly bounded $\mathcal{D}(\mathcal{D})$ valued functions on $\Omega$ which satisfy the conditions

$$\alpha P = P\alpha = \alpha, \quad \beta P = P\beta = \beta, \quad \beta^* = \beta, \quad \alpha \geq 0,$$

and let $\alpha(x) = 0$ for $x \in F$. Then we construct the operator $\tilde{T}$ in $\mathcal{H}$ as follows:

$$(\tilde{T}f)(x) = x(\beta(x)f(x) + \alpha(x) P_+(af)), \quad x \in C_1, \quad (5)$$

if $E \subset C_1$, and

$$(\tilde{T}f)(x) = (x + i\beta(x))f(x) + \alpha(x) P_+(af), \quad x \in R_1, \quad (6)$$

if $E \subset R_1$.

**Theorem 1.** The operator $\tilde{T}$ in $\mathcal{H}$ is semi-hyponormal if $E \subset C_1$ and is hyponormal if $E \subset R_1$. 
If \( T = U \|T\| \in \mathcal{L}(\mathcal{H}) \) is semi-hyponormal and \( U \) is unitary or \( T = T_1 + iT_2 \in \mathcal{L}(\mathcal{H}) \) is hyponormal and \( T_j \) is self-adjoint, then there is an operator \( \tilde{T} \) in a Hilbert space \( \mathcal{H} \) with \( E = \sigma(U) \) in the semi-hyponormal case of \( T = U \|T\| \) or with \( E = \sigma(T_j) \) in the hyponormal case \( T = T_1 + iT_2 \), respectively, and a unitary operator \( W: \mathcal{H} \to \mathcal{H} \) such that

\[
T = W^{-1} \tilde{T} W.
\]

The proof of this theorem, in the semi-hyponormal case can be found in [13] and in the hyponormal case in [3, 7, 11, 12]. But the hyponormal case can be reduced to the semi-hyponormal case by the transform

\[
T_1 + iT_2 \to [(T_1 + i\lambda)(T_1 - i\lambda)^{-1}] [T_2 - m(T_2)].
\]

Let \( B(E) \) be the set of all bounded complex Baire functions \( \phi \) on \( E \), and let \( K_\phi \) be the bounded operator

\[
(K_\phi f)(x) = P_+ (\phi) P_+ (\tilde{\phi} f), \quad f \in L^2(E, \mathfrak{B}_E, m),
\]

for \( \phi \in B(E), \ E \subset C_1, \) or \( (K_\phi f)(x) = i(\phi(x) P_+ (f) - P_+ (\phi f)) \) for real \( \phi \in B(E), \ E \subset R_1 \). The operator \( K_\phi \) is obviously self-adjoint and

\[
(K_\phi f, f) = \frac{1}{(2\pi)^2} \int_E \int_E \frac{1 - \bar{\phi}(\xi) \phi(\eta)}{1 - e^{-i\xi \eta}} f(\xi) \overline{\phi(\eta)} d\xi d\eta,
\]

for \( E \subset C_1 \) and

\[
(K_\phi f, f) = \frac{1}{(2\pi)^2} \int_E \int_E \frac{\phi(x) - \phi(y)}{x - y} f(x) \overline{f(y)} dx dy,
\]

for \( E \subset R_1 \). Let \( B_+(E) = \{ \phi \mid \phi \in B(E), m(K_\phi) > 0 \} \).

The following lemmas will be used in Sections 3 and 4.

**Lemma 1.** If \( T = U \|T\| \in \mathcal{L}(\mathcal{H}) \) is semi-hyponormal, \( U \) is unitary and \( \phi \in B(\sigma(U)) \) then

\[
|T| - \phi(U) \|T\| \phi(U)^* \geq \tilde{m}(K_\phi)(|T| - U \|T\| U^*). \tag{9}
\]

If \( T = T_1 + iT_2 \) is hyponormal, \( T_j, \ j = 1, 2, \) are self-adjoint and \( \phi_j \in B(\sigma(T_j)), j = 1, 2, \) then

\[
i[\phi_1(T_1), T_2] \geq i \tilde{m}(K_\phi)[T_1, T_2],
\]

\[
i[T_1, \phi_2(T_2)] \geq im(K_\phi)[T_1, T_2],
\]

where \( |A, B| = AB - BA \). \( \tilde{m}(K_\phi) = \sup\{c \mid m(K_\phi - CK_j) \geq 0\} \).
Proof. By Theorem 1, we only consider the singular model and use the notation $T$ and $H$ instead of $\tilde{T}$ and $\tilde{H}$, respectively. In the semi-hyponormal case it can be proved that

$$
\left(\|T\| - \phi(U) \|T\| \phi(U^*)f, f\right) = \frac{1}{(2\pi)^2} \int_E \int_E \frac{1 - \phi(e^{i\xi}) \phi(e^{i\eta})}{1 - e^{-i\xi}e^{i\eta}}
\times \left(\alpha(e^{i\xi})f(e^{i\xi}), \alpha(e^{i\eta})f(e^{i\eta})\right)_E d\xi d\eta
$$

and

$$
\left(\|T\| - U \|T\| U^*f, f\right) = \frac{1}{(2\pi)^2} \int_E \int_E \left(\alpha(e^{i\xi})f(e^{i\xi}), \alpha(e^{i\eta})f(e^{i\eta})\right)_E d\xi d\eta.
$$

Then (9) holds evidently, since $K_\omega \geq m(K_\omega)I$.

The first inequality of (10) can be proved similarly and the second can be proved by considering the hyponormal operator $-iT = T + i(\tilde{T})$.

If $E \subset \mathbb{R}$ and $\omega \in M(E)$, we define

$$
l_\omega = \sup_{x_1, x_2 \neq \omega} \frac{\phi(x_1) - \phi(x_2)}{x_1 - x_2}
$$

and

$$
m_\omega = \inf_{x_1, x_2 \neq \omega} \frac{\phi(x_1) - \phi(x_2)}{x_1 - x_2}.
$$

Let $\mathcal{L}(E) = \{\phi \mid \phi \in M(E), 0 < m_\phi \leq l_\phi < \infty\}$ and $\mathcal{L}_0(E) = M_0(E) \cap \mathcal{L}(E)$ for $E \subset [0, \infty)$.

**Theorem 2** [5]. Let $X = X_1 + iX_2$ be hyponormal.

1. If $\phi_j \in M(\sigma(X_j))$, $j = 1, 2$, and $\tau_{\phi_1, \phi_2}(X)$ is hyponormal, then (1)–(4) hold.

2. If $\phi_j \in B_+^+(\sigma(X_j))$, $j = 1, 2$, then $\tau_{\phi_1, \phi_2}(X)$ is hyponormal and (1)–(4) hold.

3. If one of $\phi_j$, $j = 1, 2$, is in $B_+^+(\sigma(X_j))$ and another is in $\mathcal{L}(\sigma(X_j))$, then (1)–(4) hold.

**Theorem 3** [6]. Let $X = U |X|$ be semi-hyponormal and $U$ unitary.

1. If $\phi \in M(\sigma(U))$, $\phi_2 \in M_0([m(|X|), M(|X|)])$ (or $\phi_2 \in M_0(\sigma(|X|))$ in the case of $\sigma(U) \neq C_1$) and $\tau_{\phi_1, \phi_2}(X)$ is semi-hyponormal then (1)–(4) hold.

2. If $\phi_1 \in B_+^+(\sigma(U))$ and $\phi_2 \in B_+^+(E)$, where $E = [m(|X|), M(|X|)]$ or $E = \sigma(|X|)$ in the case of $\sigma(U) \neq C_1$, then $\tau_{\phi_1, \phi_2}(X)$ is semi-hyponormal and (1)–(4) hold.
If $\phi_1 \in B_+(\sigma(U))$, $\phi_2 \in M_0([m(|X|), M(|X|)])$ and $t/\phi_2(t)$ is a monotonic increasing and concave function then (1)–(4) hold.

The following lemma is essentially given in [5], but we include a proof.

**Lemma 2.** Let $R$ be a set in the complex plane, $T(t)$ be an operator-valued function of $t \in [0, 1]$ which is continuous with respect to the norm topology, and $\tau_t, t \in [0, 1]$, be a family of topological mappings from $R$ to itself such that $\tau_t(z)$ is a continuous function of $t \in [0, 1]$, for every complex number $z \in R$. Suppose that $\tau_0$ is the identity mapping. If

$$\sigma_a(T(t)) \cap R = \tau_t(\sigma_a(T(0)) \cap R), \quad \text{for } t \in [0, 1],$$

then

$$\sigma_a(T(t)) \cap R = \tau_t(\sigma_a(T(0)) \cap R), \quad \text{for } t \in [0, 1],$$

and hence

$$\sigma(T(t)) \cap R = \tau_t(\sigma(T(0)) \cap R), \quad \text{for } t \in [0, 1].$$

**Proof.** For every

$$z \in \sigma_a(T(0)) \cap R$$

we have to prove that the set

$$E_z = \{t \mid t \in [0, 1], \tau_t(z) \in \sigma_a(T(s)) \text{ for } 0 \leq s \leq t\}$$

is $[0, 1]$. It is obvious that $E_z = [0, t_0)$ or $[0, t_0]$ for certain $t_0 \in [0, 1]$. If $E_z = [0, t_0)$, then $\tau_t(z) \in \sigma_a(T(t_0))$. But $z \notin \sigma_a(T(0)) \cap R$, because of (14). Then

$$\tau_t(z) \notin \sigma_a(T(t_0)) \cap R$$

according to (11). Hence $\tau_t(z) \notin \sigma_a(T(t_0))$. Thus $\tau_t(z) \in \rho(T(t_0))$, where $\rho(A)$ is the resolvent set of $A$. By the continuity of $\tau_t$ and $T(t)$, there is a positive number $\delta$ such that

$$\tau_t(z) \in \rho(T(t)) \quad \text{for } |t - t_0| < \delta.$$

Hence $\tau_t(z) \notin \sigma_a(T(t))$ for $|t - t_0| < \delta$. Thus $(t_0 - \delta, t_0) \cap E_z = \emptyset$, contrary to $E_z = [0, t_0)$.

So we must have $E_z = [0, t_0]$. We have to prove $t_0 = 1$. Since $\tau_t \in \sigma_a(T(t_0))$, there is a number $\epsilon > 0$ such that

$$\|(T(t_0) - \tau_t(z))x\| \geq \epsilon \|x\|, \quad \text{for } x \in \mathcal{H}. \quad (15)$$
If \( t_0 < 1 \), by the continuity of \( T(t) \) and \( \tau_t \), there is is positive \( \delta \) with \( t_0 + \delta < 1 \) such that
\[
\| T(t) - T(t_0) \| < \varepsilon/6, \quad |\tau_t(z) - \tau_{t_0}(z)| < \varepsilon/6, \quad \text{for} \quad t \in [t_0, t_0 + \delta). \quad (16)
\]

From (15) and (16) we have
\[
\| (T(t) - \tau_t(z)I)x \| \geq \frac{3}{2} \| x \|, \quad \text{for} \quad x \in \mathcal{H} \quad \text{and} \quad t \in [t_0, t_0 + \delta). \quad (17)
\]

If the range of the operator \( T(t) - \tau_t(z)I \) is the whole space \( \mathcal{H} \), then \( (T(t) - \tau_t(z)I)^{-1} \in \mathcal{L}(\mathcal{H}) \) and
\[
\| (T(t) - \tau_t(z)I)^{-1} \| \leq 3/2\varepsilon. \quad (18)
\]

From (18) and the first inequality of (16), we have \( \tau_{t_0}(z) \in \rho(T(t_0)) \), contrary to \( t_0 \in E_z \). Hence the range of \( T(t) - \tau_t(z)I \) must be a proper subspace of \( \mathcal{H} \). But from (17), we have
\[
\tau_t(z) \in \sigma_t(T(t)), \quad \text{for} \quad t \in [t_0, t_0 + \delta).
\]

Then \( [t_0, t_0 + \delta) \subset E_z \). This is also a contradiction.

Thus we must have \( E_z = [0, 1] \), i.e.,
\[
\sigma_s(\sigma_t(T(0)) \cap R) \subset \sigma_t(T(s)) \cap R, \quad \text{for} \quad s \in [0, 1]. \quad (19)
\]

If instead of \( T(t) \) and \( \tau_t \), we consider the functions \( T(s(1-t)) \) and \( \tau_{s(1-t)} \tau_s^{-1} \), respectively, for \( t \in [0, 1] \), where \( s \) is a fixed number in \([0, 1]\), then we have
\[
\sigma_s(T(s)) \cap R \subset \sigma_s(\sigma_t(T(0)) \cap R), \quad \text{for} \quad s \in [0, 1]. \quad (20)
\]

From (19) and (20), we obtain (12). And (13) is a direct consequence of (11) and (12).

**Lemma 3.** Let (i) \( T = T_1 + iT_2 \in \mathcal{L}(\mathcal{H}) \), \( \phi_j \in M(R_j) \), \( j = 1, 2 \), \( t \in [0, 1] \), or (ii) \( T = UT \in \mathcal{L}(\mathcal{H}) \), \( U \) be unitary, \( \phi_j \in M_0([0, \infty)) \) and \( \phi_j \in M_0(C_1) \) for \( t \in [0, 1] \). Let \( \tau_t = \tau_{t_1}, \tau_{t_2} \) for which \( \tau_0 \) is identical mapping, \( \tau_t(z) \) is a continuous function of \( t \in [0, 1] \) for every fixed complex number \( z \),
\[
\sigma_{1a}(T) = \sigma_a(T), \quad \sigma_{ja}(T(t)) = \sigma_a(T(t)), \quad t \in [0, 1], \quad (21)
\]

in case (i) and
\[
\sigma_{p1a}(T) = \sigma_a(T), \quad \sigma_{pja}(T(t)) = \sigma_a(T(t)), \quad t \in [0, 1], \quad (23)
\]

in case (ii). Then (11)–(13) hold.
Proof. We shall consider case (i) only. By Lemma 2, it suffices to prove (11). By (21) and (22), (11) is equivalent to
\[ \sigma_{j a}(T(t)) = \tau_i(\sigma_{j a}(T)), \quad t \in [0, 1]. \] (25)

If \( x = x_1 + x_2 t \in \sigma_{j a}(T) \), then by definition, there is a sequence of unit vectors \( \{f_n\} \) in \( \mathbb{H} \) such that
\[ \lim_{n \to \infty} \|(T_j - x_j I)f_n\| = 0, \quad j = 1, 2. \] (26)

Since \( \phi_{j t}(s) \) is a continuous function of \( s \), for \( \varepsilon > 0 \), there is a polynomial \( P_{j t}(s) \) such that
\[ \|\phi_{j t}(T_j) - P_{j t}(T_j)\| < \varepsilon, \quad \|\phi_{j t}(x_j) - P_{j t}(x_j)\| < \varepsilon \] (27)
for \( j = 1, 2 \) and \( t \in [0, 1] \). From (26) it is obvious that
\[ \lim_{n \to \infty} \|(P_{j t}(T_j) - P_{j t}(x_j I)f_n\| = 0, \quad j = 1, 2, \quad t \in [0, 1]. \] (28)

Then \( \lim_{n \to \infty} \|(\phi_{j t}(T_j) - \phi_{j t}(x_j I)f_n\| \leq 2\varepsilon \) follows from (27) and (28). Hence
\[ \tau_i(x) \in \sigma_{j a}(T(t)). \]

Thus
\[ \sigma_{j a}(T(t)) \supset \tau_i(\sigma_{j a}(T)). \] (29)

Similarly, we can prove that \( \sigma_{j a}(T(t)) \subset \tau_i(\sigma_{j a}(T)) \). Thus (25) holds and then (11)–(13) hold.

3

First we consider the case of hyponormal operators. Let \( E \) be a bounded closed set in \( R_1, \mathcal{L}_m(E) = \{\phi \mid \phi \in M(E), m_\phi > 0\} \). For every \( \phi \in \mathcal{L}_m(E) \), we define
\[ N(\phi) = \inf_{g \in B, \phi \in M(E)} \frac{|\min(0, \tilde{m}(K_{g^{-1}}))|}{m_{g^{-1}}}, \] (30)
where \( g^{-1} \) is the inverse mapping of \( g \).

Theorem 4. Let \( X = X_1 + iX_2 \) be a hyponormal operator. If \( \phi_j \in \mathcal{L}_m(\sigma(X_j)), j = 1, 2, \) and
\[ N(\phi_1) N(\phi_2) < 1, \] (31)
then (1)-(4) hold and

\[ \| (\tau_{\phi_1, \phi_2}(x) - \tau_{\phi_3, \phi_2}(x))^{-1} \| \leq M(\phi_1, \phi_2) \| (X - xI)^{-1} \| \]  

(32)

for every \( x \in \rho(X) \), where \( M(\phi_1, \phi_2) \) is a constant which depends only on \( \phi_1, \phi_2 \).

**Proof.** We prove the theorem in four steps.

(i) Let \( \mu_\phi = \min(0, \tilde{m}(K_\phi)) \). From (31), there are functions \( g_j \in B_+(\sigma(X_j)) \cap M(\sigma(X_j)) \) such that

\[ \prod_{j=1}^{2} \mu_{\phi_j g_j^{-1}} < \prod_{j=1}^{2} m_{\phi_j g_j^{-1}} \]  

(33)

Let \( k_j, j = 1, 2, \) be positive numbers, and \( \psi_j(s) = \phi_j(g_j^{-1}(k_j s)), j = 1, 2 \). Then

\[ \mu_{\phi_j} = k_j \mu_{\phi_j g_j^{-1}} \quad \text{and} \quad m_{\phi_j} = k_j m_{\phi_j g_j^{-1}}. \]  

(34)

We take \( k_j \) such that

\[ (\mu_{\phi_1} + \mu_{\phi_2})/2 = (\mu_{\phi_1}, \mu_{\phi_2})^{1/2}. \]  

(35)

From (33)-(35) we have \( (\mu_{\phi_1} + \mu_{\phi_2})/2 < (m_{\phi_1} m_{\phi_2})^{1/2} \); so there is a positive number \( \delta < m_{\phi_j}, j = 1, 2, \) such that

\[ \delta + (\mu_{\phi_1} + \mu_{\phi_2})/2 - (m_{\phi_1} - \delta)^{1/2}(m_{\phi_2} - \delta)^{1/2}. \]  

(36)

(ii) Let \( h_j = g_j/k_j, x_j \) be real numbers, \( x = x_1 + ix_2, Y_j = h_j(X_j), y_j = h_j(x_j), T_j = Y_j - y_j I, S_j = \psi_j(y_j), s_j = \psi_j(y_j), R_j = S_j - s_j I, Y = Y_1 + iY_2, \)

\( T = T_1 + iT_2, R = R_1 + iR_2 \) and \( S = S_1 + iS_2 \). We have to prove

\[ \mathcal{R}(RF, T\overline{F}) \geq \delta \| F \|^2, \quad \text{for } f \in \mathcal{H}. \]  

(37)

By Theorem 2, case (2), \( Y = \tau_{h_1, h_2}(Y) \) is hyponormal. Since \( S = \tau_{\phi_1, \phi_2}(Y) \), from (10), we obtain

\[ i[S_1, Y_2] \geq \mu_{\phi_1}[Y_1, Y_2], \quad i[Y_1, S_2] \geq \mu_{\phi_2}[Y_1, Y_2]. \]

But it is easy to verify that

\[ [R_1, T_2] = [S_1, Y_2], \quad [R_2, T_1] = [S_2, Y_1], \quad [Y_1, Y_2] = [T_1, T_2]. \]

Thus

\[ i[R_1, T_2] \geq \mu_{\phi_1}[T_1, T_2], \quad -i[R_2, T_1] \geq \mu_{\phi_1}[T_1, T_2]. \]  

(38)
On the other hand, by the spectral decomposition of the self-adjoint operators \( Y_j \), we can easily prove that
\[ R_j T_j = T_j R_j \geq m_{\phi_j} T_j^2, \quad j = 1, 2. \]

It is obvious that
\[
(m_{\phi_1} - \delta) T_1^2 + (m_{\phi_2} - \delta) T_2^2 - i(m_{\phi_1} - \delta)^{1/2}(m_{\phi_2} - \delta)^{1/2}[T_1, T_2] = \left[ (m_{\phi_1} - \delta)^{1/2} T_1 + i(m_{\phi_2} - \delta)^{1/2} T_2 \right] \times \left[ (m_{\phi_1} - \delta)^{1/2} T_1 - i(m_{\phi_2} - \delta)^{1/2} T_2 \right] \geq 0. \tag{39}
\]

From (36), (38) and (39) we have
\[ T_1 R_1 + T_2 R_2 + i[R_1, T_1]/2 - i[R_2, T_1]/2 \geq \delta T^* T, \]
which is just (37).

(iii) If \( x \in \rho(X) \), then \( \tau_{h_1, h_2}(x) \in \rho(\tau_{h_1, h_2}(X)) \) by Theorem 2, case (2). Since
\[ T = \tau_{h_1, h_2}(X) - \tau_{h_1, h_2}(x)I \]
and \( T^{-1} \in \mathcal{L}(\mathcal{H}) \), (37) becomes \( \mathcal{R}(RT^{-1}f, f) \geq \delta \| f \|^2 \) for \( f \in \mathcal{H} \). By Theorem 4.1 of [2], \( R^{-1} \in \mathcal{L}(\mathcal{H}) \) and
\[ \| R^{-1} \| \leq \frac{1}{\delta} \| T^{-1} \|. \tag{40} \]

Let \( y = y_1 + iy_2 = \tau_{h_1, h_2}(x) \). Since \( T \) is hyponormal,
\[ \| T^{-1} \| = \| (Y - yI)_{-1} \| = \sup_{\lambda \in \sigma(Y)} |(\lambda - y)^{-1}| = \sup_{\lambda \in \sigma(Y)} |(\tau_{h_1, h_2}(\lambda) - \tau_{h_1, h_2}(x))^{-1}| \leq \text{Max} \left( \frac{1}{m_{h_1}}, \frac{1}{m_{h_2}} \right) \| (X - xI)^{-1} \|. \tag{41} \]

However, \( R = \tau_{\phi_1, \phi_2}(Y) - \tau_{\phi_1, \phi_2}(x) = \tau_{\phi_1, \phi_2}(X) - \tau_{\phi_1, \phi_2}(x) \); (40) and (41) imply (32).

We have to prove that
\[ \phi_{J_0}(\tau_{\phi_1, \phi_2}(X)) = \sigma_{\alpha}(\tau_{\phi_1, \phi_2}(X)). \tag{42} \]

Let \( \tau_{\phi_1, \phi_2}(x) \in \sigma_{\alpha}(\tau_{\phi_1, \phi_2}(X)) \), \( \{ f_n \} \) be a sequence of unit vectors in \( \mathcal{H} \) such that
\[ \| R f_n \| = \| (\tau_{\phi_1, \phi_2}(X) - \tau_{\phi_1, \phi_2}(x)I) f_n \| \to 0. \]
Then, from (37) we have $\|T'_n\| \to 0$. But $T$ is hyponormal,

$$\|(Y_j - y_j)T_n\| \to 0.$$  

By the same method as that used in Lemma 3, we have

$$\|\phi_j(X_j) - \phi_j(x_j)\|_f = \|\psi_j(Y_j) - \psi_j(y_j)\|_f \to 0;$$

thus $\tau_{\psi_{1,0},\phi_2}(x) \in \sigma_{\lambda}(\tau_{\phi_{1,0},\phi_2}(X))$ and therefore (42).

(iv) We consider $\phi_j$ as topological mapping in $R$, and define

$$\phi_j(s) = s\phi_j(s) + (1-t)s, \quad j = 1, 2, t \in [0, 1].$$

It is obvious that

$$\mu_{\phi_j(x)} \leq \mu_{\phi_j(x)}^{-1}, m_{\phi_j(x)} \geq t m_{\phi_j(x)}^{-1}, \quad t \in [0, 1].$$

Thus $N(\phi_j) \leq N(\phi_j)$, for $t \in [0, 1]$, and (42) can be generalized to

$$\sigma_{\lambda}(\tau_{\phi_{1,0},\phi_2}(X)) = \sigma_{\lambda}(\tau_{\phi_{1,0},\phi_2}(X)).$$  (43)

Then (1)-(4) follow from Lemma 3 and (43).

**Theorem 5.** If $X = X_1 + iX_2$ is hyponormal, $\phi_j$, $j = 1, 2$, are bounded real Baire functions on $\sigma(X_j)$, $j = 1, 2$, and

$$m(\phi_1, \phi_2) = \sup_{\phi_j \in \sigma(X_j)} \frac{\phi(j) - \phi(0)}{s_1 - s_2} < \infty,$$

then

$$\|\tau_{\phi_1,\phi_2}(X) - \tau_{\phi_1,\phi_2}(x)I^{-1}\| \geq m(\phi_1, \phi_2) \|X - xI^{-1}\|$$  (44)

for every $x \in \rho(X) \cap \tau_{\phi_1,\phi_2}(\rho(\tau_{\phi_1,\phi_2}(X)))$, where

$$m(\phi_1, \phi_2) = (\sqrt{2} \max(t_{\phi_1}, t_{\phi_2}))^{-1}.$$

**Proof.** It is obvious that

$$\|\tau_{\phi_1,\phi_2}(X) - \tau_{\phi_1,\phi_2}(x)I\| \leq 2 \sum_{j=1}^{2} \|\phi_j(x_j) - \phi_j(x)I\| \leq 2 \sum_{j=1}^{2} i_{\phi_j} \|X_j - x_jI\| \leq m(\phi_1, \phi_2)^{-2} \sum_{j=1}^{2} \|X_j - x_jI\|^2.$$  (45)
But it is well known that if \( A = A_1 + iA_2 \) is hyponormal then
\[
A^*A \geq A_1^2 + A_2^2.
\]
Thus \( \|(X - xI)f\|^2 \geq \sum_{j=1}^n \|(X_j - x_j I)f\|^2 \). Combining this inequality and (45), we obtain (44).

**Corollary 1.** Under the hypotheses of Theorems 4 and 5, let \( T = \tau_{\phi_1, \phi_2}(X) \). Then there are two finite positive constants \( M' \) and \( m' \) depending on \( \phi_1 \) and \( \phi_2 \) such that
\[
m' \inf_{\lambda \in \sigma(T)} \frac{1}{|\lambda - z|} \leq \|(T - zI)^{-1}\| \leq M' \inf_{\lambda \in \sigma(T)} \frac{1}{|\lambda - z|},
\]
for \( z \in (T) \).

**Proof.** Let \( z = \tau_{\phi_1, \phi_2}(x) \). Then
\[
\inf_{\lambda \in \sigma(T)} |\lambda - z| = \inf_{\lambda \in \sigma(X)} |\tau_{\phi_1, \phi_2}(\lambda) - \tau_{\phi_1, \phi_2}(x)|
\]
\[
= \inf_{\lambda_1 + i\lambda_2 \in \sigma(X)} \sqrt{\sum_{j=1}^2 (\phi_j(\lambda_j) - \phi_j(x_j))^2}
\]
But since \( 0 < m_{\phi_j} \leq t_{\phi_j} < \infty \), we have
\[
\min(m_{\phi_1}, m_{\phi_2}) \inf_{\lambda \in \sigma(X)} |\lambda - x| \leq \inf_{\lambda \in \sigma(T)} |\lambda - z| \leq \max(t_{\phi_1}, t_{\phi_2}) \inf_{\lambda \in \sigma(X)} |\lambda - x|.
\]
It is obvious that (32), (44) and (47) imply (46).

**Theorem 6.** Let \( X = X_1 + iX_2 \) be hyponormal operator. If (i) \( \phi_1 \in M(\sigma(X_1)) \), \( \phi_2 \in B_+(\sigma(X_2)) \) or (ii) \( \phi_1 \in B_+(\sigma(X_1)) \), \( \phi_2 \in M(\sigma(X_2)) \) then (1)–(4) hold.

**Proof.** We consider case (i) only. Let \( \phi_0(s) \equiv s \). We notice that
\[
\tau_{\phi_1, \phi_2} = \tau_{\phi_1, \phi_0} \tau_{\phi_0, \phi_2}
\]
and the mapping \( \tau_{\phi_0, \phi_2} \) satisfies the condition of Theorem 2, case (2). Thus we can suppose that \( \phi_2 = \phi_0 \).

If \( \tilde{m}(K_{\phi_1}) \geq 0 \), then \( \phi_1 \in B_+(\sigma(X_1)) \), so we are reduced to Theorem 2, case (2). Let us now suppose that \( \tilde{m}(K_{\phi_1}) < 0 \). Let
\[
k = (-2\tilde{m}(K_{\phi_1}))^{-1},
\]
$X'_1 = X_1/k$, and $x'_i = x_i/k$. By Lemma 1, we have

$$i[\phi(X_1), X_2] \geq -i[X'_1, X_2]/2.$$ 

Let $R = \tau_{a_0}(X) - \tau_{a_0}(x)I, T = X'_1 + iX_2 - (x'_1 + ix_2)I$. We have

$$\frac{1}{2}(R^*T + T^*R) = (X'_1 - x'_1I)(\phi_1(x_1)I)$$

$$+ (X_2 - ix_2I)^2 + \frac{i}{2} [\phi_1(x_1) + X_1, X_2] \geq (X_2 - ix_2I)^2,$$

i.e.,

$$\mathcal{A}(\mathcal{A}(Rf, Tf)) \geq \|(X_2 - ix_2I)f\|^2. \quad (48)$$

If $\tau_{a_0}(x) \in \sigma_a(\tau_{a_0}(x)), \text{ then there is a sequence of unit vectors } \{f_n\} \text{ such that } \|Rf_n\| \to 0. \text{ From (48), we have }$$

$$\|(X_2 - ix_2I)f_n\| \to 0$$

immediately, and hence $\|(\phi_1(x_1) - i\phi_1(x_1))f_n\| \to 0$. Thus

$$\sigma_a(\tau_{a_0}(x)) \subset \sigma_{a_0}(\tau_{a_0}(x)).$$

Then by the same method as that used in Theorem 4, we can prove (1)–(4).

Theorem 6 is more general than case (3) in Theorem 2.

4

Now we consider the case of semi-hyponormal operators. Let $\Omega_0 = (C_1, B_{C_1}, m), E$ be a closed subset in $C_1$, and

$$L^2(E) = \{f \mid f \in L^2(\Omega_0) \text{ and } f(z) = 0 \text{ for } z \in C_1 - E\}$$

be the subspace of Hilbert space $L^2(\Omega_0)$. Let $\phi \in M_0(E)$. If there is a finite non-negative number $a$ such that

$$|(P_+(\phi g), g) + a(\phi g, g)| \leq a\|g\|^2 + \|P_+(g)\|^2, \quad g \in L^2(E), \quad (49)$$

then we define $\delta_\phi$ as the minimum of $a(1 + a)^{-1}$, when $a$ varies over all non-negative numbers satisfying (49).

If $X = U|X|$ is semi-hyponormal and $U$ is unitary, then there exist the polar symbols [13]

$$|X|_\pm = st - \lim_{t \to \pm \infty} U^{-n}|X|U^n.$$
It can be proved that if $X^{-1} \in \mathcal{L}(\mathcal{H})$ then $|X|^{-1/2} \in \mathcal{L}(\mathcal{H})$ and

$$m(|X|^{-1/2} |X|^{-1/2}) \geq (\|X\| \|X^{-1}\|)^{-1}.$$  

Let $CA$ be the class of all functions $f$ analytic in $|z| < 1$ and continuous in $|z| \leq 1$. Let

$$\|\phi\| = \sup_{z \in C_1} |\phi(z)|$$

for any bounded function $C_1$.

**Theorem 7.** Let $X = U |X|$ be semi-hyponormal, where $U$ is unitary. If $\phi_1 \in M_0(\sigma(U))$, $t/\phi_2(t)$ is a monotonic increasing function,

$$\delta_{\phi_1} \leq m(|X|^{-1/2} |X|^{-1/2})$$  \hspace{1cm} (50)

or

$$\inf_{f \in CA} \| (\phi_1 - W)^{-2} (k \phi_1 W^{-1} + k^{-1} W/\phi_1 - f) \| \leq 1$$  \hspace{1cm} (51)

for all $|W| = 1$, where $k = (m(|X|^{-1/2} |X|^{-1/2}))^{1/2}$; and $\phi_2 \in M_0(E)(E = [m(|X|)], M(|X|))$ or $\sigma(|X|)$ when $\sigma(U) \neq C_1$ then (1) and (4) hold.

If furthermore there is a family $\phi_1(z) \in M_0(\sigma(U))$, $t \in [0, 1]$, such that $\phi_1(t)$ is a continuous function of $t$ for every fixed $z \in C_1$ and $\phi_1$ satisfies (50) or (51) (with $\phi_1 = \phi_1$) for all $t \in [0, 1]$, $\phi_{10}(z) \equiv z$ and $\phi_{11} = \phi_1$, then (2)–(3) also hold.

**Proof:** (i) In the first step, we have to prove that (51) implies (50). If $k = a^{1/2}(1 + a)^{-1/2}$, $g_+ = P_+(g)$, and $g_- = g - g_+$ then (49) is equivalent to

$$\inf_{|W| = 1} \left\{ k^{-1} \| (\phi_1 - W) g_+ \|^2 + k \| (\phi_1 - W) g_- \|^2 ight.$$  

$$- 2 \Re(W(k(\bar{\phi}_1 g_+ + g_-) + k^{-1}(\bar{\phi}_1 g_+ + g_-))) \right\} \geq 0.$$  \hspace{1cm} (52)

We construct the analytic function

$$q(z) = \exp \left\{ \frac{1}{\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \ln |\phi_1(e^{i\theta}) - W| \, d\theta \right\}.$$  \hspace{1cm}

Then

$$|q(z)| = |\phi_1(z) - W|^2,$$  \hspace{1cm} (53)
Put \( h_0(z) = q(z) g_-(\bar{z}^{-1})z^{-1}, h_1(z) = q(z) g_+(z) \). We can prove that (52) is equivalent to

\[
\inf_{|W|=1} \left\{ k \| h_0 \|^2 + k^{-1} \| h_1 \|^2 \right\} - 2 \mathcal{R} \left( \frac{1}{2\pi i} \int_{|z|=1} q(z)^{-1} \psi(z) h_0(z) h_1(z) \, dz \right) \geq 0, \quad (54)
\]

where \( \psi(z) = k \phi_1(z) W^{-1} + k^{-1} W \phi_1(z)^{-1} \). By virtue of (53), (51) implies (54) and hence (50) also.

(ii) We have to prove that

\[
\mathcal{R}(\|X| - W|X| \phi_1(U^*)) f, f \geq 0 \quad (55)
\]

for \( f \in \mathcal{H} \) and \( |W| = 1 \). Put \( a = \delta_{\phi_1} (1 - \delta_{\phi_1})^{-1} \). We consider the singular integral model in Theorem 1. From (49), we have

\[
\mathcal{R} \{ (P_+ (1 - W \phi_1(\cdot)) \alpha(\cdot) f(\cdot), \alpha(\cdot) f(\cdot)) \}
+ a(1 - W \phi_1(\cdot)) \alpha(\cdot) f(\cdot), \alpha(\cdot) f(\cdot)) \} \geq 0.
\]

Thus

\[
\mathcal{R}(\|T| - W|T| \phi_1(U^*)) f, f 
\geq \mathcal{R}[\beta(\cdot)(1 - W \phi_1(\cdot)) f(\cdot), f(\cdot)] 
- a((1 - W \phi_1(\cdot)) \alpha(\cdot) f(\cdot), \alpha(\cdot) f(\cdot))]. \quad (56)
\]

Let \( g(z) = (\mathcal{R}(1 - W \phi(z)))^{1/2} f(z) \). The right-hand side of inequality (56) is greater than or equal to

\[
((X_-, g, g) - a((X_+ - X_-) g, g)) = (1 + a) ((X_-, g, g) - \delta_{\phi_1}(X_+, g, g)) \geq 0,
\]

where \( X_+ g = (\beta + a^* a) g \) and \( X_- g = \beta g \). Thus (55) is proved.

(iii) Let \( T = \phi_1(U) X \) and \( \mathcal{U} = \phi_1(e^{i\theta}) \rho \). It is obvious that

\[
\| (T - \mathcal{U}) f \|^2 = \| (X - \rho I) f \|^2 + 2 \rho \mathcal{R}(\|X - \phi(e^{i\theta}) X \phi(U^*) f, f). \quad (57)
\]

Hence

\[
\| (T - \mathcal{U}) f \| \geq \| (X - \rho I) f \|, \quad (58)
\]

from (55) and (57).
Let $|X| = \int_{[m,M]} s \, dQ_s$ be the spectral decomposition of $|X|$, where $0 < m \leq M < \infty$. From (58), we obtain that

$$\|(\tau_{\phi_1,\phi_2}(X) - \tau_{\phi_1,\phi_2}(pe^{i\theta}I))f\| = \|((\phi_1(U)(\rho\phi_2(|X|)/\phi_2(\rho) - |X|)f + (T - \mathcal{H}I)f\| \phi_2(\rho)/\rho$$

$$\geq \{\|(\phi_2(|X|)/\phi_2(\rho) - |X|)f\|\} \phi_2(\rho)/\rho$$

$$\geq C \int_{[m,M]} \mathcal{H}(s, \rho) d(Q_s,f,f)/\|f\|,$$  (59)

where $C$ is a positive constant depending on $\rho$ and

$$\mathcal{H}(s, \rho) = \phi_2(s)/\phi_2(\rho)(1 - 2s/\rho + \phi_2(s)/\phi_2(\rho)).$$

We notice that, for every $\delta > 0$, there is a positive $\varepsilon$ such that

$$\inf_{|s - \rho| > \delta} \mathcal{H}(s, \rho) > \varepsilon.  \quad (60)$$

If $\|f\| = 1$ then (59) and (60) imply

$$\|(|X| - \rho I)f\|^2 \leq \delta^2 \|(\tau_{\phi_1,\phi_2}(X) - \tau_{\phi_1,\phi_2}(pe^{i\theta}I))f\|/(Ce) + \delta.  \quad (61)$$

(iv) If $\tau_{\phi_1,\phi_2}(pe^{i\theta}) \in \sigma_p(\tau_{\phi_1,\phi_2}(X))$ and $\rho \neq 0$, $\{f_n\}$ is the sequence of unit vectors in $\mathcal{H}$ such that

$$\|(\tau_{\phi_1,\phi_2}(X) - \tau_{\phi_1,\phi_2}(pe^{i\theta}I))f_n\| \to 0.  \quad (62)$$

Then by (61), we have $\lim_{n \to \infty} \|(|X| - \rho I)f_n\| \leq \delta$ for arbitrary $\delta > 0$. Thus

$$\|(|X| - \rho I)f_n\| \to 0.  \quad (63)$$

From (62) and (63) we can prove $\tau_{\phi_1,\phi_2}(pe^{i\theta}) \in \sigma_p(e(\tau_{\phi_1,\phi_2}(X)))$. Thus (1) and (4) hold.

The proof of the remaining part of Theorem 7 can be performed by a method similar to that used in Theorem 4.

Theorem 7 is a generalization of Theorem 3, case (3).

**Example.** Let $a_j$, $j = 0, 1, 2, \ldots, n$, and $b_j$, $j = 1, 2, \ldots, n$, be the points in the unit circle. If $r = \max_j(|a_j|, |b_j|)$ is sufficiently small, then

$$\phi_1(z) = e^{i\lambda} \prod_{j=0}^{n} \frac{z - a_j}{1 - \bar{a}_j z} \prod_{j=1}^{n} \frac{1 - b_j z}{z - b_j}$$

($\lambda$ is a real constant) satisfies the hypothesis of Theorem 7.
It is well known that if $X = X_1 + iX_2 \in \mathcal{L}(\mathscr{H})$ is self-adjoint, then Putnam's inequality \[8,9\] holds.

Thus if (3) holds for $T = T_1 + iT_2 = \tau_{\psi_1, \psi_2}(X)$, then we denote $\phi_j^{-1}$ by $\psi_j$ and we have

$$
\| [\psi_1(T_1), \psi_2(T_2)] \| \leq \frac{1}{2\pi} \int_{\sigma(T)} d\psi_1(x_1) d\psi_2(x_2),
$$

from (64). This generalizes Putnam's inequality (64).

**Corollary 2.** Let $T = T_1 + iT_2 \in \mathcal{L}(\mathscr{H})$, $T_j, j = 1, 2$, be self-adjoint. If $\psi_j \in M(\sigma(T_j)), j = 1, 2$, such that

and (i) $\psi_j^{-1} \in L^m(\sigma(\psi_j(T_j))), j = 1, 2$, $N(\psi_1^{-1})N(\psi_2^{-1}) < 1$ or (ii) one of $\psi_j^{-1} \in B_+(\sigma(\psi(T_j)))$, then (65) holds. If, furthermore, the planar Lebesgue measure of $\sigma(T)$ is zero, then $T$ is normal.

In the previous papers the author \[13\] and Li \[4,6\] obtained an inequality of Putnam's type for the semi-hyponormal operators.

**Theorem 8.** If $X = U |X|$ is semi-hyponormal, then

$$
\| |X| - U |X| U^* \| \leq \frac{1}{\pi} \int_{\rho e^{i\theta} \in \sigma(X)} dp d\theta.
$$

Similarly, we can obtain a generalization of (66) as follows.

**Corollary 3.** Let $T = U |T| \in \mathcal{L}(\mathscr{H})$, where $U$ is unitary. Suppose $\psi_1 \in M_0(\sigma(U)), \psi_2 \in M_0([m(|T|), M(|T|)])$ and

$$
\psi_2(|T|) - \psi_1(U) \psi_2(|T|) \psi_1(U)^* \geq 0.
$$

If

$$
\rho \geq m(|\psi_2(|T|)|)^{-1/2} |\psi_2(|T|)| - |\psi_2(|T|)|^{-1/2})
$$

(67)
or
\[ \inf_{f \in C^1(A)} \|(\psi_1^{-1} - W)^{-1}(k\psi_1^{-1}W - 1) + k^{-1}W\psi_1^{-1} - f\| \leq 1 \]
for all \( |W| = 1 \), where \( k \) is the square root of the right-hand side of (67), then
\[ \|\psi_2(T) - \psi_1(U)\psi_2(T)\psi_1(U)^*\| \leq \frac{1}{\pi} \int_{\rho \in \sigma(T)} d\psi_1(e^{i\theta}) d\psi_2(\rho). \]

If furthermore the planar Lebesgue measure of \( \sigma(T) \) is zero, then \( T \) is normal.

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